KEY

TO

TODHUNTER AND HOGG'S

PLANE TRIGONOMETRY

BY

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KEY

TO

PLANE TRIGONOMETRY.

CHAPTER I.

1. Let x denote the number of degrees in the larger angle, and y the number of degrees in the smaller angle. Then, since 10 grades are equal to 9 degrees, $x-y=9$, also $x+y=45$: hence we obtain $x=27$ and $y=18$.

2. In two-thirds of a right angle there are 60 degrees; let x denote the number of degrees in one part, then $60-x$ denotes the number of degrees in the other part, therefore the number of grades in this part is $\frac{10}{9}(60-x)$. Hence

$$x : \frac{10}{9}(60-x) :: 3 : 10; \text{ therefore } 10x=\frac{30}{9}(60-x),$$

therefore $9x=3(60-x)$, *therefore* $12x=180$, *therefore* $x=15$.

3. In half a right angle there are 45 degrees; let x denote the number of degrees in one part, then $45-x$ denotes the number of degrees in the other part, therefore the number of grades in this part is $\frac{10}{9}(45-x)$. Hence

$$x : \frac{10}{9}(45-x) :: 9 : 5, \text{ therefore } 5x=10(45-x),$$

therefore $\quad 15x=450$, therefore $x=30$.

4. $1' 5'' = \cdot 0105$ of a grade, $\frac{9}{10}$ of $\cdot 0105 = \cdot 00945$.

5. Let x denote the number of degrees in one part, then $n-x$ denotes the number of degrees in the other part. In x degrees there are $60x$ English minutes. In $n-x$ degrees there are $\frac{10}{9}(n-x)$ grades, and therefore $\frac{10}{9}\times 100(n-x)$ French minutes. Therefore

$$60x=\frac{1000}{9}(n-x),$$

therefore $\quad 1540x=1000n$, therefore $77x=50n$,

therefore $\quad x=\frac{50n}{77}$, and $n-x=\frac{27n}{77}$.

6. In one-third of a right angle there are 30 degrees; if this be taken as the unit of measurement an angle of 75 degrees must be denoted by $\frac{75}{30}$, that is by $\frac{5}{2}$, that is by $2\frac{1}{2}$.

7 Let x denote the number of grades in the unit Then an angle of $66\frac{2}{3}$ grades is denoted by $\frac{66\frac{2}{3}}{x}$, and this is equal to 20 Therefore

$$20x = 66\tfrac{2}{3} = \frac{200}{3}, \quad \text{therefore } x = \frac{10}{3}$$

Hence the number of degrees in the unit is $\frac{9}{10} \times \frac{10}{3}$, that is 3

8 Let $3x$ denote the number of sides in the equiangular polygon which has the greater number of sides, then $2x$ denotes the number of sides in the other equiangular polygon All the angles of the polygon of $2x$ sides are equal to $(4x-4)$ right angles, that is to $(4x-4)\,100$ grades, therefore each angle contains $\frac{(4x-4)\,100}{2x}$ grades All the angles of the polygon of $3x$ sides are equal to $(6x-4)$ right angles, that is to $(6x-4)\,90$ degrees, therefore each angle contains $\frac{(6x-4)\,90}{3x}$ degrees, therefore

$$\frac{(4x-4)\,100}{2x} = \frac{(6x-4)\,90}{3x},$$

therefore $(4x-4)\,5 = (6x-4)\,3$, therefore $2x = 8$, therefore $x = 4$ Thus one polygon has 8 sides and the other polygon has 12 sides

9 It is shewn in Art 9 that an angle expressed in centesimal seconds is transformed to English seconds by multiplying by $\frac{81}{250}$, and $\frac{81}{250} = \frac{324}{1000}$

10 Suppose one angle to contain x English seconds, and another to contain x French minutes The second angle then contains $100x$ French seconds, and therefore $\frac{81}{250} \times 100x$ English seconds Hence the ratio of the former angle to the latter is that of 1 to $\frac{8100}{250}$, or of 1 to $\frac{162}{5}$, or of 5 to 162

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$$\begin{array}{r|l} 60 & \underline{\;3\;00} \\ 60 & \underline{10\;05} \\ & \;\;1675 \end{array}$$

Thus $35^\circ\,10'\,3'' = 35^\circ\,1675$

$$\begin{array}{r} 35\;1675 \\ 3\;9075 \\ \hline 39\;0750 \end{array}$$

And $39^g\,0750 = 39^g\,7^`\,50^{``}$.

12 $69^g\,22^`\,50^{``} = 69^g{\cdot}225$

$$\begin{array}{r} 69\;225 \\ 6\;9225 \\ \hline 62\;3025 \\ 60 \\ \hline 18\;1500 \\ 60 \\ \hline 9\;00 \end{array}$$

CHAPTER II. ? 13

1 It is shewn in Art 8 that $\frac{D}{90}=\frac{G}{100}$, and it is shewn in Art 22 that $\frac{D}{180}=\frac{C}{\pi}$, so that $\frac{D}{90}=\frac{2C}{\pi}$ Therefore $\frac{D}{90}=\frac{G}{100}=\frac{2C}{\pi}$

In fact the three expressions denote the same thing, namely the ratio of the angle considered to a right angle.

2 The circular measure of the angle is $\frac{9}{10 \times 12}$, that is $\frac{3}{40}$ Therefore, by Art 22, the number of degrees in the angle is $\frac{3}{40}$ of $\frac{180}{\pi}$.

3 $5^0\,37'\,30''=337\frac{1}{2}$ minutes Thus the circular measure

$$=\frac{337\frac{1}{2}}{180 \times 60}\pi=\frac{675}{180 \times 60 \times 2}\pi=\frac{135}{180 \times 12 \times 2}\pi$$

$$=\frac{27}{36 \times 12 \times 2}\pi=\frac{\pi}{32}.$$

4. The angle contains 1 01 grades, therefore, by Art 24, the circular measure is $\frac{1\ 01}{200}\pi$, that is $\pi \times\ 00505$

5 Let x denote the number of degrees in the first angle, y the number in the second, and z the number in the third

The circular measure of the first angle is $\frac{x\pi}{180}$, and the circular measure of the second is $\frac{y\pi}{180}$, therefore $\frac{x\pi}{180}-\frac{y\pi}{180}=\frac{\pi}{10}$, therefore $x-y=18$

The number of grades in the second angle is $\frac{10y}{9}$, and the number of grades in the third is $\frac{10z}{9}$; therefore $\frac{10y}{9}+\frac{10z}{9}=30$, therefore $y+z=27$

Also $x+y=36$

From these three equations we have $x=27$, $y=9$, $z=18$

6 The circular measure of a right angle is $\frac{\pi}{2}$, and therefore the circular measure of five-sixteenths of a right angle is $\frac{5}{16}$ of $\frac{\pi}{2}$, that is $\frac{5\pi}{32}$.

The number of degrees is $\frac{5}{16}$ of 90, that is $\frac{450}{16}$, that is 28 125

The number of grades is $\frac{5}{16}$ of 100, that is $\frac{500}{16}$, that is 31 25

7. Let the numbers of degrees in the three angles be denoted respectively by $x-y$, x, and $x+y$ Then $x-y+x+x+y=180$, that is $3x=180$, therefore $x=60$

Also $x+y=2(x-y)$, therefore $3y=x=60$, therefore $y=20$

Hence in degrees the angles are denoted by 40, 60, and 80 Therefore in grades they will be denoted by $\frac{400}{9}$, $\frac{600}{9}$, and $\frac{800}{9}$. And in circular measure they will be denoted by $\frac{40\pi}{180}$, $\frac{60\pi}{180}$, and $\frac{80\pi}{180}$, that is by $\frac{2\pi}{9}$, $\frac{\pi}{3}$, and $\frac{4\pi}{9}$

8 Let the numbers of degrees in the three angles be denoted respectively by $x-y$, x, and $x+y$. Then $x-y+x+x+y=180$, that is $3x=180$, therefore $x=60$

The circular measure of the greatest angle is $\frac{(x+y)\pi}{180}$; thus

$$x-y \quad \frac{(x+y)\pi}{180} \quad 60 \quad \pi, \text{ therefore } (x-y)\pi=\frac{(x+y)\pi}{3}$$

therefore $3(x-y)=x+y$, therefore $y=\frac{x}{2}=30$

Thus the angles are 30^0, 60^0, and 90^0.

9. All the angles of the polygon are equal to $(2n-4)$ right angles, that is to $(2n-4)\frac{\pi}{2}$ in circular measure, that is to $(n-2)\pi$ Hence the circular measure of each angle is $\frac{(n-2)\pi}{n}$

10 During the quarter of an hour since twelve the long hand has described one-fourth of four right angles, that is a right angle The short hand has described one-twelfth of this, that is $\frac{1}{12}$ of a right angle Hence the angle between the hands at a quarter past twelve is $\frac{11}{12}$ of a right angle

The measure in degrees $=\frac{11}{12}$ of $90=\frac{11\times 15}{2}=\frac{165}{2}=82\frac{1}{2}$

The measure in grades $=\frac{11}{12}$ of $100=\frac{11\times 25}{3}=\frac{275}{3}=91\frac{2}{3}$.

The circular measure $=\frac{11}{12}$ of $\frac{\pi}{2}=\frac{11\pi}{24}$.

CHAPTER III

1 Let $\sin A = \frac{3}{5}$. Then we have

$$\cos A = \sqrt{(1-\sin^2 A)} = \sqrt{\left(1-\frac{9}{25}\right)} = \sqrt{\frac{16}{25}} = \frac{4}{5};$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{3}{5} \div \frac{4}{5} = \frac{3}{5} \times \frac{5}{4} = \frac{3}{4},$$

$$\cot A = \frac{1}{\tan A} = \frac{4}{3},$$

$$\sec A = \frac{1}{\cos A} = \frac{5}{4}, \quad \operatorname{cosec} A = \frac{1}{\sin A} = \frac{5}{3},$$

$$\operatorname{vers} A = 1 - \cos A = 1 - \frac{4}{5} = \frac{1}{5}$$

2 Let $\tan A = \frac{4}{3}$ Then we have

$$\sin A = \frac{\tan A}{\sqrt{(1+\tan^2 A)}} = \frac{\frac{4}{3}}{\sqrt{\left(1+\frac{16}{9}\right)}} = \frac{4}{3} \div \frac{5}{3} = \frac{4}{5},$$

$$\cos A = \frac{1}{\sqrt{(1+\tan^2 A)}} = \frac{1}{\sqrt{\left(1+\frac{16}{9}\right)}} = 1 \div \frac{5}{3} = \frac{3}{5},$$

$$\sec A = \frac{1}{\cos A} = \frac{5}{3}, \quad \operatorname{cosec} A = \frac{1}{\sin A} = \frac{5}{4},$$

$$\operatorname{vers} A = 1 - \cos A = 1 - \frac{3}{5} = \frac{2}{5}$$

3 Let $\cos A = \sqrt{\frac{2}{3}}$ Then we have

$$\sin A = \sqrt{(1-\cos^2 A)} = \sqrt{\left(1-\frac{2}{3}\right)} = \sqrt{\frac{1}{3}},$$

$$\tan A = \frac{\sin A}{\cos A} = \sqrt{\frac{1}{3}} \div \sqrt{\frac{2}{3}} = \frac{1}{\sqrt{2}},$$

$$\cot A = \frac{1}{\tan A} = \sqrt{2},$$

$$\sec A = \frac{1}{\cos A} = \sqrt{\frac{3}{2}}, \quad \operatorname{cosec} A = \frac{1}{\sin A} = \sqrt{3},$$

$$\operatorname{vers} A = 1 - \cos A = 1 - \sqrt{\frac{2}{3}}$$

4 $\sec^2\theta \operatorname{cosec}^2\theta = (1+\tan^2\theta)(1+\cot^2\theta) = 1+\tan^2\theta+\cot^2\theta+(\tan\theta\cot\theta)^2$
$$=1+\tan^2\theta+\cot^2\theta+1=\tan^2\theta+\cot^2\theta+2$$

5 $\sin^2\theta\tan\theta+\cos^2\theta\cot\theta+2\sin\theta\cos\theta=\dfrac{\sin^3\theta}{\cos\theta}+\dfrac{\cos^3\theta}{\sin\theta}+2\sin\theta\cos\theta$

$$=\frac{\sin^4\theta+\cos^4\theta+2\sin^2\theta\cos^2\theta}{\sin\theta\cos\theta}=\frac{(\sin^2\theta+\cos^2\theta)^2}{\sin\theta\cos\theta}=\frac{1}{\sin\theta\cos\theta}$$

$$=\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}=\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}=\tan\theta+\cot\theta$$

6 $2(\sin^6\theta+\cos^6\theta)=2(\sin^2\theta+\cos^2\theta)(\sin^4\theta-\sin^2\theta\cos^2\theta+\cos^4\theta)$
$$=2(\sin^4\theta-\sin^2\theta\cos^2\theta+\cos^4\theta),$$

therefore $\quad 2(\sin^6\theta+\cos^6\theta)-3(\sin^4\theta+\cos^4\theta)+1$
$$=-2\sin^2\theta\cos^2\theta-\sin^4\theta-\cos^4\theta+1$$
$$=1-(\sin^2\theta+\cos^2\theta)^2=1-1=0$$

7. $\sin^2\theta=\frac{3}{2}\cos\theta$, therefore $1-\cos^2\theta=\frac{3}{2}\cos\theta$,

therefore
$$\cos^2\theta+\frac{3}{2}\cos\theta=1$$

By solving this quadratic in the usual way we obtain $\cos\theta=\frac{1}{2}$ or -2, but only the former value is applicable, for $\cos\theta$ cannot be numerically greater than unity Hence $\cos\theta=\frac{1}{2}$, and therefore $\theta=\frac{\pi}{3}$.

8 $\sin\theta+\cos\theta=1$, therefore $\cos\theta=1-\sin\theta$, therefore $\cos^2\theta=(1-\sin\theta)^2$, therefore $1-\sin^2\theta=(1-\sin\theta)^2$, that is $(1-\sin\theta)(1+\sin\theta)=(1-\sin\theta)^2$ Therefore either $1-\sin\theta=0$, or $1+\sin\theta=1-\sin\theta$

Take $1-\sin\theta=0$, thus $\sin\theta=1$, therefore $\theta=\frac{\pi}{2}$.

Next take $1+\sin\theta=1-\sin\theta$, thus $\sin\theta=0$, therefore $\theta=0$

9 $\cot\theta=2\cos\theta$, therefore $\dfrac{\cos\theta}{\sin\theta}=2\cos\theta$

Therefore either $\cos\theta=0$, or $\dfrac{1}{\sin\theta}=2$

Take $\cos\theta=0$, then $\theta=\frac{\pi}{2}$. Next take $\dfrac{1}{\sin\theta}=2$, thus $\sin\theta=\frac{1}{2}$; therefore $\theta=\frac{\pi}{6}$.

10 $\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$, therefore $1 - \cos^2\theta - 2\cos\theta + \frac{1}{4} = 0$, therefore $\cos^2\theta + 2\cos\theta = \frac{5}{4}$ By solving this quadratic in the ordinary way we obtain $\cos\theta = \frac{1}{2}$, or $-\frac{5}{2}$; but only the former value is applicable, therefore $\theta = \frac{\pi}{3}$

11 $3\sec^4\theta + 8 = 10\sec^2\theta$; therefore $3\sec^4\theta - 10\sec^2\theta + 8 = 0$ By solving this quadratic in the ordinary way we obtain $\sec^2\theta = 2$ or $\frac{4}{3}$; therefore $\sec\theta = \sqrt{2}$ or $\frac{2}{\sqrt{3}}$, therefore $\theta = \frac{\pi}{4}$ or $\frac{\pi}{6}$

12. $\tan\theta + \cot\theta = 2$; therefore $\tan\theta + \frac{1}{\tan\theta} = 2$,

therefore $\tan^2\theta - 2\tan\theta + 1 = 0$, that is $(\tan\theta - 1)^2 = 0$,

therefore $\tan\theta = 1$, therefore $\theta = \frac{\pi}{4}$

13 $\sin(A - B) = \frac{1}{2}$, therefore $A - B = 30^\circ$,

$\cos(A + B) = \frac{1}{2}$, therefore $A + B = 60^\circ$,

from these two equations we obtain $A = 45^\circ$, and $B = 15^\circ$

14. $\tan(A + B) = \sqrt{3}$, therefore $A + B = 60^\circ$,

$\tan(A - B) = 1$; therefore $A - B = 45^\circ$,

from these two equations we obtain $A = 52\frac{1}{2}^\circ$, $B = 7\frac{1}{2}^\circ$

CHAPTER IV

1 $585^\circ = 360^\circ + 225^\circ$. Thus the Trigonometrical Ratios are the same as for an angle of 225°

$$\sin 225^\circ = \sin(180^\circ + 45^\circ) = -\sin 45^\circ = -\frac{1}{\sqrt{2}},$$

$$\cos 225^\circ = \cos(180^\circ + 45^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}}$$

2 $690^\circ = 360^\circ + 330^\circ$. Thus the Trigonometrical Ratios are the same as for an angle of 330°

$$\sin 330^\circ = \sin(180^\circ + 150^\circ) = -\sin 150^\circ = -\sin 30^\circ = -\frac{1}{2},$$

$$\cos 330^\circ = \cos(180^\circ + 150^\circ) = -\cos 150^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

3 $930^\circ = 720^\circ + 210^\circ$ Thus the Trigonometrical Ratios are the same as for an angle of 210°

$$\sin 210^\circ = \sin(180^\circ + 30^\circ) = -\sin 30^\circ = -\frac{1}{2};$$

$$\cos 210^\circ = \cos(180^\circ + 30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

4 $6420^\circ = 17 \times 360^\circ + 300^\circ$. Thus the Trigonometrical Ratios are the same as for an angle of 300°

$$\sin 300^\circ = \sin(180^\circ + 120^\circ) = -\sin 120^\circ = -\sin 60^\circ = -\frac{\sqrt{3}}{2},$$

$$\cos 300^\circ = \cos(180^\circ + 120^\circ) = -\cos 120^\circ = \cos 60^\circ = \frac{1}{2}$$

5 The smallest angle is 45°, the other angles are found by increasing successively by 180° thus all the angles are 45°, 225°, 405°, 585°, 765°

6 Since $\cos^2\theta = \frac{1}{2}$, we have $\cos\theta = \pm\frac{1}{\sqrt{2}}$

Take the upper sign, then the smallest value is 45°, and the others are $360^\circ - 45^\circ$, $360^\circ + 45^\circ$, $720^\circ - 45^\circ$, $720^\circ + 45^\circ$.

Take the lower sign, then the smallest value is 135°, and the others are $360^\circ - 135^\circ$, $360^\circ + 135^\circ$, $720^\circ - 135^\circ$, $720^\circ + 135^\circ$

7 $\operatorname{vers}\frac{n\pi}{4} = 1 - \cos\frac{n\pi}{4}$

Suppose $n=0$, then we have $1-\cos 0$, that is $1-1$, that is 0, next suppose $n=1$, then we have $1-\cos\frac{\pi}{4}$, that is $1-\frac{1}{\sqrt{2}}$, next suppose $n=2$, then we have $1-\cos\frac{\pi}{2}$, that is $1-0$, that is 1, next suppose $n=3$, then we have $1-\cos\frac{3\pi}{4}$, that is $1+\frac{1}{\sqrt{2}}$, next suppose $n=4$, then we have $1-\cos\pi$, that is $1+1$, that is 2 Then the values begin to recur in the inverse order, for $\cos\frac{5\pi}{4} = \cos\frac{3\pi}{4}$, $\cos\frac{6\pi}{4} = \cos\frac{2\pi}{4}$, $\cos\frac{7\pi}{4} = \cos\frac{\pi}{4}$, $\cos\frac{8\pi}{4} = \cos 2\pi = \cos 0$

Then the whole series recurs For $\cos\frac{9\pi}{4} = \cos\frac{\pi}{4}$, and so on.

8 Suppose $n=0$, then we have $\sin\frac{\pi}{6}$, that is $\frac{1}{2}$, next suppose $n=1$, then we have $\sin\left(\frac{\pi}{2}-\frac{\pi}{6}\right)$, that is $\sin\frac{\pi}{3}$, that is $\frac{\sqrt{3}}{2}$, next suppose $n=2$, then we have $\sin\left(\pi+\frac{\pi}{6}\right)$, that is $-\sin\frac{\pi}{6}$, that is $-\frac{1}{2}$, next suppose $n=3$,

then we have $\sin\left(\frac{3\pi}{2}-\frac{\pi}{6}\right)$, that is $-\sin\left(\frac{\pi}{2}-\frac{\pi}{6}\right)$, that is $-\sin\frac{\pi}{3}$, that is $-\frac{\sqrt{3}}{2}$

Then the values recur, for suppose $n=4$; then we have $\sin\left(2\pi+\frac{\pi}{6}\right)$, that is $\sin\frac{\pi}{6}$, and so on.

9 $\sin^3\theta=-\cos^3\theta$. Extract the cube root of both sides, thus $\sin\theta=-\cos\theta$, therefore $\frac{\sin\theta}{\cos\theta}=-1$, that is $\tan\theta=-1$, therefore $\theta=\frac{3\pi}{4}$

10 $2\sin^2\theta-5\cos\theta-4=0$, therefore $2(1-\cos^2\theta)-5\cos\theta-4=0$, therefore $2\cos^2\theta+5\cos\theta+2=0$ By solving this quadratic in the usual way we obtain $\cos\theta=-\frac{1}{2}$ or -2, but only the former value is applicable, therefore $\theta=\frac{2\pi}{3}$.

11 When $\theta=0$ we have $\cos\theta=1$ and $\sin\theta=0$, so that $\cos\theta-\sin\theta=1$ Let θ change from 0 to $\frac{\pi}{2}$, then $\cos\theta$ changes from 1 to 0, and $\sin\theta$ from 0 to 1; therefore $\cos\theta-\sin\theta$ changes from 1 to -1, vanishing when $\theta=\frac{\pi}{4}$

Let θ change from $\frac{\pi}{2}$ to π, then $\cos\theta$ changes from 0 to -1 and $\sin\theta$ from 1 to 0; thus $\cos\theta-\sin\theta$ remains negative It has its greatest numerical value, namely $-\sqrt{2}$, when $\theta=\frac{3\pi}{4}$. For we have

$$(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2=2(\cos^2\theta+\sin^2\theta)=2;$$

and thus $(\cos\theta-\sin\theta)^2$ has its greatest value when $\cos\theta+\sin\theta$ vanishes, that is when $\tan\theta=-1$, that is when $\theta=\frac{3\pi}{4}$

Let θ change from π to $\frac{3\pi}{2}$, then $\cos\theta-\sin\theta$ goes through the same numerical values, with a *contrary* sign, as when θ changes from 0 to $\frac{\pi}{2}$ this follows from Art 50.

Let θ change from $\frac{3\pi}{2}$ to 2π; then $\cos\theta-\sin\theta$ goes through the same numerical values, with a *contrary* sign, as when θ changes from $\frac{\pi}{2}$ to π this follows from Art 50

12 Let θ change from 0 to $\frac{\pi}{2}$, then $\cos^2\theta$ changes from 1 to 0, and $\sin^2\theta$ from 0 to 1, therefore $\cos^2\theta-\sin^2\theta$ changes from 1 to -1

Let θ change from $\frac{\pi}{2}$ to π, then $\cos^2\theta-\sin^2\theta$ changes from -1 to 1

Let θ change from π to $\frac{3\pi}{2}$, then $\cos^2\theta-\sin^2\theta$ goes through the same values as when θ changes from 0 to $\frac{\pi}{2}$

Let θ change from $\frac{3\pi}{2}$ to 2π, then $\cos^2\theta-\sin^2\theta$ goes through the same values as when θ changes from $\frac{\pi}{2}$ to π

13 $\tan\theta+\cot\theta=\tan\theta+\frac{1}{\tan\theta}$ Let θ change from 0 to $\frac{\pi}{2}$, then $\tan\theta$ changes from 0 to infinity Thus $\tan\theta+\frac{1}{\tan\theta}$ is always positive, and is infinite both when $\theta=0$, and when $\theta=\frac{\pi}{2}$ The least value is when $\theta=\frac{\pi}{4}$, for we have

$$\left(\tan\theta+\frac{1}{\tan\theta}\right)^2=\left(\tan\theta-\frac{1}{\tan\theta}\right)^2+4,$$

and thus the least value is when $\tan\theta-\frac{1}{\tan\theta}$ vanishes, that is when $\tan^2\theta=1$ Thus $\tan\theta+\cot\theta$ diminishes from infinity to 2, as θ changes from 0 to $\frac{\pi}{4}$, and then increases from 2 to infinity, as θ changes from $\frac{\pi}{4}$ to $\frac{\pi}{2}$.

Let θ change from $\frac{\pi}{2}$ to π, then $\tan\theta+\cot\theta$ goes in reverse order through the same numerical values, with a *contrary* sign, as when θ changes from 0 to $\frac{\pi}{2}$ this follows from Art 48

Let θ change from π to 2π, then $\tan\theta+\cot\theta$ goes through the same values as when θ changes from 0 to π this follows from Art 50

14 We know by Algebra that if a and b are unequal $2ab$ is less than a^2+b^2, and therefore $4ab$ is less than a^2+b^2+2ab, that is $4ab$ is less than $(a+b)^2$ Therefore $\frac{4ab}{(a+b)^2}$ is less than unity, and cannot be equal to the secant of any angle, for a secant is never less than unity

15 $\tan(A+90^0)=\frac{\sin(A+90^0)}{\cos(A+90^0)}=\frac{\cos A}{-\sin A}$, by Art 52, $=-\cot A$,

$$\cot(A+90^0)=\frac{1}{\tan(A+90^0)}=-\frac{1}{\cot A}=-\tan A,$$

$$\sec(A+90^\circ)=\frac{1}{\cos(A+90^\circ)}=\frac{1}{-\sin A},\text{ by Art. 52, }=-\operatorname{cosec}A,$$

$$\operatorname{cosec}(A+90^\circ)=\frac{1}{\sin(A+90^\circ)}=\frac{1}{\cos A},\text{ by Art 52, }=\sec A,$$

$$\operatorname{vers}(A+90^\circ)=1-\cos(A+90^\circ)=1+\sin A,\text{ by Art 52}$$

16 $\sin(270^\circ-A)=-\sin(90^\circ-A)$, by Art 50, $=-\cos A$

$\cos(270^\circ-A)=-\cos(90^\circ-A)$, by Art 50, $=-\sin A$.

17. $\sin(270^\circ-A)=-\sin(90^\circ+A)$, by Art 50,

$=-\cos A$, by Art. 52

$\cos(270^\circ+A)=-\cos(90^\circ+A)$, by Art. 50,

$=-(-\sin A)$, by Art. 52, $=\sin A$

18. $\sin(360^\circ-A)=-\sin(180^\circ-A)$, by Art 50,

$=-\sin A$, by Art 48

$\cos(360^\circ-A)=-\cos(180^\circ-A)$, by Art 50,

$=-(-\cos A)$, by Art 48, $=\cos A$

CHAPTER V

1 $\tan\theta=1$, the smallest value of θ is $\frac{\pi}{4}$, and the general value is $n\pi+\frac{\pi}{4}$, by Art 68

2 $\sin\theta=1$, the smallest value of θ is $\frac{\pi}{2}$, and the general value is $n\pi+(-1)^n\frac{\pi}{2}$, by Art. 66 This expression may be simplified, for first suppose n even, denote it by $2m$, so that we have $2m\pi+\frac{\pi}{2}$, next suppose n odd, denote it by $2m+1$, so that we have $(2m+1)\pi-\frac{\pi}{2}$, that is $2m\pi+\frac{\pi}{2}$ Hence both cases are included in the expression $2m\pi+\frac{\pi}{2}$, that is $(4m+1)\frac{\pi}{2}$

3. $\cos\theta=1$, the smallest value of θ is 0, and the general value is $2n\pi$, by Art 67.

4 $\cos\theta=-\frac{1}{2}$, the smallest value of θ is $\frac{2\pi}{3}$, and the general value is $2n\pi\pm\frac{2\pi}{3}$, by Art 67.

5 $\sin^2\theta=\sin^2\alpha$, therefore $\sin\theta=\pm\sin\alpha$ Take the upper sign, then the simplest solution is $\theta=\alpha$, and the general solution is $\theta=n\pi+(-1)^n\alpha$ Take the lower sign, then the simplest solution is $\theta=-\alpha$, and the general solution is $\theta=n\pi-(-1)^n\alpha$ The two expressions are included in the single expression $\theta=n\pi\pm\alpha$

This might also be obtained from a diagram in the manner of Arts 66, 67, and 68

6 Since $\operatorname{cosec}^2\theta=\frac{4}{3}$ we have $\sin^2\theta=\frac{3}{4}=\sin^2\frac{\pi}{3}$ hence, by Example 5, the general solution is $\theta=n\pi\pm\frac{\pi}{3}$

7 $\cos^2\theta=\cos^2\alpha$, therefore $\cos\theta=\pm\cos\alpha$ Take the upper sign, then the simplest solution is $\theta=\alpha$, and the general solution is $\theta=2n\pi\pm\alpha$ Take the lower sign, then the simplest solution is $\theta=\pi-\alpha$, and the general solution is $\theta=2n\pi\pm(\pi-\alpha)$ The two expressions are included in the single expression $\theta=m\pi\pm\alpha$

It will be seen that the result is the same as for Example 5, and this should be the case, for if $\cos^2\theta=\cos^2\alpha$, then $1-\cos^2\theta=1-\cos^2\alpha$, that is $\sin^2\theta=\sin^2\alpha$

8 Since $\sec^2\theta=2$, we have $\cos^2\theta=\frac{1}{2}=\cos^2\frac{\pi}{4}$, hence, by Example 7, the general solution is $\theta=n\pi\pm\frac{\pi}{4}$

9 $\tan^2\theta=\tan^2\alpha$, therefore $\tan\theta=\pm\tan\alpha$ Take the upper sign, then the simplest solution is $\theta=\alpha$, and the general solution is $\theta=n\pi+\alpha$ Take the lower sign, then the simplest solution is $\theta=-\alpha$, and the general solution is $\theta=n\pi-\alpha$ The two expressions are included in the single expression $\theta=n\pi\pm\alpha$.

The result is the same as for Example 7, and this should be the case, for if $\tan^2\theta=\tan^2\alpha$ then $1+\tan^2\theta=1+\tan^2\alpha$, therefore $\sec^2\theta=\sec^2\alpha$, by Art 34, therefore $\cos^2\theta=\cos^2\alpha$

10 $\tan^2\theta=\frac{1}{3}=\tan^2\frac{\pi}{6}$, hence, by Example 9, the general solution is $\theta=n\pi\pm\frac{\pi}{6}$

11 All the angles included in the expression $2n\pi\pm\alpha$ have the same cosine as α, by Art 67

Now by Art 45 $\sin(2n\pi+\alpha)=\sin\alpha$, and $\sin(2n\pi-\alpha)=\sin(-\alpha)=-\sin\alpha$. Thus the angles which have both the same sine and the same cosine as α are all comprised in the expression $2n\pi+\alpha$

12 $-\frac{1}{2}=\sin\left(\pi+\frac{\pi}{6}\right)=\sin\frac{7\pi}{6}$, and $-\frac{\sqrt{3}}{2}=\cos\left(\pi+\frac{\pi}{6}\right)=\cos\frac{7\pi}{6}$, hence, by Example 11, the required general value is $\theta=2n\pi+\frac{7\pi}{6}$

CHAPTER VI.

1. $$\frac{\cos A+\sin A}{\cos A-\sin A}=\frac{(\cos A+\sin A)^2}{(\cos A-\sin A)(\cos A+\sin A)}$$
$$=\frac{\cos^2 A+\sin^2 A+2\sin A\cos A}{\cos^2 A-\sin^2 A}=\frac{1+\sin 2A}{\cos 2A}$$
$$=\frac{\sin 2A}{\cos 2A}+\frac{1}{\cos 2A}=\tan 2A+\sec 2A$$

2. $$2\sin^2 A\sin^2 B+2\cos^2 A\cos^2 B$$
$$=\frac{(1-\cos 2A)(1-\cos 2B)}{2}+\frac{(1+\cos 2A)(1+\cos 2B)}{2}$$
$$=\frac{1-\cos 2A-\cos 2B+\cos 2A\cos 2B}{2}+\frac{1+\cos 2A+\cos 2B+\cos 2A\cos 2B}{2}$$
$$=1+\cos 2A\cos 2B$$

3. $$\tan(45^\circ+A)-\tan(45^\circ-A)$$
$$=\frac{\tan 45^\circ+\tan A}{1-\tan 45^\circ\tan A}-\frac{\tan 45^\circ-\tan A}{1+\tan 45^\circ\tan A}=\frac{1+\tan A}{1-\tan A}-\frac{1-\tan A}{1+\tan A}$$
$$=\frac{(1+\tan A)^2-(1-\tan A)^2}{1-\tan^2 A}=\frac{4\tan A}{1-\tan^2 A}=2\tan 2A$$

4. $$\sin 3A\operatorname{cosec} A-\cos 3A\sec A$$
$$=\frac{\sin 3A}{\sin A}-\frac{\cos 3A}{\cos A}=\frac{3\sin A-4\sin^3 A}{\sin A}-\frac{4\cos^3 A-3\cos A}{\cos A}$$
$$=3-4\sin^2 A-(4\cos^2 A-3)=6-4(\sin^2 A+\cos^2 A)=6-4=2$$

5. $$3\sin A-\sin 3A=3\sin A-(3\sin A-4\sin^3 A)$$
$$=4\sin^3 A=2\sin A\times 2\sin^2 A=2\sin A(1-\cos 2A)$$

6. $$\frac{\sin A+2\sin 3A+\sin 5A}{\sin 3A+2\sin 5A+\sin 7A}=\frac{\sin A+\sin 5A+2\sin 3A}{\sin 3A+\sin 7A+2\sin 5A}$$
$$=\frac{2\sin 3A\cos 2A+2\sin 3A}{2\sin 5A\cos 2A+2\sin 5A},\text{ by Art 84,}$$
$$=\frac{2\sin 3A(1+\cos 2A)}{2\sin 5A(1+\cos 2A)}=\frac{\sin 3A}{\sin 5A}$$

7. $$\frac{\sin(2A+B)}{\sin A}-2\cos(A+B)=\frac{\sin(A+B+A)-2\sin A\cos(A+B)}{\sin A}$$
$$=\frac{\sin(A+B)\cos A+\cos(A+B)\sin A-2\sin A\cos(A+B)}{\sin A}$$
$$=\frac{\sin(A+B)\cos A-\cos(A+B)\sin A}{\sin A}=\frac{\sin(A+B-A)}{\sin A}=\frac{\sin B}{\sin A}$$

8 $$4\sin A\cos^3 A - 4\cos A\sin^3 A = 4\sin A\cos A(\cos^2 A - \sin^2 A)$$
$$= 2\sin 2A\cos 2A = \sin 4A$$

9 $$\frac{\cos A - \cos 3A}{\sin 3A - \sin A} = \frac{2\sin 2A\sin A}{2\cos 2A\sin A}, \text{ by Art } 84,$$
$$= \frac{\sin 2A}{\cos 2A} = \tan 2A$$

10. $$\frac{\cos 2A - \cos 4A}{\sin 4A - \sin 2A} = \frac{2\sin 3A\sin A}{2\cos 3A\sin A}, \text{ by Art } 84,$$
$$= \frac{\sin 3A}{\cos 3A} = \tan 3A$$

11 $$\operatorname{cosec} 2A + \cot 4A = \frac{1}{\sin 2A} + \frac{\cos 4A}{\sin 4A}$$
$$= \frac{2\cos 2A}{2\cos 2A\sin 2A} + \frac{\cos 4A}{\sin 4A} = \frac{2\cos 2A + \cos 4A}{\sin 4A}$$
$$= \frac{2\cos 2A + 2\cos^2 2A - 1}{\sin 4A} = \frac{2\cos 2A(1+\cos 2A) - 1}{\sin 4A}$$
$$= \frac{2\cos 2A(1+\cos 2A)}{2\sin 2A\cos 2A} - \frac{1}{\sin 4A} = \frac{1+\cos 2A}{\sin 2A} - \frac{1}{\sin 4A}$$
$$= \frac{2\cos^2 A}{2\sin A\cos A} - \frac{1}{\sin 4A} = \frac{\cos A}{\sin A} - \frac{1}{\sin 4A} = \cot A - \operatorname{cosec} 4A$$

12 $$\cos^2(A-B) + \cos^2 B - 2\cos(A-B)\cos A\cos B$$
$$= \cos(A-B)\{\cos(A-B) - \cos A\cos B\}$$
$$+ \cos B\{\cos B - \cos(A-B)\cos A\}$$
$$= \cos(A-B)\sin A\sin B$$
$$+ \cos B\{\cos(A - \overline{A-B}) - \cos(A-B)\cos A\}$$
$$= \cos(A-B)\sin A\sin B + \cos B\sin A\sin(A-B)$$
$$= \sin A\{\cos(A-B)\sin B + \sin(A-B)\cos B\}$$
$$= \sin A\sin(A-B+B) = \sin A\sin A = \sin^2 A$$

13 $$\sin^2(A-B) + \sin^2 B + 2\sin(A-B)\sin B\cos A$$
$$= \sin(A-B)\{\sin(A-B) + \sin B\cos A\}$$
$$+ \sin B\{\sin B + \sin(A-B)\cos A\}$$
$$= \sin(A-B)\sin A\cos B$$
$$+ \sin B\{\sin(A - \overline{A-B}) + \sin(A-B)\cos A\}$$
$$= \sin(A-B)\sin A\cos B + \sin B\sin A\cos(A-B)$$
$$= \sin A\{\sin(A-B)\cos B + \cos(A-B)\sin B\}$$
$$= \sin A\sin(A-B+B) = \sin A\sin A = \sin^2 A$$

14 $$\frac{1-\tan^2(45^0-A)}{1+\tan^2(45^0-A)}=\frac{1-\dfrac{\sin^2(45^0-A)}{\cos^2(45^0-A)}}{1+\dfrac{\sin^2(45^0-A)}{\cos^2(45^0-A)}}$$

$$=\frac{\cos^2(45^0-A)-\sin^2(45^0-A)}{\cos^2(45^0-A)+\sin^2(45^0-A)}=\frac{\cos 2(45^0-A)}{1}$$

$$=\cos(90^0-2A)=\sin 2A.$$

15 $$\frac{4\tan A(1-\tan^2 A)}{(1+\tan^2 A)^2}=\frac{\dfrac{4\sin A}{\cos A}\left(1-\dfrac{\sin^2 A}{\cos^2 A}\right)}{\left(1+\dfrac{\sin^2 A}{\cos^2 A}\right)^2}$$

$$=\frac{4\sin A\cos A(\cos^2 A-\sin^2 A)}{(\cos^2 A+\sin^2 A)^2}=2\sin 2A\cos 2A$$

$$=\sin 4A$$

16 $$\sin A(1+\tan A)+\cos A(1+\cot A)$$

$$=\sin A\left(1+\frac{\sin A}{\cos A}\right)+\cos A\left(1+\frac{\cos A}{\sin A}\right)$$

$$=\sin A+\frac{\sin^2 A}{\cos A}+\cos A+\frac{\cos^2 A}{\sin A}$$

$$=\sin A+\frac{1-\cos^2 A}{\cos A}+\cos A+\frac{1-\sin^2 A}{\sin A}$$

$$=\sin A+\frac{1}{\cos A}-\cos A+\cos A+\frac{1}{\sin A}-\sin A$$

$$=\frac{1}{\cos A}+\frac{1}{\sin A}=\sec A+\operatorname{cosec} A$$

17. $$\frac{\sin 3A+\cos 3A}{\sin 3A-\cos 3A}=\frac{3\sin A-4\sin^3 A+4\cos^3 A-3\cos A}{3\sin A-4\sin^3 A-4\cos^3 A+3\cos A}$$

$$=\frac{3(\sin A-\cos A)-4(\sin^3 A-\cos^3 A)}{3(\sin A+\cos A)-4(\sin^3 A+\cos^3 A)}$$

$$=\frac{\sin A-\cos A}{\sin A+\cos A}\times\frac{3-4(\sin^2 A+\cos^2 A+\sin A\cos A)}{3-4(\sin^2 A+\cos^2 A-\sin A\cos A)}$$

$$=\frac{\sin A-\cos A}{\sin A+\cos A}\times\frac{-1-4\sin A\cos A}{-1+4\sin A\cos A}$$

$$=\frac{\dfrac{\sin A}{\cos A}-1}{\dfrac{\sin A}{\cos A}+1}\ \frac{1+2\sin 2A}{1-2\sin 2A}$$

$$=\frac{\tan A-1}{\tan A+1}\,\frac{1+2\sin 2A}{1-2\sin 2A}$$
$$=\tan(A-45^\circ)\,\frac{1+2\sin 2A}{1-2\sin 2A}$$

18 $\cos A+\cos(120^\circ-A)+\cos(120^\circ+A)$
$=\cos A+\cos 120^\circ\cos A+\sin 120^\circ\sin A+\cos 120^\circ\cos A-\sin 120^\circ\sin A$
$=\cos A+2\cos 120^\circ\cos A=\cos A-\cos A=0$

19 $4\sin A\sin(60^\circ-A)\sin(60^\circ+A)$
$=4\sin A(\sin^2 60^\circ-\sin^2 A)$, by Art 83,
$=4\sin A\left(\frac{3}{4}-\sin^2 A\right)$
$=3\sin A-4\sin^3 A=\sin 3A$

20 $4\cos A\cos(60^\circ+A)\cos(60^\circ-A)$
$=4\cos A(\cos^2 A-\sin^2 60^\circ)$, by Art 83,
$=4\cos A\left(\cos^2 A-\frac{3}{4}\right)$
$=4\cos^3 A-3\cos A=\cos 3A$

21 $\tan A\tan(60^\circ+A)\tan(120^\circ+A)$
$$=\frac{\sin A\sin(60^\circ+A)\sin(120^\circ+A)}{\cos A\cos(60^\circ+A)\cos(120^\circ+A)}$$
$$=-\frac{\sin A\sin(60^\circ+A)\sin(60^\circ-A)}{\cos A\cos(60^\circ+A)\cos(60^\circ-A)},\text{ by Art 48,}$$
$$=-\frac{\sin 3A}{\cos 3A},\text{ by Examples 19 and 20, }=-\tan 3A.$$

22 $\tan A+\tan(60^\circ+A)+\tan(120^\circ+A)$
$=\tan A+\tan(60^\circ+A)-\tan(60^\circ-A)$, by Art 48,
$$=\tan A+\frac{\tan 60^\circ+\tan A}{1-\tan 60^\circ\tan A}-\frac{\tan 60^\circ-\tan A}{1+\tan 60^\circ\tan A}$$
$$=\tan A+\frac{(\tan 60^\circ+\tan A)(1+\tan 60^\circ\tan A)-(\tan 60^\circ-\tan A)(1-\tan 60^\circ\tan A)}{1-\tan^2 60^\circ\tan^2 A}$$
$$=\tan A+\frac{2\tan^2 60^\circ\tan A+2\tan A}{1-\tan^2 60^\circ\tan^2 A}$$
$$=\tan A+\frac{8\tan A}{1-3\tan^2 A}=\frac{9\tan A-3\tan^3 A}{1-3\tan^2 A}$$
$=3\tan 3A$

23 $\cot A+\cot(60^\circ+A)+\cot(120^\circ+A)$

$$=\frac{1}{\tan A}+\frac{1}{\tan(60^\circ+A)}-\frac{1}{\tan(60^\circ-A)}$$

$$=\frac{1}{\tan A}+\frac{1-\tan 60^\circ\tan A}{\tan 60^\circ+\tan A}-\frac{1+\tan 60^\circ\tan A}{\tan 60^\circ-\tan A}$$

$$=\frac{1}{\tan A}+\frac{(1-\tan 60^\circ\tan A)(\tan 60^\circ-\tan A)-(1+\tan 60^\circ\tan A)(\tan 60^\circ+\tan A)}{\tan^2 60^\circ-\tan^2 A}$$

$$=\frac{1}{\tan A}-\frac{2\tan^2 60^\circ\tan A+2\tan A}{\tan^2 60^\circ-\tan^2 A}$$

$$=\frac{1}{\tan A}-\frac{8\tan A}{3-\tan^2 A}=\frac{3-9\tan^2 A}{3\tan A-\tan^3 A}$$

$$=\frac{3}{\tan 3A}=3\cot 3A$$

24 $\cot A\cot(60^\circ+A)+\cot(60^\circ+A)\cot(120^\circ+A)+\cot(120^\circ+A)\cot A$

$$=\frac{1}{\tan A\tan(60^\circ+A)}+\frac{1}{\tan(60^\circ+A)\tan(120^\circ+A)}+\frac{1}{\tan(120^\circ+A)\tan A}$$

$$=\frac{\tan(120^\circ+A)+\tan A+\tan(60^\circ+A)}{\tan A\tan(60^\circ+A)\tan(120^\circ+A)}$$

$=\dfrac{3\tan 3A}{-\tan 3A}$, by Examples 21 and 22, $=-3$

25 $\sin^3 A=\frac{1}{4}\{3\sin A-\sin 3A\}$,

$$\sin^3(120^\circ+A)=\frac{1}{4}\{3\sin(120^\circ+A)-\sin 3(120^\circ+A)\}$$

$$=\frac{1}{4}\{3\sin(120^\circ+A)-\sin 3A\},$$

$$\sin^3(240^\circ+A)=\frac{1}{4}\{3\sin(240^\circ+A)-\sin 3(240^\circ+A)\}$$

$$=\frac{1}{4}\{3\sin(240^\circ+A)-\sin 3A\}$$

By addition we obtain

$$\frac{3}{4}\{\sin A+\sin(120^\circ+A)+\sin(240^\circ+A)\}-\frac{3}{4}\sin 3A,$$

that is $-\frac{3}{4}\sin 3A$, for

$$\sin A+\sin(120^\circ+A)+\sin(240^\circ+A)$$
$$=\sin A+\sin(60^\circ-A)-\sin(60^\circ+A)$$
$$=\sin A+\sin 60^\circ\cos A-\cos 60^\circ\sin A-\sin 60^\circ\cos A-\cos 60^\circ\sin A$$
$$=\sin A-2\cos 60^\circ\sin A=\sin A-\sin A=0$$

26 $\sin 3A \sin^3 A + \cos 3A \cos^3 A$

$= (3 \sin A - 4 \sin^3 A) \sin^3 A + (4 \cos^3 A - 3 \cos A) \cos^3 A$

$= 3 (\sin^4 A - \cos^4 A) - 4 \sin^6 A + 4 \cos^6 A$

$= 3 (\sin^4 A - \cos^4 A) (\sin^2 A + \cos^2 A) - 4 \sin^6 A + 4 \cos^6 A$

$= \cos^6 A - 3 \cos^4 A \sin^2 A + 3 \cos^2 A \sin^4 A - \sin^6 A$

$= (\cos^2 A - \sin^2 A)^3 = \cos^3 2A$

27 $\dfrac{\cos^3 A \sin 3A}{3} + \dfrac{\sin^3 A \cos 3A}{3}$

$= \dfrac{1}{12} (3 \cos A + \cos 3A) \sin 3A + \dfrac{1}{12} (3 \sin A - \sin 3A) \cos 3A$

$= \dfrac{1}{4} (\sin 3A \cos A + \cos 3A \sin A)$

$= \dfrac{1}{4} \sin (3A + A) = \dfrac{1}{4} \sin 4A$

28 $\cos nA \cos (n+2) A = \cos \{(n+1) A - A\} \cos \{(n+1) A + A\}$

$= \cos^2 (n+1) A - \sin^2 A$, by Art 83,

therefore $\cos nA \cos (n+2) A - \cos^2 (n+1) A + \sin^2 A = 0$

29 $\dfrac{\sin A \pm \sin nA + \sin (2n-1) A}{\cos A \pm \cos nA + \cos (2n-1) A}$

$= \dfrac{\sin A + \sin (2n-1) A \pm \sin nA}{\cos A + \cos (2n-1) A \pm \cos nA}$

$= \dfrac{2 \sin nA \cos (n-1) A \pm \sin nA}{2 \cos nA \cos (n-1) A \pm \cos nA}$, by Art 84,

$= \dfrac{\sin nA \{2 \cos (n-1) A \pm 1\}}{\cos nA \{2 \cos (n-1) A \pm 1\}} = \dfrac{\sin nA}{\cos nA} = \tan nA$

30 $\sin nA \operatorname{cosec}^2 A \sec A - \cos nA \sec^2 A \operatorname{cosec} A$

$= \dfrac{\sin nA}{\cos A \sin^2 A} - \dfrac{\cos nA}{\cos^2 A \sin A}$

$= \dfrac{\sin nA \cos A - \cos nA \sin A}{\sin^2 A \cos^2 A} = \dfrac{4 \sin (nA - A)}{4 \sin^2 A \cos^2 A}$

$= \dfrac{4 \sin (nA - A)}{\sin^2 2A} = 4 \sin (nA - A) \operatorname{cosec}^2 2A$

31. $\cos 10A + \cos 8A + 3 \cos 4A + 3 \cos 2A$

$= 2 \cos 9A \cos A + 6 \cos 3A \cos A$, by Art 84,

$= 2 \cos A (\cos 9A + 3 \cos 3A)$

$= 2 \cos A (4 \cos^3 3A - 3 \cos 3A + 3 \cos 3A)$

$= 8 \cos A \cos^3 3A$

32 $\cot A+\cot 2A+\cot 4A=\dfrac{\cos A}{\sin A}+\dfrac{\cos 2A}{\sin 2A}+\dfrac{\cos 4A}{\sin 4A}$

$$=\frac{2\cos^2 A}{2\sin A\cos A}+\frac{\cos 2A}{\sin 2A}+\frac{\cos 4A}{\sin 4A}=\frac{1+2\cos 2A}{\sin 2A}+\frac{\cos 4A}{\sin 4A}$$

$$=\frac{2\cos 2A(1+2\cos 2A)}{2\sin 2A\cos 2A}+\frac{\cos 4A}{\sin 4A}$$

$$=\frac{1}{\sin 4A}\{2\cos 2A+4\cos^2 2A+\cos 4A\}$$

$$=\frac{1}{\sin 4A}\{2\cos 2A+2(1+\cos 4A)+\cos 4A\}$$

$$=\operatorname{cosec} 4A\,\{2+2\cos 2A+3\cos 4A\}$$

33 $\dfrac{2\sin 2A+2\cos 2A}{\cos A-\sin A-\cos 3A+\sin 3A}=\dfrac{2(\sin 2A+\cos 2A)}{\cos A-\cos 3A+\sin 3A-\sin A}$

$$=\frac{2(\sin 2A+\cos 2A)}{2\sin 2A\sin A+2\cos 2A\sin A}\text{, by Art 84,}$$

$$=\frac{2(\sin 2A+\cos 2A)}{2(\sin 2A+\cos 2A)\sin A}=\frac{1}{\sin A}$$

34 $(\cos A-\sin 3A)^2+2\cos A\sin 3A(\cos A-\sin A)^2$

$$=\cos^2 A+\sin^2 3A-2\cos A\sin 3A+2\cos A\sin 3A(1-2\sin A\cos A)$$

$$=\cos^2 A+\sin^2 3A-2\cos A\sin 3A\sin 2A$$

$$=\cos A\{\cos A-\sin 3A\sin 2A\}+\sin 3A\{\sin 3A-\cos A\sin 2A\}$$

$$=\cos A\{\cos(3A-2A)-\sin 3A\sin 2A\}$$
$$+\sin 3A\{\sin(2A+A)-\cos A\sin 2A\}$$

$$=\cos A\cos 3A\cos 2A+\sin 3A\sin A\cos 2A$$

$$=\cos 2A\{\cos 3A\cos A+\sin 3A\sin A\}$$

$$=\cos 2A\cos(3A-A)=\cos 2A\cos 2A=\cos^2 2A$$

35 $\cos^6 A-\sin^6 A=(\cos^2 A-\sin^2 A)(\cos^4 A+\sin^4 A+\sin^2 A\cos^2 A)$

$$=\cos 2A(\cos^4 A+\sin^4 A+\sin^2 A\cos^2 A)$$

$$=\cos 2A\{(\cos^2 A+\sin^2 A)^2-\sin^2 A\cos^2 A\}$$

$$=\cos 2A\{1-\sin^2 A\cos^2 A\}=\cos 2A\left\{1-\frac{\sin^2 2A}{4}\right\}$$

36 $\sin 5A=\sin(3A+2A)=\sin 3A\cos 2A+\cos 3A\sin 2A$

$$=(3\sin A-4\sin^3 A)(1-2\sin^2 A)+(4\cos^3 A-3\cos A)2\sin A\cos A$$

$$=(3\sin A-4\sin^3 A)(1-2\sin^2 A)+(4\cos^2 A-3)2\sin A\cos^2 A$$

$$=(3\sin A-4\sin^3 A)(1-2\sin^2 A)+(1-4\sin^2 A)2\sin A(1-\sin^2 A)$$

$$=5\sin A-20\sin^3 A+16\sin^5 A$$

37 $$\tan\left(\frac{\pi}{4}-\theta\right)+\cot\left(\frac{\pi}{4}-\theta\right)=4,$$

therefore $$\frac{\sin\left(\frac{\pi}{4}-\theta\right)}{\cos\left(\frac{\pi}{4}-\theta\right)}+\frac{\cos\left(\frac{\pi}{4}-\theta\right)}{\sin\left(\frac{\pi}{4}-\theta\right)}=4,$$

therefore $$\sin^2\left(\frac{\pi}{4}-\theta\right)+\cos^2\left(\frac{\pi}{4}-\theta\right)=4\sin\left(\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}-\theta\right),$$

therefore $$1=2\sin\left(\frac{\pi}{2}-2\theta\right)=2\cos 2\theta,$$

therefore $$\cos 2\theta=\frac{1}{2},$$

therefore $$2\theta=2n\pi\pm\frac{\pi}{3},$$

therefore $$\theta=n\pi\pm\frac{\pi}{6}$$

38. $\sin 4\theta+\sin\theta=0$, therefore $2\sin\frac{5\theta}{2}\cos\frac{3\theta}{2}=0$ by Art 84, therefore either $\sin\frac{5\theta}{2}=0$, or $\cos\frac{3\theta}{2}=0$ The former gives $\frac{5\theta}{2}=n\pi$, and the latter gives $\frac{3\theta}{2}=2n\pi\pm\frac{\pi}{2}$, which may be expressed more simply as $\frac{3\theta}{2}=m\pi+\frac{\pi}{2}$

Or we might proceed thus $\sin 4\theta=-\sin\theta$, therefore $\sin 4\theta=\sin(\pi+\theta)$ Thus 4θ and $\pi+\theta$ must be angles which have the same sine, and therefore all the solutions are contained in $4\theta=n\pi+(-1)^n(\pi+\theta)$

39 $\sin 7\theta-\sin\theta=\sin 3\theta$, therefore $2\sin 3\theta\cos 4\theta=\sin 3\theta$, therefore either $\sin 3\theta=0$, or $2\cos 4\theta=1$ The former gives $3\theta=n\pi$; and the latter gives $4\theta=2n\pi\pm\frac{\pi}{3}$

40 $\sin\theta+\cos\theta=\frac{1}{\sqrt{2}}$, therefore $\frac{\cos\theta}{\sqrt{2}}+\frac{\sin\theta}{\sqrt{2}}=\frac{1}{2}$,

therefore $\cos\left(\theta-\frac{\pi}{4}\right)=\frac{1}{2}$, therefore $\theta-\frac{\pi}{4}=2n\pi\pm\frac{\pi}{3}$

41 By Example 36 we have

$$\sin 5\theta=5\sin\theta-20\sin^3\theta+16\sin^5\theta.$$

Thus $$5 \sin\theta - 20 \sin^3\theta + 16 \sin^5\theta = 16 \sin^5\theta,$$
therefore $$5 \sin\theta - 20 \sin^3\theta = 0,$$
therefore $$\text{either } \sin\theta = 0 \text{ or } \sin^2\theta = \frac{1}{4}$$

The former gives $\theta = n\pi$, the latter gives $\sin^2\theta = \sin^2\frac{\pi}{6}$, and therefore $\theta = n\pi \pm \frac{\pi}{6}$, by Example V 5

42 $\cos 3\theta + \cos 2\theta + \cos\theta = 0$, therefore $\cos 2\theta + 2\cos 2\theta\cos\theta = 0$, therefore either $\cos 2\theta = 0$, or $\cos\theta = -\frac{1}{2}$ The former gives $2\theta = n\pi + \frac{\pi}{2}$, as in Example 38, and the latter gives $\theta = 2n\pi \pm \frac{2\pi}{3}$.

43 $\sin 3\theta + \sin 2\theta + \sin\theta = 0$, therefore $\sin 2\theta + 2\sin 2\theta\cos\theta = 0$, therefore either $\sin 2\theta = 0$, or $\cos\theta = -\frac{1}{2}$ The former gives $2\theta = n\pi$, and the latter gives $\theta = 2n\pi \pm \frac{2\pi}{3}$

44 $\tan\theta + \tan\left(\frac{\pi}{4} + \theta\right) = 2$, therefore $\tan\theta + \frac{1 + \tan\theta}{1 - \tan\theta} = 2$,

therefore $$\tan\theta - \tan^2\theta + 1 + \tan\theta = 2 - 2\tan\theta,$$
therefore $$\tan^2\theta - 4\tan\theta + 1 = 0,$$
therefore $$\frac{\sin^2\theta}{\cos^2\theta} - \frac{4\sin\theta}{\cos\theta} + 1 = 0,$$
therefore $$\sin^2\theta + \cos^2\theta = 4\sin\theta\cos\theta,$$
therefore $$1 = 4\sin\theta\cos\theta = 2\sin 2\theta,$$
therefore $$\sin 2\theta = \frac{1}{2}, \text{ therefore } 2\theta = n\pi + (-1)^n\frac{\pi}{6}$$

45 $\tan 2\theta + \cot\theta = 8\cos^2\theta$, therefore $\frac{\sin 2\theta}{\cos 2\theta} + \frac{\cos\theta}{\sin\theta} = 8\cos^2\theta$,

therefore $$\sin 2\theta\sin\theta + \cos 2\theta\cos\theta = 8\cos^2\theta\sin\theta\cos 2\theta,$$
therefore $$\cos(2\theta - \theta) = 8\cos^2\theta\sin\theta\cos 2\theta,$$
therefore $$\text{either } \cos\theta = 0, \text{ or } 1 = 8\cos\theta\sin\theta\cos 2\theta$$

The former gives $\theta = n\pi + \frac{\pi}{2}$, the latter gives
$$1 = 4\sin 2\theta\cos 2\theta = 2\sin 4\theta,$$
so that $$\sin 4\theta = \frac{1}{2}, \text{ and } 4\theta = n\pi + (-1)^n\frac{\pi}{6}$$

46 $\tan\left(\frac{\pi}{4}+\theta\right)=3\tan\left(\frac{\pi}{4}-\theta\right)$,

therefore $$\tan\left(\frac{\pi}{4}+\theta\right)=3\cot\left(\frac{\pi}{4}+\theta\right)=\frac{3}{\tan\left(\frac{\pi}{4}+\theta\right)},$$

therefore $$\tan^2\left(\frac{\pi}{4}+\theta\right)=3=\tan^2\frac{\pi}{3},$$

therefore $$\frac{\pi}{4}+\theta=n\pi\pm\frac{\pi}{3},\text{ by Example V 9}$$

CHAPTER VII. 1/0

1 Here $\frac{A}{2}$ lies between 225° and 315°, thus $\sin\frac{A}{2}$ is negative, and is numerically greater than $\cos\frac{A}{2}$, hence

$$\sin\frac{A}{2}+\cos\frac{A}{2}=-\sqrt{(1+\sin A)},\quad \sin\frac{A}{2}-\cos\frac{A}{2}=-\sqrt{(1-\sin A)}$$

therefore $$2\sin\frac{A}{2}=-\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)}$$

2 Here $\frac{A}{2}$ lies between 405° and 495°, thus $\sin\frac{A}{2}$ is positive, and is numerically greater than $\cos\frac{A}{2}$, hence

$$\sin\frac{A}{2}+\cos\frac{A}{2}=\sqrt{(1+\sin A)},\quad \sin\frac{A}{2}-\cos\frac{A}{2}=\sqrt{(1-\sin A)}$$

therefore $$2\cos\frac{A}{2}=\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)}$$

3 Here $\frac{A}{2}$ lies between -45° and -135°, thus $\sin\frac{A}{2}$ is negative, and is numerically greater than $\cos\frac{A}{2}$, hence

$$\sin\frac{A}{2}+\cos\frac{A}{2}=-\sqrt{(1+\sin A)},\quad \sin\frac{A}{2}-\cos\frac{A}{2}=-\sqrt{(1-\sin A)}$$

therefore $$2\sin\frac{A}{2}=-\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)}$$

4 The proposed formula must have arisen from

$$\sin A+\cos A=-\sqrt{(1+\sin 2A)},\quad \sin A-\cos A=\sqrt{(1-\sin 2A)},$$

the former shews that A must lie between $2n\pi+\frac{3\pi}{4}$ and $2n\pi+\frac{7\pi}{4}$, and the

latter shews that A must lie between $2m\pi+\frac{\pi}{4}$ and $2m\pi+\frac{5\pi}{4}$, hence, by combining these results, it follows that A must lie between $2n\pi+\frac{3\pi}{4}$ and $2n\pi+\frac{5\pi}{4}$ See Art 101.

5 The proposed formula must have arisen from

$$\sin A+\cos A=-\sqrt{(1+\sin 2A)},\quad \sin A-\cos A=-\sqrt{(1-\sin 2A)},$$

the former shews that A must lie between $2n\pi+\frac{3\pi}{4}$ and $2n\pi+\frac{7\pi}{4}$, and the latter shews that A must lie between $2m\pi+\frac{5\pi}{4}$ and $2m\pi+\frac{9\pi}{4}$, hence, by combining these results, it follows that A must lie between $2n\pi+\frac{5\pi}{4}$ and $2n\pi+\frac{7\pi}{4}$.

6 The proposed formula must have arisen from

$$\sin A+\cos A=\sqrt{(1+\sin 2A)},\quad \sin A-\cos A=-\sqrt{(1-\sin 2A)},$$

the former shews that A must lie between $2n\pi-\frac{\pi}{4}$ and $2n\pi+\frac{3\pi}{4}$, and the latter shews that A must lie between $2m\pi+\frac{5\pi}{4}$ and $2m\pi+\frac{9\pi}{4}$, that is, between $2(m+1)\pi-\frac{3\pi}{4}$ and $2(m+1)\pi+\frac{\pi}{4}$ hence, by combining these results, it follows that A must lie between $2n\pi-\frac{\pi}{4}$ and $2n\pi+\frac{\pi}{4}$

7 Let A denote the given angle, and m the given ratio Let x denote one of the two parts, and therefore $A-x$ the other Then $\sin x=m\sin(A-x)$, thus $\sin x=m(\sin A\cos x-\cos A\sin x)$ Divide by $\cos x$; thus

$$\tan x=m(\sin A-\cos A\tan x),$$

therefore
$$\tan x=\frac{m\sin A}{1+m\cos A}$$

Thus $\tan x$ is known, and therefore x is known

8 Let A denote the given angle, and m the given ratio Let x denote one of the two parts, and therefore $A-x$ the other Then $\cos x=m\cos(A-x)$, thus $\cos x=m(\cos A\cos x+\sin A\sin x)$ Divide by $\cos x$, thus

$$1=m(\cos A+\sin A\tan x),$$

therefore
$$\tan x=\frac{1-m\cos A}{m\sin A}$$

Thus $\tan x$ is known, and therefore x is known

9 Let A denote the given angle, and m the given ratio Let x denote one of the two parts, and therefore $A-x$ the other Then $\tan x = m\tan(A-x)$,

thus $$\tan x = \frac{m(\tan A - \tan x)}{1+\tan A \tan x},$$

therefore $$\tan x(1+\tan A \tan x) = m(\tan A - \tan x)$$

Thus we have a quadratic equation from which the value of $\tan x$ may be found

Or we may proceed thus,

$$\tan x = m\tan(A-x), \text{ therefore } \frac{\sin x}{\cos x} = \frac{m\sin(A-x)}{\cos(A-x)},$$

therefore $$2\sin x\cos(A-x) = 2m\sin(A-x)\cos x,$$

therefore $$\sin A + \sin(2x-A) = m\{\sin A + \sin(A-2x)\}$$
$$= m\{\sin A - \sin(2x-A)\},$$

therefore $$(m+1)\sin(2x-A) = (m-1)\sin A$$

Thus $\sin(2x-A)$ is known, and therefore $2x-A$ is known, and therefore x is known

10 By Art 87, $$\sin A = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}} = \frac{2(2-\sqrt{3})}{1+(2-\sqrt{3})^2}$$
$$= \frac{2(2-\sqrt{3})}{1+4+3-4\sqrt{3}} = \frac{2(2-\sqrt{3})}{4(2-\sqrt{3})} = \frac{1}{2}$$

11 $\sin 105^\circ + \cos 105^\circ = \sqrt{(1+\sin 210^\circ)}$,

and $\sin 105^\circ - \cos 105^\circ = \sqrt{(1-\sin 210^\circ)}$,

therefore $$2\cos 105^\circ = \sqrt{(1+\sin 210^\circ)} - \sqrt{(1-\sin 210^\circ)}$$
$$= \sqrt{\left(1-\frac{1}{2}\right)} - \sqrt{\left(1+\frac{1}{2}\right)} = \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{\sqrt{2}},$$

thus $$2\cos 105^\circ = \frac{1-\sqrt{3}}{\sqrt{2}}, \text{ and } \cos 105^\circ = \frac{1-\sqrt{3}}{2\sqrt{2}} = -\frac{\sqrt{3}-1}{2\sqrt{2}}$$

12 $$\tan 2A = \frac{2\tan A}{1-\tan^2 A}, \text{ thus } \frac{2\tan A}{1-\tan^2 A} = -\frac{24}{7},$$

therefore $14\tan A = -24(1-\tan^2 A)$, therefore $24\tan^2 A - 14\tan A = 24$.

By solving this quadratic in the ordinary way we obtain

$$\tan A = \frac{4}{3}, \text{ or } -\frac{3}{4}$$

Also $$\sin A = \frac{\tan A}{\sqrt{(1+\tan^2 A)}}, \text{ and } \cos A = \frac{1}{\sqrt{(1+\tan^2 A)}}$$

If $\tan A = \frac{4}{3}$ we get $\sin A = \pm\frac{4}{5}$, and $\cos A = \pm\frac{3}{5}$

If $\tan A = -\frac{3}{4}$ we get $\sin A = \pm\frac{3}{5}$, and $\cos A = \mp\frac{4}{5}$

13 $\quad \tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$ Let $2A = 330^0$, then $\tan 2A = -\dfrac{1}{\sqrt{3}}$,

therefore
$$-\frac{1}{\sqrt{3}} = \frac{2\tan A}{1-\tan^2 A},$$

therefore $-1+\tan^2 A = 2\sqrt{3}\tan A$, therefore $\tan^2 A - 2\sqrt{3}\tan A = 1$

By solving this quadratic in the ordinary way we obtain $\tan A = \sqrt{3}\pm 2$ But $\tan 165^0$ must be a negative quantity, and is therefore equal to $\sqrt{3}-2$

14 $$\frac{2\sin A - \sin 2A}{2\sin A + \sin 2A} = \frac{2\sin A - 2\sin A\cos A}{2\sin A + 2\sin A\cos A} = \frac{2\sin A\,(1-\cos A)}{2\sin A\,(1+\cos A)}$$
$$= \frac{1-\cos A}{1+\cos A} = \frac{2\sin^2\frac{A}{2}}{2\cos^2\frac{A}{2}} = \tan^2\frac{A}{2}$$

15 $$2\operatorname{vers}\frac{1}{2}(180^0+A)\operatorname{vers}\frac{1}{2}(180^0-A)$$
$$= 2\left\{1-\cos\left(90^0+\frac{A}{2}\right)\right\}\left\{1-\cos\left(90^0-\frac{A}{2}\right)\right\}$$
$$= 2\left(1+\sin\frac{A}{2}\right)\left(1-\sin\frac{A}{2}\right) = 2\left(1-\sin^2\frac{A}{2}\right) = 2\cos^2\frac{A}{2},$$

and
$$\operatorname{vers}(180^0-A) = 1-\cos(180^0-A) = 1+\cos A = 2\cos^2\frac{A}{2}$$

Thus the proposed expressions are equal

16 $$(\cos A+\cos B)^2+(\sin A+\sin B)^2$$
$$= \cos^2 A+\cos^2 B+2\cos A\cos B+\sin^2 A+\sin^2 B+2\sin A\sin B$$
$$= 2+2(\cos A\cos B+\sin A\sin B) = 2+2\cos(A-B)$$
$$= 2\{1+\cos(A-B)\} = 4\cos^2\frac{1}{2}(A-B)$$

17 $$(\cos A-\cos B)^2+(\sin A-\sin B)^2$$
$$= \cos^2 A+\cos^2 B-2\cos A\cos B+\sin^2 A+\sin^2 B-2\sin A\sin B$$
$$= 2-2(\cos A\cos B+\sin A\sin B) = 2-2\cos(A-B)$$
$$= 2\{1-\cos(A-B)\} = 4\sin^2\frac{1}{2}(A-B)$$

18 $\quad 2\sin^2 22\tfrac{1}{2}^0 = 1-\cos 45^0$, therefore
$$4\sin^2 22\tfrac{1}{2}^0 = 2-2\cos 45^0 = 2-\frac{2}{\sqrt{2}} = 2-\sqrt{2},$$

therefore
$$2\sin 22\tfrac{1}{2}^0 = \sqrt{(2-\sqrt{2})}$$

And $2\cos^2 22\frac{1}{2}^0 = 1 + \cos 45^0$, therefore

$$4\cos^2 22\tfrac{1}{2}^0 = 2 + 2\cos 45^0 = 2 + \frac{2}{\sqrt{2}} = 2 + \sqrt{2},$$

therefore $$2\cos 22\tfrac{1}{2}^0 = \sqrt{(2+\sqrt{2})}$$

Hence $$\frac{\sin 22\frac{1}{2}^0}{\cos 22\frac{1}{2}^0} = \frac{\sqrt{(2-\sqrt{2})}}{\sqrt{(2+\sqrt{2})}} = \frac{\sqrt{(2-\sqrt{2})}}{\sqrt{(2+\sqrt{2})}} \cdot \frac{\sqrt{(2-\sqrt{2})}}{\sqrt{(2-\sqrt{2})}}$$

$$= \frac{2-\sqrt{2}}{\sqrt{(4-2)}} = \frac{2-\sqrt{2}}{\sqrt{2}} = \sqrt{2} - 1,$$

that is $$\tan 22\tfrac{1}{2}^0 = \sqrt{2} - 1$$

19 $$(\tan A + \cot A)\, 2\tan\frac{A}{2}\left(1 - \tan^2\frac{A}{2}\right)$$

$$= \left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right) 2\tan\frac{A}{2}\left(1 - \tan^2\frac{A}{2}\right)$$

$$= \frac{\sin^2 A + \cos^2 A}{\sin A\cos A}\ \frac{2\sin\frac{A}{2}}{\cos\frac{A}{2}}\ \frac{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}}{\cos^2\frac{A}{2}}$$

$$= \frac{1}{\sin A\cos A} \cdot \frac{2\sin\frac{A}{2}}{\cos\frac{A}{2}}\ \frac{\cos A}{\cos^2\frac{A}{2}}$$

$$= \frac{2\sin\frac{A}{2}}{\sin A\cos^3\frac{A}{2}} = \frac{2\sin\frac{A}{2}}{2\sin\frac{A}{2}\cos^4\frac{A}{2}} = \frac{1}{\cos^4\frac{A}{2}}$$

$$= \left\{\frac{1}{\cos^2\frac{A}{2}}\right\}^2 = \left(1 + \tan^2\frac{A}{2}\right)^2$$

20 $$\tan^2\left(\frac{\pi}{4} + \frac{A}{2}\right) = \left\{\frac{1+\tan\frac{A}{2}}{1-\tan\frac{A}{2}}\right\}^2 = \left\{\frac{\cos\frac{A}{2} + \sin\frac{A}{2}}{\cos\frac{A}{2} - \sin\frac{A}{2}}\right\}^2$$

$$= \frac{\cos^2\frac{A}{2} + \sin^2\frac{A}{2} + 2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{2} + \sin^2\frac{A}{2} - 2\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{1+\sin A}{1-\sin A}$$

$$= \frac{\frac{1}{\cos A} + \frac{\sin A}{\cos A}}{\frac{1}{\cos A} - \frac{\sin A}{\cos A}} = \frac{\sec A + \tan A}{\sec A - \tan A}$$

21 $\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$

$$=\sin\frac{\pi}{4}\cos\frac{\theta}{2}-\cos\frac{\pi}{4}\sin\frac{\theta}{2}+\cos\frac{\pi}{4}\cos\frac{\theta}{2}+\sin\frac{\pi}{4}\sin\frac{\theta}{2}$$

$$=2\sin\frac{\pi}{4}\cos\frac{\theta}{2}=\frac{2\cos\frac{\theta}{2}}{\sqrt{2}}=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sqrt{2}\sin\frac{\theta}{2}}$$

$$=\frac{\sin\theta}{\sqrt{\left(2\sin^2\frac{\theta}{2}\right)}}=\frac{\sin\theta}{\sqrt{(1-\cos\theta)}}=\frac{\sin\theta}{\sqrt{(\operatorname{vers}\theta)}}$$

22 $4\sin^2\frac{\theta}{4}\left(1-\sin\frac{\theta}{2}\right)=4\sin^2\frac{\theta}{4}\left(\sin^2\frac{\theta}{4}+\cos^2\frac{\theta}{4}-2\sin\frac{\theta}{4}\cos\frac{\theta}{4}\right)$

$$=4\sin^2\frac{\theta}{4}\left(\sin\frac{\theta}{4}-\cos\frac{\theta}{4}\right)^2$$

$$=\left(2\sin^2\frac{\theta}{4}-2\sin\frac{\theta}{4}\cos\frac{\theta}{4}\right)^2$$

$$=\left(1-\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2$$

And $\sqrt{(1+\sin\theta)}=\sqrt{\left(\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}=\sin\frac{\theta}{2}+\cos\frac{\theta}{2}$,

therefore $\{1-\sqrt{(1+\sin\theta)}\}^2=\left(1-\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)^2$

23 $2\cos^2\frac{\theta}{2}=1+\cos\theta$, therefore

$4\cos^4\frac{\theta}{2}=(1+\cos\theta)^2=1+2\cos\theta+\cos^2\theta$

$$=1+2\cos\theta+\frac{1+\cos 2\theta}{2}=\frac{1}{2}(3+4\cos\theta+\cos 2\theta),$$

therefore $\cos^4\frac{\theta}{2}=\frac{1}{8}(3+4\cos\theta+\cos 2\theta)$

Use this formula for each of the terms, thus

$$\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}$$

$$=\frac{12}{8}+\frac{1}{2}\left(\cos\frac{\pi}{4}+\cos\frac{3\pi}{4}+\cos\frac{5\pi}{4}+\cos\frac{7\pi}{4}\right)$$

$$+\frac{1}{8}\left(\cos\frac{\pi}{2}+\cos\frac{3\pi}{2}+\cos\frac{5\pi}{2}+\cos\frac{7\pi}{2}\right)$$

$=\frac{3}{2}$: see Art. 50.

24. $$\tan 7\tfrac{1}{2}^0 = \frac{\sin 15^0}{1+\cos 15^0} = \frac{\dfrac{\sqrt{3}-1}{2\sqrt{2}}}{1+\dfrac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{\sqrt{3}-1}{2\sqrt{2}+\sqrt{3}+1}$$

$$= \frac{(\sqrt{3}-1)(2\sqrt{2}+1-\sqrt{3})}{(2\sqrt{2}+\sqrt{3}+1)(2\sqrt{2}+1-\sqrt{3})} = \frac{2\sqrt{6}-2\sqrt{2}-4+2\sqrt{3}}{6+4\sqrt{2}}$$

$$= \frac{\sqrt{6}-\sqrt{2}-2+\sqrt{3}}{3+2\sqrt{2}}$$

Multiply both numerator and denominator by $3-2\sqrt{2}$, then we obtain unity for denominator, and for numerator $\sqrt{6}-\sqrt{3}+\sqrt{2}-2$

25. $$\tan 142\tfrac{1}{2}^0 = \frac{\sin 285^0}{1+\cos 285^0} = -\frac{\sin 105^0}{1-\cos 105^0} = -\frac{\cos 15^0}{1+\sin 15^0}$$

$$= -\frac{\dfrac{\sqrt{3}+1}{2\sqrt{2}}}{1+\dfrac{\sqrt{3}-1}{2\sqrt{2}}} = -\frac{\sqrt{3}+1}{2\sqrt{2}-1+\sqrt{3}}$$

$$= -\frac{(\sqrt{3}+1)(2\sqrt{2}-1-\sqrt{3})}{(2\sqrt{2}-1+\sqrt{3})(2\sqrt{2}-1-\sqrt{3})}$$

$$= -\frac{\sqrt{6}+\sqrt{2}-2-\sqrt{3}}{3-2\sqrt{2}} = \frac{2+\sqrt{3}-\sqrt{2}-\sqrt{6}}{3-2\sqrt{2}}.$$

Multiply both numerator and denominator by $3+2\sqrt{2}$, then we obtain unity for denominator, and for numerator $2+\sqrt{2}-\sqrt{3}-\sqrt{6}$

26 $$\tan x = \frac{3\tan\dfrac{x}{3}-\tan^3\dfrac{x}{3}}{1-3\tan^2\dfrac{x}{3}},$$

and since this is equal to $(2+\sqrt{3})\tan\dfrac{x}{3}$ we obtain

$$\frac{3-\tan^2\dfrac{x}{3}}{1-3\tan^2\dfrac{x}{3}} = 2+\sqrt{3},$$

therefore $$3-\tan^2\frac{x}{3} = (2+\sqrt{3})\left(1-3\tan^2\frac{x}{3}\right),$$

therefore $$(6+3\sqrt{3}-1)\tan^2\frac{x}{3} = 2+\sqrt{3}-3,$$

therefore $$\tan^2\frac{x}{3}=\frac{\sqrt{3}-1}{5+3\sqrt{3}}=\frac{(\sqrt{3}-1)(5-3\sqrt{3})}{(5+3\sqrt{3})(5-3\sqrt{3})}$$
$$=\frac{8\sqrt{3}-14}{25-27}=7-4\sqrt{3},$$

therefore $$\tan\frac{x}{3}=\sqrt{(7-4\sqrt{3})}=\pm(2-\sqrt{3})$$

Hence $$\tan x=\pm(2+\sqrt{3})(2-\sqrt{3})=\pm 1.$$

27. $$\tan\alpha+\cot\alpha=\frac{\sin\alpha}{\cos\alpha}+\frac{\cos\alpha}{\sin\alpha}=\frac{\sin^2\alpha+\cos^2\alpha}{\sin\alpha\cos\alpha}=\frac{1}{\sin\alpha\cos\alpha}$$
$$=\frac{2}{2\sin\alpha\cos\alpha}=\frac{2}{\sin 2\alpha}.$$

Put for α its value; then the expression

$$=\frac{2}{\sin 2\left(n+\frac{1}{4}\pm\frac{1}{6}\right)\pi}=\frac{2}{\sin\left(\frac{\pi}{2}\pm\frac{\pi}{3}\right)}=\frac{2}{\cos\frac{\pi}{3}}$$
$$=2\div\frac{1}{2}=4$$

28. $$\frac{\cos\alpha\cos 13\alpha}{\cos 3\alpha+\cos 5\alpha}=\frac{\cos\alpha\cos 13\alpha}{2\cos\alpha\cos 4\alpha}=\frac{\cos 13\alpha}{2\cos 4\alpha}=-\frac{1}{2},$$

for $13\alpha+4\alpha=\pi$, and therefore $\cos 13\alpha=-\cos 4\alpha$

29. $\sec(\phi+\alpha)+\sec(\phi-\alpha)=2\sec\phi$, therefore

$$\frac{1}{\cos(\phi+\alpha)}+\frac{1}{\cos(\phi-\alpha)}=\frac{2}{\cos\phi},$$

therefore $$\frac{\cos(\phi-\alpha)+\cos(\phi+\alpha)}{\cos(\phi+\alpha)\cos(\phi-\alpha)}=\frac{2}{\cos\phi},$$

therefore $$\frac{2\cos\phi\cos\alpha}{\cos^2\phi-\sin^2\alpha}=\frac{2}{\cos\phi},$$

therefore $$\cos^2\phi\cos\alpha=\cos^2\phi-\sin^2\alpha,$$

therefore $$\cos^2\phi=\frac{\sin^2\alpha}{1-\cos\alpha}=\frac{1-\cos^2\alpha}{1-\cos\alpha}=1+\cos\alpha=2\cos^2\frac{\alpha}{2},$$

therefore $$\cos\phi=\sqrt{2}\cos\frac{\alpha}{2}$$

30 $\tan^2\frac{\theta}{2}=\frac{1+c}{1-c}\tan^2\frac{\phi}{2}$, therefore

$$\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}=\frac{1-\frac{1+c}{1-c}\,\dfrac{\sin^2\frac{\phi}{2}}{\cos^2\frac{\phi}{2}}}{1+\frac{1+c}{1-c}\,\dfrac{\sin^2\frac{\phi}{2}}{\cos^2\frac{\phi}{2}}}=\frac{(1-c)\cos^2\frac{\phi}{2}-(1+c)\sin^2\frac{\phi}{2}}{(1-c)\cos^2\frac{\phi}{2}+(1+c)\sin^2\frac{\phi}{2}},$$

therefore, by Art 87,

$$\cos\theta=\frac{\cos^2\frac{\phi}{2}-\sin^2\frac{\phi}{2}-c\left(\cos^2\frac{\phi}{2}+\sin^2\frac{\phi}{2}\right)}{\cos^2\frac{\phi}{2}+\sin^2\frac{\phi}{2}-c\left(\cos^2\frac{\phi}{2}-\sin^2\frac{\phi}{2}\right)}=\frac{\cos\phi-c}{1-c\cos\phi}$$

CHAPTER VIII

1 By Art 113 we have

$$\cos(\alpha+\beta+\gamma)=\cos\alpha\cos\beta\cos\gamma-\cos\alpha\sin\beta\sin\gamma-\cos\beta\sin\gamma\sin\alpha-\cos\gamma\sin\alpha\sin\beta,$$

divide both sides by $\cos\alpha\cos\beta\cos\gamma$, thus

$$\frac{\cos(\alpha+\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma}=1-\tan\beta\tan\gamma-\tan\gamma\tan\alpha-\tan\alpha\tan\beta$$

2 By Art 113 we have

$$\sin(\alpha+\beta+\gamma)=\sin\alpha\cos\beta\cos\gamma+\sin\beta\cos\gamma\cos\alpha+\sin\gamma\cos\alpha\cos\beta-\sin\alpha\sin\beta\sin\gamma$$

divide both sides by $\cos\alpha\cos\beta\cos\gamma$, thus

$$\frac{\sin(\alpha+\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma}=\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma$$

3 $\sin(\alpha-\beta)+\sin(\beta-\gamma)=2\sin\frac{\alpha-\gamma}{2}\cos\frac{\alpha-2\beta+\gamma}{2}$

$$=-2\sin\frac{\gamma-\alpha}{2}\cos\frac{\alpha-2\beta+\gamma}{2},$$

$$\sin(\gamma-\alpha)=2\sin\frac{\gamma-\alpha}{2}\cos\frac{\gamma-\alpha}{2},$$

therefore $\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)$

$$=2\sin\frac{\gamma-\alpha}{2}\left\{\cos\frac{\gamma-\alpha}{2}-\cos\frac{\alpha-2\beta+\gamma}{2}\right\}$$

$$=2\sin\frac{\gamma-\alpha}{2}\,2\sin\frac{\gamma-\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$=-4\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2},$$

therefore $\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)$

$$+4\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}=0.$$

4. $4\sin(\theta-\alpha)\sin(m\theta-\alpha)\cos(\theta-m\theta)$

$=2\cos(\theta-m\theta)\{\cos(\theta-m\theta)-\cos(\theta+m\theta-2\alpha)\}$, by Art 84,

$=2\cos^2(\theta-m\theta)-2\cos(\theta-m\theta)\cos(\theta+m\theta-2\alpha)$

$=1+\cos 2(\theta-m\theta)-\{\cos(2\theta-2\alpha)+\cos(2m\theta-2\alpha)\}$

$=1+\cos 2(\theta-m\theta)-\cos(2\theta-2\alpha)-\cos(2m\theta-2\alpha)$

5. $\sin(\alpha+\beta)\cos\beta=\sin(\alpha+\beta+\gamma-\gamma)\cos\beta$

$=\{\sin(\alpha+\beta+\gamma)\cos\gamma-\cos(\alpha+\beta+\gamma)\sin\gamma\}\cos\beta,$

$\sin(\alpha+\gamma)\cos\gamma=\sin(\alpha+\beta+\gamma-\beta)\cos\gamma$

$=\{\sin(\alpha+\beta+\gamma)\cos\beta-\cos(\alpha+\beta+\gamma)\sin\beta\}\cos\gamma,$

therefore $\sin(\alpha+\beta)\cos\beta-\sin(\alpha+\gamma)\cos\gamma$

$=\cos(\alpha+\beta+\gamma)\{\sin\beta\cos\gamma-\sin\gamma\cos\beta\}$

$=\cos(\alpha+\beta+\gamma)\sin(\beta-\gamma)$

6. $\cos(\alpha+\beta+\gamma)-\cos(\alpha+\beta-\gamma)=2\cos(\alpha+\beta)\cos\gamma,$

$\cos(\alpha-\beta+\gamma)+\cos(\beta+\gamma-\alpha)=2\cos(\alpha-\beta)\cos\gamma;$

hence the sum $=2\cos\gamma\{\cos(\alpha+\beta)+\cos(\alpha-\beta)\}$

$=4\cos\alpha\cos\beta\cos\gamma$

7. $\cos 2\alpha+\cos 2\beta=2\cos(\alpha+\beta)\cos(\alpha-\beta),$

$\cos 2\gamma+\cos 2(\alpha+\beta+\gamma)=2\cos(2\gamma+\alpha+\beta)\cos(\alpha+\beta),$

hence the sum $=2\cos(\alpha+\beta)\{\cos(\alpha-\beta)+\cos(2\gamma+\alpha+\beta)\}$

$=2\cos(\alpha+\beta)\,2\cos(\alpha+\gamma)\cos(\beta+\gamma)$

$=4\cos(\alpha+\beta)\cos(\beta+\gamma)\cos(\gamma+\alpha).$

8. Reduce the three fractions to have the common denominator

$$\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha),$$

then the whole numerator

$=-\sin\alpha\sin(\beta-\gamma)-\sin\beta\sin(\gamma-\alpha)-\sin\gamma\sin(\alpha-\beta)$

$=-\frac{1}{2}\{\cos(\alpha-\beta+\gamma)-\cos(\alpha+\beta-\gamma)\}-\frac{1}{2}\{\cos(\beta+\alpha-\gamma)-\cos(\beta+\gamma-\alpha)\}$

$-\frac{1}{2}\{\cos(\gamma-\alpha+\beta)-\cos(\gamma+\alpha-\beta)\}=0$

9 $\cos(\alpha+\beta)\sin\beta-\cos(\alpha+\gamma)\sin\gamma$

$$=\frac{1}{2}\{\sin(\alpha+\beta+\beta)-\sin(\alpha+\beta-\beta)\}-\frac{1}{2}\{\sin(\alpha+\gamma+\gamma)-\sin(\alpha+\gamma-\gamma)\}$$

$$=\frac{1}{2}\sin(\alpha+2\beta)-\frac{1}{2}\sin(\alpha+2\gamma),$$

$\sin(\alpha+\beta)\cos\beta-\sin(\alpha+\gamma)\cos\gamma$

$$=\frac{1}{2}\{\sin(\alpha+\beta+\beta)+\sin(\alpha+\beta-\beta)\}-\frac{1}{2}\{\sin(\alpha+\gamma+\gamma)+\sin(\alpha+\gamma-\gamma)\}$$

$$=\frac{1}{2}\sin(\alpha+2\beta)-\frac{1}{2}\sin(\alpha+2\gamma)$$

Thus the two expressions are equal

10 $\sin(\alpha+\beta-2\gamma)\cos\beta-\sin(\alpha+\gamma-2\beta)\cos\gamma$

$$=\frac{1}{2}\{\sin(\alpha+2\beta-2\gamma)+\sin(\alpha-2\gamma)-\sin(\alpha+2\gamma-2\beta)-\sin(\alpha-2\beta)\},$$

$\sin(\beta-\gamma)\{\cos(\beta+\gamma-\alpha)+\cos(\alpha+\gamma-\beta)+\cos(\alpha+\beta-\gamma)\}$

$$=\frac{1}{2}\{\sin(2\beta-\alpha)+\sin(\alpha-2\gamma)\}+\frac{1}{2}\{\sin\alpha+\sin(2\beta-2\gamma-\alpha)\}$$
$$+\frac{1}{2}\{-\sin\alpha+\sin(2\beta-2\gamma+\alpha)\}$$

$$=\frac{1}{2}\{\sin(2\beta-\alpha)+\sin(\alpha-2\gamma)+\sin(2\beta-2\gamma-\alpha)+\sin(2\beta-2\gamma+\alpha)\}$$

Thus the two expressions are equal

11 $\sin(\alpha+\beta+\gamma)\sin\beta=\frac{1}{2}\{\cos(\alpha+\gamma)-\cos(\alpha+2\beta+\gamma)\},$

$$\sin(\alpha+\beta)\sin(\beta+\gamma)=\frac{1}{2}\{\cos(\alpha-\gamma)-\cos(\alpha+2\beta+\gamma)\},$$

$$\sin\alpha\sin\gamma=\frac{1}{2}\{\cos(\alpha-\gamma)-\cos(\alpha+\gamma)\}$$

Hence $\sin(\alpha+\beta)\sin(\beta+\gamma)-\sin\alpha\sin\gamma$

$$=\frac{1}{2}\{\cos(\alpha+\gamma)-\cos(\alpha+2\beta+\gamma)\}$$

$$=\sin(\alpha+\beta+\gamma)\sin\beta$$

12 $\sin\alpha\sin\beta\sin(\beta-\alpha)=\frac{1}{2}\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}\sin(\beta-\alpha)$

$$=\frac{1}{2}\cos(\beta-\alpha)\sin(\beta-\alpha)-\frac{1}{4}\{\sin 2\beta-\sin 2\alpha\}$$

$$=\frac{1}{4}\sin 2(\beta-\alpha)-\frac{1}{4}\sin 2\beta+\frac{1}{4}\sin 2\alpha$$

Similarly we may transform $\sin\beta\sin\gamma\sin(\gamma-\beta)$ and $\sin\gamma\sin\alpha\sin(\alpha-\gamma)$.

Also, by Example 3, we have

$$\sin(\beta-\alpha)\sin(\gamma-\beta)\sin(\alpha-\gamma)=\frac{1}{4}\{\sin 2(\alpha-\beta)+\sin 2(\beta-\gamma)+\sin 2(\gamma-\alpha)\}.$$

Hence the sum of the four expressions is zero.

13. $$\cos(\alpha+\beta)\sin(\alpha-\beta)=\frac{1}{2}(\sin 2\alpha-\sin 2\beta),$$

$$\cos(\beta+\gamma)\sin(\beta-\gamma)=\frac{1}{2}(\sin 2\beta-\sin 2\gamma),$$

$$\cos(\gamma+\delta)\sin(\gamma-\delta)=\frac{1}{2}(\sin 2\gamma-\sin 2\delta),$$

$$\cos(\delta+\alpha)\sin(\delta-\alpha)=\frac{1}{2}(\sin 2\delta-\sin 2\alpha),$$

hence the sum of the four expressions is zero

14. $$\sin(\delta-\beta)\sin(\alpha-\gamma)=\frac{1}{2}\{\cos(\alpha+\beta-\gamma-\delta)-\cos(\alpha-\beta-\gamma+\delta)\},$$

$$\sin(\beta-\gamma)\sin(\alpha-\delta)=\frac{1}{2}\{\cos(\alpha-\beta+\gamma-\delta)-\cos(\alpha+\beta-\gamma-\delta)\},$$

$$\sin(\gamma-\delta)\sin(\alpha-\beta)=\frac{1}{2}\{\cos(\alpha-\beta-\gamma+\delta)-\cos(\alpha-\beta+\gamma-\delta)\};$$

since the sum of the three expressions is zero

15. $$\cot\frac{A}{2}+\cot\frac{B}{2}=\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}+\frac{\cos\frac{B}{2}}{\sin\frac{B}{2}}=\frac{\sin\frac{B}{2}\cos\frac{A}{2}+\sin\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}}$$

$$=\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{A}{2}\sin\frac{B}{2}}=\frac{\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}};$$

$$\frac{\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}}+\cot\frac{C}{2}=\cos\frac{C}{2}\left\{\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\right\}$$

$$=\frac{\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\left\{\sin\frac{C}{2}+\sin\frac{A}{2}\sin\frac{B}{2}\right\}$$

$$=\frac{\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\left\{\cos\frac{1}{2}(A+B)+\sin\frac{A}{2}\sin\frac{B}{2}\right\}$$

$$=\frac{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

16 $\sin A+\sin B=2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)=2\cos\frac{C}{2}\cos\frac{1}{2}(A-B)$

$$\sin C=2\sin\frac{C}{2}\cos\frac{C}{2}=2\cos\frac{C}{2}\cos\frac{1}{2}(A+B),$$

therefore $\sin A+\sin B+\sin C=2\cos\frac{C}{2}\left\{\cos\frac{1}{2}(A-B)+\cos\frac{1}{2}(A+B)\right\}$

$$=2\cos\frac{C}{2}\;2\cos\frac{A}{2}\cos\frac{B}{2}$$

$$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

17. $\sin A+\sin C=2\sin\frac{1}{2}(A+C)\cos\frac{1}{2}(A-C)=2\cos\frac{B}{2}\cos\frac{1}{2}(A-C),$

$$\sin B=2\sin\frac{B}{2}\cos\frac{B}{2}=2\cos\frac{B}{2}\cos\frac{1}{2}(A+C),$$

therefore $\sin A-\sin B+\sin C=2\cos\frac{B}{2}\left\{\cos\frac{1}{2}(A-C)-\cos\frac{1}{2}(A+C)\right\}$

$$=2\cos\frac{B}{2}\;2\sin\frac{A}{2}\sin\frac{C}{2}$$

$$=4\sin\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

18. $\cos 2A+\cos 2B=2\cos(A+B)\cos(A-B)=-2\cos C\cos(A-B)$

$$\cos 2C=2\cos^2 C-1=-2\cos C\cos(A+B)-1,$$

therefore $\cos 2A+\cos 2B+\cos 2C=-2\cos C\{\cos(A-B)+\cos(A+B)\}-1$

$$=-2\cos C\;\;2\cos A\cos B-1$$

$$=-4\cos A\cos B\cos C-1,$$

therefore $\cos 2A+\cos 2B+\cos 2C+4\cos A\cos B\cos C+1=0$

19 $\cos 4A+\cos 4B=2\cos 2(A+B)\cos 2(A-B)=2\cos 2C\cos 2(A-B)$,

$$\cos 4C=2\cos^2 2C-1=2\cos 2C\cos 2(A+B)-1,$$

therefore $\cos 4A+\cos 4B+\cos 4C=2\cos 2C\{\cos 2(A-B)+\cos 2(A+B)\}-1$

$$=2\cos 2C\ \ 2\cos 2A\cos 2B-1$$

$$=4\cos 2A\cos 2B\cos 2C-1,$$

therefore $\cos 4A+\cos 4B+\cos 4C+1=4\cos 2A\cos 2B\cos 2C$

20 Let $\alpha=\frac{1}{2}(\pi-A),\ \beta=\frac{1}{2}(\pi-B),\ \gamma=\frac{1}{2}(\pi-C)$,

therefore $\alpha+\beta+\gamma=\frac{1}{2}(3\pi-A-B-C)=\frac{1}{2}2\pi=\pi$,

hence, by Example 16,

$$\sin\alpha+\sin\beta+\sin\gamma=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2},$$

that is $\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}$.

21 Let $\alpha=\frac{1}{2}(\pi-A),\ \beta=\frac{1}{2}(\pi-B),\ \gamma=\frac{1}{2}(\pi-C)$,

therefore $\alpha+\beta+\gamma=\frac{1}{2}(3\pi-A-B-C)=\frac{1}{2}2\pi=\pi$,

hence, by Example 17,

$$\sin\alpha-\sin\beta+\sin\gamma=4\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2},$$

that is $\cos\frac{A}{2}-\cos\frac{B}{2}+\cos\frac{C}{2}=4\sin\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\sin\frac{\pi-C}{4}$.

22 Let $\alpha=\frac{1}{2}(\pi-A),\ \beta=\frac{1}{2}(\pi-B),\ \gamma=\frac{1}{2}(\pi-C)$,

therefore $\alpha+\beta+\gamma=\frac{1}{2}(3\pi-A-B-C)=\frac{1}{2}2\pi=\pi$,

hence, by Art 114,

$$\cos\alpha+\cos\beta+\cos\gamma-1=4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2},$$

that is $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-1=4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}$.

23 $\sin^2 A+\sin^2 B+\sin^2 C=\frac{1}{2}\{1-\cos 2A+1-\cos 2B+1-\cos 2C\}$

$$=\frac{3}{2}-\frac{1}{2}\{\cos 2A+\cos 2B+\cos 2C\}$$

$$=\frac{3}{2}+\frac{1}{2}\{1+4\cos A\cos B\cos C\},\text{ by Example 18,}$$

$$=2+2\cos A\cos B\cos C,$$

therefore $\sin^2 A+\sin^2 B+\sin^2 C-2\cos A\cos B\cos C=2$

24 $\sin^2 2A+\sin^2 2B+\sin^2 2C=\frac{1}{2}\{3-\cos 4A-\cos 4B-\cos 4C\}$

$$=\frac{3}{2}-\frac{1}{2}\{4\cos 2A\cos 2B\cos 2C-1\},\text{ by Example 19,}$$

$$=2-2\cos 2A\cos 2B\cos 2C,$$

therefore $\sin^2 2A+\sin^2 2B+\sin^2 2C+2\cos 2A\cos 2B\cos 2C=2$

25 $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}$

$$=\frac{1}{\cot\frac{A}{2}\cot\frac{B}{2}}+\frac{1}{\cot\frac{B}{2}\cot\frac{C}{2}}+\frac{1}{\cot\frac{C}{2}\cot\frac{A}{2}}$$

$$=\frac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}=1,\text{ by Example 15}$$

26 $\sin A+\sin B-\sin C=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$, by Example 17,

$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$, by Example 16,

therefore, by division,

$$\frac{\sin A+\sin B-\sin C}{\sin A+\sin B+\sin C}=\frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}=\tan\frac{A}{2}\tan\frac{B}{2}$$

27 $\cos A\sin B\sin C+\cos B\sin A\sin C+\cos C\sin A\sin B$

$$=\sin C(\cos A\sin B+\cos B\sin A)+\cos C\sin A\sin B$$
$$=\sin C\sin(A+B)+\cos C\sin A\sin B$$
$$=\sin^2 C+\cos C\sin A\sin B$$
$$=1-\cos^2 C+\cos C\sin A\sin B$$
$$=1+\cos C\{\cos(A+B)+\sin A\sin B\}$$
$$=1+\cos C\cos A\cos B.$$

28 Take Example 27, and divide by $\sin A \sin B \sin C$,

therefore $$\frac{1}{\sin A \sin B \sin C} + \frac{\cos A \cos B \cos C}{\sin A \sin B \sin C} = \frac{\cos A}{\sin A} + \frac{\cos B}{\sin B} + \frac{\cos C}{\sin C}$$

thus we obtain the required result

29 By Example 17 we have

$$\frac{(\sin B + \sin C - \sin A)(\sin C + \sin A - \sin B)}{4 \sin A \sin B}$$

$$= \frac{16 \sin \frac{B}{2} \sin \frac{C}{2} \cos \frac{A}{2} \sin \frac{C}{2} \sin \frac{A}{2} \cos \frac{B}{2}}{16 \sin \frac{A}{2} \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{B}{2}} = \sin^2 \frac{C}{2}$$

30 $$\cot A + \frac{\sin A}{\sin B \sin C} = \frac{\cos A}{\sin A} + \frac{\sin A}{\sin B \sin C}$$

$$= \frac{\cos A \sin B \sin C + \sin^2 A}{\sin A \sin B \sin C} = \frac{1 - \cos^2 A + \cos A \sin B \sin C}{\sin A \sin B \sin C}$$

$$= \frac{1 + \cos A \{\cos (B + C) + \sin B \sin C\}}{\sin A \sin B \sin C} = \frac{1 + \cos A \cos B \cos C}{\sin A \sin B \sin C}$$

We have thus an expression which involves A, B, and C symmetrically, and we shall in the same manner obtain the same result if in the original expression any two of the quantities A, B, C be interchanged.

31 By Art 114, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$,

by Example 16, $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$,

therefore, by division,

$$\frac{\tan A + \tan B + \tan C}{(\sin A + \sin B + \sin C)^2} = \frac{\tan A \tan B \tan C}{16 \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2}}$$

$$= \frac{8 \sin \frac{A}{2} \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{B}{2} \sin \frac{C}{2} \cos \frac{C}{2}}{16 \cos A \cos B \cos C \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2}} = \frac{\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}}{2 \cos A \cos B \cos C}.$$

32. $$\sin nA + \sin nB = 2 \sin \frac{n}{2}(A + B) \cos \frac{n}{2}(A - B)$$

$$= 2 \sin \frac{n}{2}(\pi - C) \cos \frac{n}{2}(A - B)$$

$$= 2 \left\{\sin \frac{n\pi}{2} \cos \frac{nC}{2} - \cos \frac{n\pi}{2} \sin \frac{nC}{2}\right\} \cos \frac{n}{2}(A - B)$$

$$= 2 \sin \frac{n\pi}{2} \cos \frac{nC}{2} \cos \frac{n}{2}(A - B), \text{ since } \cos \frac{n\pi}{2} = 0$$

Also $\sin nC = 2\sin\frac{nC}{2}\cos\frac{nC}{2} = 2\sin\frac{n}{2}(\pi - A - B)\cos\frac{nC}{2}$

$$= 2\left\{\sin\frac{n\pi}{2}\cos\frac{n}{2}(A+B) - \cos\frac{n\pi}{2}\sin\frac{n}{2}(A+B)\right\}\cos\frac{nC}{2}$$

$$= 2\sin\frac{n\pi}{2}\cos\frac{n}{2}(A+B)\cos\frac{nC}{2}$$

Therefore $\sin nA + \sin nB + \sin nC$

$$= 2\sin\frac{n\pi}{2}\cos\frac{nC}{2}\left\{\cos\frac{n}{2}(A-B) + \cos\frac{n}{2}(A+B)\right\}$$

$$= 4\sin\frac{n\pi}{2}\cos\frac{nA}{2}\cos\frac{nB}{2}\cos\frac{nC}{2}$$

33 Proceed as in Example 32 Thus

$$\sin nA + \sin nB = 2\left\{\sin\frac{n\pi}{2}\cos\frac{nC}{2} - \cos\frac{n\pi}{2}\sin\frac{nC}{2}\right\}\cos\frac{n}{2}(A-B)$$

$$= -2\cos\frac{n\pi}{2}\sin\frac{nC}{2}\cos\frac{n}{2}(A-B), \text{ since } \sin\frac{n\pi}{2} = 0$$

Also $\sin nC = 2\sin\frac{nC}{2}\cos\frac{nC}{2} = 2\cos\frac{n}{2}(\pi - A - B)\sin\frac{nC}{2}$

$$= 2\left\{\cos\frac{n\pi}{2}\cos\frac{n}{2}(A+B) + \sin\frac{n\pi}{2}\sin\frac{n}{2}(A+B)\right\}\sin\frac{nC}{2}$$

$$= 2\cos\frac{n\pi}{2}\sin\frac{nC}{2}\cos\frac{n}{2}(A+B)$$

Therefore $\sin nA + \sin nB + \sin nC$

$$= -2\cos\frac{n\pi}{2}\sin\frac{nC}{2}\left\{\cos\frac{n}{2}(A-B) - \cos\frac{n}{2}(A+B)\right\}$$

$$= -4\cos\frac{n\pi}{2}\sin\frac{nC}{2}\sin\frac{nA}{2}\sin\frac{nB}{2}$$

34 By Example 20,

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = 4\cos\frac{\pi - A}{4}\cos\frac{\pi - B}{4}\cos\frac{\pi - C}{4}$$

$$= 4\cos\frac{B+C}{4}\cos\frac{C+A}{4}\cos\frac{A+B}{4}$$

35 $\dfrac{\tan B}{\tan A} + \dfrac{\tan C}{\tan A} = \dfrac{1}{\tan A}\left(\dfrac{\sin B}{\cos B} + \dfrac{\sin C}{\cos C}\right) = \dfrac{\sin(B+C)}{\tan A\cos B\cos C}$

$$= \frac{\sin A}{\tan A\cos B\cos C} = \frac{\cos A}{\cos B\cos C}$$

In this way we see that the given expression

$$=\frac{\cos A}{\cos B\cos C}+\frac{\cos B}{\cos C\cos A}+\frac{\cos C}{\cos A\cos B}$$

$$=\frac{\cos^2 A+\cos^2 B+\cos^2 C}{\cos A\cos B\cos C}=\frac{3-\sin^2 A-\sin^2 B-\sin^2 C}{\cos A\cos B\cos C}$$

$$=\frac{1-2\cos A\cos B\cos C}{\cos A\cos B\cos C}, \text{ by Example 23,}$$

$$=\sec A\sec B\sec C-2$$

36. Suppose $A+B+C+D=180^\circ$, then $A+B=180^\circ-C-D$,

therefore $\tan(A+B)=-\tan(C+D)$, by Art 48,

therefore $$\frac{\tan A+\tan B}{1-\tan A\tan B}=-\frac{\tan C+\tan D}{1-\tan C\tan D},$$

therefore $(\tan A+\tan B)(1-\tan C\tan D)=-(\tan C+\tan D)(1-\tan A\tan B)$,

therefore $\tan A+\tan B+\tan C+\tan D$

$$-(\tan A+\tan B)\tan C\tan D+(\tan C+\tan D)\tan A\tan B$$
$$=\tan B\tan C\tan D+\tan A\tan C\tan D+\tan A\tan B\tan D$$
$$+\tan A\tan B\tan C.$$

37. $$\frac{\sin^2 C}{\sin^2 A}=1-\frac{\tan(A-B)}{\tan A}=1-\frac{\sin(A-B)\cos A}{\cos(A-B)\sin A}$$

$$=\frac{\sin A\cos(A-B)-\cos A\sin(A-B)}{\cos(A-B)\sin A}=\frac{\sin\{A-(A-B)\}}{\cos(A-B)\sin A}$$

$$=\frac{\sin B}{\cos(A-B)\sin A}, \quad \text{therefore } \sin^2 C=\frac{\sin A\sin B}{\cos(A-B)}$$

Hence $$\cos^2 C=1-\sin^2 C=1-\frac{\sin A\sin B}{\cos(A-B)}$$

$$=\frac{\cos(A-B)-\sin A\sin B}{\cos(A-B)}=\frac{\cos A\cos B}{\cos(A-B)}$$

Therefore $$\frac{\sin^2 C}{\cos^2 C}=\frac{\sin A\sin B}{\cos(A-B)}\div\frac{\cos A\cos B}{\cos(A-B)}=\frac{\sin A\sin B}{\cos A\cos B},$$

that is $$\tan^2 C=\tan A\tan B$$

38 $$\frac{\tan^2\alpha}{\tan^2\beta}=\frac{\cos\beta(\cos x-\cos\alpha)}{\cos\alpha(\cos x-\cos\beta)},$$

therefore $$\frac{\cos x-\cos\alpha}{\cos x-\cos\beta}=\frac{\tan^2\alpha\cos\alpha}{\tan^2\beta\cos\beta}=\frac{\sin^2\alpha\cos\beta}{\sin^2\beta\cos\alpha};$$

therefore $\cos x = \dfrac{\sin^2\beta\cos^2\alpha - \sin^2\alpha\cos^2\beta}{\sin^2\beta\cos\alpha - \sin^2\alpha\cos\beta}$

$$= \frac{(1-\cos^2\beta)\cos^2\alpha - (1-\cos^2\alpha)\cos^2\beta}{(1-\cos^2\beta)\cos\alpha - (1-\cos^2\alpha)\cos\beta}$$

$$= \frac{\cos^2\alpha - \cos^2\beta}{(\cos\alpha-\cos\beta)(1+\cos\alpha\cos\beta)} = \frac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}$$

Hence $\dfrac{1-\cos x}{1+\cos x} = \dfrac{1+\cos\alpha\cos\beta-\cos\alpha-\cos\beta}{1+\cos\alpha\cos\beta+\cos\alpha+\cos\beta} = \dfrac{(1-\cos\alpha)(1-\cos\beta)}{(1+\cos\alpha)(1+\cos\beta)}$,

therefore $\tan^2\dfrac{x}{2} = \tan^2\dfrac{\alpha}{2}\tan^2\dfrac{\beta}{2}$, by Art 82

39 $\dfrac{\tan^2\theta}{\tan^2\theta'} = \dfrac{\tan^2\alpha}{\tan^2\alpha'}$, but $\tan^2\theta = \dfrac{1-\cos^2\theta}{\cos^2\theta} = \dfrac{\cos\beta-\cos\alpha}{\cos\alpha}$,

and $\tan^2\theta' = \dfrac{1-\cos^2\theta'}{\cos^2\theta'} = \dfrac{\cos\beta-\cos\alpha'}{\cos\alpha'}$,

therefore $\dfrac{\cos\beta-\cos\alpha}{\cos\beta-\cos\alpha'}\ \dfrac{\cos\alpha'}{\cos\alpha} = \dfrac{\tan^2\alpha}{\tan^2\alpha'}$,

therefore $\dfrac{\cos\beta-\cos\alpha}{\cos\beta-\cos\alpha'} = \dfrac{\sin^2\alpha\cos\alpha'}{\sin^2\alpha'\cos\alpha}$,

therefore $\cos\beta = \dfrac{\sin^2\alpha'\cos^2\alpha - \sin^2\alpha\cos^2\alpha'}{\sin^2\alpha'\cos\alpha - \sin^2\alpha\cos\alpha'}$

$$= \frac{(1-\cos^2\alpha')\cos^2\alpha - (1-\cos^2\alpha)\cos^2\alpha'}{(1-\cos^2\alpha')\cos\alpha - (1-\cos^2\alpha)\cos\alpha'}$$

$$= \frac{\cos^2\alpha - \cos^2\alpha'}{(\cos\alpha-\cos\alpha')(1+\cos\alpha\cos\alpha')} = \frac{\cos\alpha+\cos\alpha'}{1+\cos\alpha\cos\alpha'}.$$

Hence $\dfrac{1-\cos\beta}{1+\cos\beta} = \dfrac{1+\cos\alpha\cos\alpha'-\cos\alpha-\cos\alpha'}{1+\cos\alpha\cos\alpha'+\cos\alpha+\cos\alpha'} = \dfrac{(1-\cos\alpha)(1-\cos\alpha')}{(1+\cos\alpha)(1+\cos\alpha')}$,

therefore $\tan^2\dfrac{\beta}{2} = \tan^2\dfrac{\alpha}{2}\tan^2\dfrac{\alpha'}{2}$

40 $\cos\phi = \dfrac{\cos\alpha}{\cos\beta}$, $\cos\phi' = \dfrac{\cos\alpha}{\cos\beta'}$;

therefore $1-\cos\phi = \dfrac{\cos\beta-\cos\alpha}{\cos\beta}$, $1-\cos\phi' = \dfrac{\cos\beta'-\cos\alpha}{\cos\beta'}$,

therefore $2\sin^2\dfrac{\phi}{2} = \dfrac{\cos\beta-\cos\alpha}{\cos\beta}$, $2\sin^2\dfrac{\phi'}{2} = \dfrac{\cos\beta'-\cos\alpha}{\cos\beta'}$;

therefore $4\sin^2\dfrac{\phi}{2}\sin^2\dfrac{\phi'}{2} = \dfrac{(\cos\beta-\cos\alpha)(\cos\beta'-\cos\alpha)}{\cos\beta\cos\beta'}$.

Thus $$\sin^2\alpha=\frac{(\cos\beta-\cos\alpha)(\cos\beta'-\cos\alpha)}{\cos\beta\cos\beta'},$$

therefore $$\cos\beta\cos\beta'\sin^2\alpha=\cos\beta\cos\beta'-\cos\alpha(\cos\beta+\cos\beta')+\cos^2\alpha,$$

therefore $$\cos\beta\cos\beta'\cos^2\alpha=\cos\alpha(\cos\beta+\cos\beta')-\cos^2\alpha,$$

therefore $$\cos\alpha(1+\cos\beta\cos\beta')=\cos\beta+\cos\beta';$$

therefore $$\cos\alpha=\frac{\cos\beta+\cos\beta'}{1+\cos\beta\cos\beta'}$$

Hence $$\frac{1-\cos\alpha}{1+\cos\alpha}=\frac{(1-\cos\beta)(1-\cos\beta')}{(1+\cos\beta)(1+\cos\beta')};$$

therefore $$\tan^2\frac{\alpha}{2}=\tan^2\frac{\beta}{2}\tan^2\frac{\beta'}{2}$$

41 The proposed result is true if
$$\cot\beta-\cot(\alpha+\theta)=\cot\theta+\cot(\alpha-\beta),$$
that is if
$$\frac{\cos\beta}{\sin\beta}-\frac{\cos(\alpha+\theta)}{\sin(\alpha+\theta)}=\frac{\cos\theta}{\sin\theta}+\frac{\cos(\alpha-\beta)}{\sin(\alpha-\beta)},$$
that is if
$$\frac{\sin(\alpha+\theta)\cos\beta-\cos(\alpha+\theta)\sin\beta}{\sin\beta\sin(\alpha+\theta)}=\frac{\sin(\alpha-\beta)\cos\theta+\cos(\alpha-\beta)\sin\theta}{\sin\theta\sin(\alpha-\beta)},$$
that is if
$$\frac{\sin(\alpha+\theta-\beta)}{\sin\beta\sin(\alpha+\theta)}=\frac{\sin(\alpha-\beta+\theta)}{\sin\theta\sin(\alpha-\beta)},$$
that is if
$$\sin\theta\sin(\alpha-\beta)=\sin\beta\sin(\alpha+\theta),$$
and this is true by supposition

42 $$\left(\frac{\tan\alpha-\cos\theta\tan\beta}{\sin\theta}\right)^2=\tan^2\alpha-\tan^2\beta,$$ therefore
$$(\tan\alpha-\cos\theta\tan\beta)^2=(1-\cos^2\theta)(\tan^2\alpha-\tan^2\beta),$$
therefore
$$\tan^2\alpha-2\cos\theta\tan\alpha\tan\beta+\cos^2\theta\tan^2\beta=(1-\cos^2\theta)(\tan^2\alpha-\tan^2\beta),$$
therefore $$\tan^2\beta-2\cos\theta\tan\alpha\tan\beta+\cos^2\theta\tan^2\alpha=0,$$
that is $$(\tan\beta-\cos\theta\tan\alpha)^2=0,$$
therefore $$\tan\beta-\cos\theta\tan\alpha=0,\quad \text{therefore}\quad \cos\theta=\frac{\tan\beta}{\tan\alpha}$$

43 $$\cos\theta=\frac{\tan\phi}{\tan\alpha};\quad \text{therefore}\quad \tan^2\theta=\frac{\tan^2\alpha-\tan^2\phi}{\tan^2\phi};$$

therefore $$\frac{\tan^2\alpha-\tan^2\phi}{\tan^2\phi}=\frac{\tan^2\alpha'}{\sin^2\phi};$$

therefore $$\frac{\cos^2\phi\tan^2\alpha-\sin^2\phi}{\sin^2\phi}=\frac{\tan^2\alpha'}{\sin^2\phi},$$

therefore $$\cos^2\phi\tan^2\alpha-(1-\cos^2\phi)=\tan^2\alpha',$$

therefore $$\cos^2\phi=\frac{1+\tan^2\alpha'}{1+\tan^2\alpha}=\frac{\cos^2\alpha}{\cos^2\alpha'},$$

therefore $$\cos\phi=\pm\frac{\cos\alpha}{\cos\alpha'}$$

Take the upper sign, thus $\cos\phi=\frac{\cos\alpha}{\cos\alpha'}$, therefore

$$\frac{1-\cos\phi}{1+\cos\phi}=\frac{\cos\alpha'-\cos\alpha}{\cos\alpha+\cos\alpha}=\frac{2\sin\frac{1}{2}(\alpha-\alpha')\sin\frac{1}{2}(\alpha+\alpha')}{2\cos\frac{1}{2}(\alpha-\alpha')\cos\frac{1}{2}(\alpha+\alpha')}$$
$$=\tan\frac{1}{2}(\alpha-\alpha')\tan\frac{1}{2}(\alpha+\alpha')$$

44 $1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma$
$$=1-(\cos\alpha-\cos\beta\cos\gamma)^2+\cos^2\beta\cos^2\gamma-\cos^2\beta-\cos^2\gamma$$
$$=(1-\cos^2\beta)(1-\cos^2\gamma)-(\cos\alpha-\cos\beta\cos\gamma)^2$$
$$=\sin^2\beta\sin^2\gamma-(\cos\alpha-\cos\beta\cos\gamma)^2$$
$$=(\sin\beta\sin\gamma-\cos\alpha+\cos\beta\cos\gamma)(\sin\beta\sin\gamma+\cos\alpha-\cos\beta\cos\gamma)$$
$$=\{-\cos\alpha+\cos(\beta-\gamma)\}\{\cos\alpha-\cos(\beta+\gamma)\}$$
$$=4\sin\frac{\alpha+\beta-\gamma}{2}\sin\frac{\alpha-\beta+\gamma}{2}\sin\frac{\alpha+\beta+\gamma}{2}\sin\frac{\beta+\gamma-\alpha}{2}$$

Hence in order that the proposed expression may be zero one of the fou sines last written must be zero, and thus one of the four angles must be zer or a multiple of two right angles

45 Let $\frac{1}{k}$ denote the common value of the three fractions, so that

$$x=k\tan(\theta+\alpha),\quad y=k\tan(\theta+\beta),\quad z=k\tan(\theta+\gamma)$$

Then $$\frac{x+y}{x-y}\sin^2(\alpha-\beta)=\frac{\tan(\theta+\alpha)+\tan(\theta+\beta)}{\tan(\theta+\alpha)-\tan(\theta+\beta)}\sin^2(\alpha-\beta)$$
$$=\frac{\sin(\theta+\alpha)\cos(\theta+\beta)+\sin(\theta+\beta)\cos(\theta+\alpha)}{\sin(\theta+\alpha)\cos(\theta+\beta)-\sin(\theta+\beta)\cos(\theta+\alpha)}\sin^2(\alpha-\beta)$$
$$=\frac{\sin(2\theta+\alpha+\beta)}{\sin(\alpha-\beta)}\sin^2(\alpha-\beta)=\sin(2\theta+\alpha+\beta)\sin(\alpha-\beta)$$
$$=\frac{1}{2}\{\cos(2\theta+2\beta)-\cos(2\theta+2\alpha)\}$$

Similarly $\dfrac{y+z}{y-z}\sin^2(\beta-\gamma)=\dfrac{1}{2}\{\cos(2\theta+2\gamma)-\cos(2\theta+2\beta)\}$,

and $$\frac{z+x}{z-x}\sin^2(\gamma-\alpha)=\frac{1}{2}\{\cos(2\theta+2\alpha)-\cos(2\theta+2\gamma)\}$$

Thus the sum of the three terms is zero

46. From the second given equation

$$\sin^2\phi=\frac{\sin^2\beta\sin^2\theta}{\sin^2\alpha},$$

therefore $$\tan^2\phi=\frac{\sin^2\beta\sin^2\theta}{\sin^2\alpha-\sin^2\beta\sin^2\theta}$$

Substitute in the first given equation, thus

$$\frac{\tan^2\theta}{\tan^2\alpha}+\frac{\cos^2\beta\sin^2\theta}{\sin^2\alpha-\sin^2\beta\sin^2\theta}=1,$$

therefore $$\frac{\tan^2\theta}{\tan^2\alpha}=\frac{\sin^2\alpha-\sin^2\beta\sin^2\theta-\cos^2\beta\sin^2\theta}{\sin^2\alpha-\sin^2\beta\sin^2\theta}$$

$$=\frac{\sin^2\alpha-\sin^2\theta}{\sin^2\alpha-\sin^2\beta\sin^2\theta};$$

therefore $$\frac{\sin^2\theta\cos^2\alpha}{(1-\sin^2\theta)\sin^2\alpha}=\frac{\sin^2\alpha-\sin^2\theta}{\sin^2\alpha-\sin^2\beta\sin^2\theta};$$

therefore

$$\sin^2\theta\cos^2\alpha(\sin^2\alpha-\sin^2\beta\sin^2\theta)=(\sin^2\alpha-\sin^2\theta)(1-\sin^2\theta)\sin^2\alpha,$$

therefore

$$\sin^4\theta(\sin^2\alpha+\cos^2\alpha\sin^2\beta)-\sin^2\theta(\cos^2\alpha\sin^2\alpha+\sin^2\alpha+\sin^4\alpha)+\sin^4\alpha=0,$$

therefore $$\sin^4\theta(1-\cos^2\alpha\cos^2\beta)-2\sin^2\theta\sin^2\alpha+\sin^4\alpha=0$$

By solving this quadratic in the ordinary way we obtain

$$\sin^2\theta=\frac{1\pm\cos\alpha\cos\beta}{1-\cos^2\alpha\cos^2\beta}\sin^2\alpha=\frac{\sin^2\alpha}{1\mp\cos\alpha\cos\beta}$$

47 $$\frac{\sin\{\theta-\beta-(\alpha-\beta)\}}{\sin(\theta-\beta)}=\frac{a}{b},$$

therefore $$\frac{\sin(\theta-\beta)\cos(\alpha-\beta)-\cos(\theta-\beta)\sin(\alpha-\beta)}{\sin(\theta-\beta)}=\frac{a}{b},$$

therefore $$\cos(\alpha-\beta)-\sin(\alpha-\beta)\cot(\theta-\beta)=\frac{a}{b}.$$

Again, $$\frac{\cos\{\theta-\beta-(\alpha-\beta)\}}{\cos(\theta-\beta)}=\frac{a'}{b'},$$

therefore $$\frac{\cos(\theta-\beta)\cos(\alpha-\beta)+\sin(\theta-\beta)\sin(\alpha-\beta)}{\cos(\theta-\beta)}=\frac{a'}{b'},$$

therefore $$\cos(\alpha-\beta)+\tan(\theta-\beta)\sin(\alpha-\beta)=\frac{a'}{b'}$$

Hence $\sin(\alpha-\beta)\cot(\theta-\beta)\sin(\alpha-\beta)\tan(\theta-\beta)$

$$=\left\{\cos(\alpha-\beta)-\frac{a}{b}\right\}\left\{\frac{a'}{b'}-\cos(\alpha-\beta)\right\},$$

therefore $$\sin^2(\alpha-\beta)=-\frac{aa'}{bb'}+\left(\frac{a}{b}+\frac{a'}{b'}\right)\cos(\alpha-\beta)-\cos^2(\alpha-\beta);$$

therefore $$1+\frac{aa'}{bb'}=\left(\frac{a}{b}+\frac{a'}{b'}\right)\cos(\alpha-\beta),$$

therefore $$\cos(\alpha-\beta)=\frac{aa'+bb'}{ab'+a'b}.$$

48 $\tan\phi=\dfrac{2\tan\dfrac{\phi}{2}}{1-\tan^2\dfrac{\phi}{2}}$, thus

$$\frac{2\tan\dfrac{\phi}{2}}{1-\tan^2\dfrac{\phi}{2}}=\frac{\sin\theta\cos\theta'}{\sin\theta'+\cos\theta},$$

therefore $$2\tan\frac{\phi}{2}(\sin\theta'+\cos\theta)=\left(1-\tan^2\frac{\phi}{2}\right)\sin\theta\cos\theta';$$

therefore $$\sin\theta\cos\theta'\tan^2\frac{\phi}{2}+2\tan\frac{\phi}{2}(\sin\theta'+\cos\theta)=\sin\theta\cos\theta'.$$

By solving this quadratic in the ordinary way we obtain

$$\tan\frac{\phi}{2}=\frac{-(\sin\theta'+\cos\theta)\pm(1+\sin\theta'\cos\theta)}{\sin\theta\cos\theta'}$$

Take the upper sign; thus $\tan\dfrac{\phi}{2}=\dfrac{(1-\sin\theta')(1-\cos\theta)}{\sin\theta\cos\theta'}$.

Now $$\frac{1-\cos\theta}{\sin\theta}=\frac{2\sin^2\dfrac{\theta}{2}}{2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}=\tan\frac{\theta}{2},$$

and similarly $$\frac{1-\sin\theta'}{\cos\theta'}=\frac{1-\cos\left(\frac{\pi}{2}-\theta'\right)}{\sin\left(\frac{\pi}{2}-\theta'\right)}=\tan\left(\frac{\pi}{4}-\frac{\theta'}{2}\right),$$

thus $$\tan\frac{\phi}{2}=\tan\frac{\theta}{2}\tan\left(\frac{\pi}{4}-\frac{\theta'}{2}\right)$$

In like manner with the lower sign we shall find that

$$\tan\frac{\phi}{2}=-\cot\frac{\theta}{2}\cot\left(\frac{\pi}{4}-\frac{\theta'}{2}\right)$$

The product of the two values of $\tan\frac{\phi}{2}$ is -1, as it should be by the nature of quadratic equations

49 $\cos\theta=\cos\alpha\cos\beta$,

therefore $$\frac{1-\cos\theta}{1+\cos\theta}=\frac{1-\cos\alpha\cos\beta}{1+\cos\alpha\cos\beta};$$

therefore $$\tan^2\frac{\theta}{2}=\frac{1-\cos\alpha\cos\beta}{1+\cos\alpha\cos\beta}.$$

Similarly $$\tan^2\frac{\theta'}{2}=\frac{1-\cos\alpha'\cos\beta}{1+\cos\alpha'\cos\beta}$$

Hence $$\frac{(1-\cos\alpha\cos\beta)(1-\cos\alpha'\cos\beta)}{(1+\cos\alpha\cos\beta)(1+\cos\alpha'\cos\beta)}=\tan^2\frac{\beta}{2}=\frac{1-\cos\beta}{1+\cos\beta},$$

therefore $$\frac{1-(\cos\alpha+\cos\alpha')\cos\beta+\cos\alpha\cos\alpha'\cos^2\beta}{1+(\cos\alpha+\cos\alpha')\cos\beta+\cos\alpha\cos\alpha'\cos^2\beta}=\frac{1-\cos\beta}{1+\cos\beta},$$

therefore $$\frac{(\cos\alpha+\cos\alpha')\cos\beta}{1+\cos\alpha\cos\alpha'\cos^2\beta}=\cos\beta,$$

therefore $$\cos\alpha+\cos\alpha'=1+\cos\alpha\cos\alpha'(1-\sin^2\beta),$$

therefore $$\sin^2\beta\cos\alpha\cos\alpha'=1-\cos\alpha-\cos\alpha'+\cos\alpha\cos\alpha'$$
$$=(1-\cos\alpha)(1-\cos\alpha'),$$

therefore $$\sin^2\beta=\left(\frac{1}{\cos\alpha}-1\right)\left(\frac{1}{\cos\alpha'}-1\right)$$
$$=(\sec\alpha-1)(\sec\alpha'-1)$$

50. Here

$$\sin(C+A-B)-\sin(B+C-A)=\sin(A+B-C)-\sin(C+A-B),$$

therefore $$2\sin(A-B)\cos C=2\sin(B-C)\cos A,$$

therefore $(\sin A\cos B-\cos A\sin B)\cos C=(\sin B\cos C-\cos B\sin C)\cos A$

Divide by $\cos A \cos B \cos C$, thus

$$\tan A - \tan B = \tan B - \tan C,$$

therefore $\tan A$, $\tan B$, and $\tan C$ are in Arithmetical Progression

51 Suppose $\sin A$, $\sin B$, and $\sin C$ to be in Arithmetical Progression, so that $\sin B - \sin A = \sin C - \sin B$

Thus $$2 \sin \frac{B-A}{2} \cos \frac{B+A}{2} = 2 \sin \frac{C-B}{2} \cos \frac{C+B}{2},$$

therefore $$\sin \frac{B-A}{2} \sin \frac{C}{2} = \sin \frac{C-B}{2} \sin \frac{A}{2},$$

therefore $$\left(\sin \frac{B}{2} \cos \frac{A}{2} - \cos \frac{B}{2} \sin \frac{A}{2}\right) \sin \frac{C}{2}$$
$$= \left(\sin \frac{C}{2} \cos \frac{B}{2} - \cos \frac{C}{2} \sin \frac{B}{2}\right) \sin \frac{A}{2}$$

Divide by $\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$, thus

$$\cot \frac{A}{2} - \cot \frac{B}{2} = \cot \frac{B}{2} - \cot \frac{C}{2},$$

thus $\cot \frac{C}{2}$, $\cot \frac{B}{2}$ and $\cot \frac{A}{2}$ are in Arithmetical Progression

52 Suppose $$\cos^2 A + \cos^2 B + \cos^2 C = 1,$$

therefore $$3 - \sin^2 A - \sin^2 B - \sin^2 C = 1,$$

therefore $$\sin^2 A + \sin^2 B + \sin^2 C = 2,$$

therefore by Example 23 we have $\cos A \cos B \cos C = 0$, therefore one of the three angles is a right angle, and this will be the largest angle Suppose it to be A, so that $A = 90^\circ$, therefore $B + C = 90^\circ = A$, therefore $A - C = B$

53 $$\sin\left(A + \frac{C}{2}\right) = \sin\left(A + \frac{180^\circ - A - B}{2}\right)$$
$$= \sin\left(90^\circ - \frac{B-A}{2}\right) = \cos \frac{B-A}{2},$$

and $$\sin \frac{C}{2} = \cos \frac{A+B}{2},$$

thus $$\cos \frac{B-A}{2} = n \cos \frac{A+B}{2},$$

therefore $$\cos \frac{A}{2} \cos \frac{B}{2} + \sin \frac{A}{2} \sin \frac{B}{2} = n\left(\cos \frac{A}{2} \cos \frac{B}{2} - \sin \frac{A}{2} \sin \frac{B}{2}\right),$$

therefore $$(n+1) \sin \frac{A}{2} \sin \frac{B}{2} = (n-1) \cos \frac{A}{2} \cos \frac{B}{2},$$

therefore $$\frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}=\frac{n-1}{n+1},$$

therefore $$\tan\frac{A}{2}\tan\frac{B}{2}=\frac{n-1}{n+1}$$

54 Suppose $\frac{1}{k}$ to denote the value of $\frac{\sin A}{x}$, $\frac{\sin B}{y}$ and $\frac{\sin C}{z}$, then

$$x=k\sin A,\quad y=k\sin B,\quad z=k\sin C$$

Therefore $$(x-y)\cot\frac{C}{2}=k(\sin A-\sin B)\cot\frac{C}{2}$$

$$=2k\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B)\cot\frac{C}{2}$$

$$=2k\sin\frac{1}{2}(A-B)\sin\frac{C}{2}\cot\frac{C}{2}$$

$$=2k\sin\frac{1}{2}(A-B)\cos\frac{C}{2}$$

$$=2k\sin\frac{1}{2}(A-B)\sin\frac{1}{2}(A+B)$$

$$=2k\{\sin^2\frac{1}{2}A-\sin^2\frac{1}{2}B\},$$ by Art 83

Similarly $$(y-z)\cot\frac{A}{2}=2k\{\sin^2\frac{1}{2}B-\sin^2\frac{1}{2}C\},$$

and $$(z-x)\cot\frac{B}{2}=2k\{\sin^2\frac{1}{2}C-\sin^2\frac{1}{2}A\}$$

Thus the sum of the three terms is zero

55 $\tan(A+B+C)=\tan m\pi=0$, and therefore, by Art. 113,

$$\tan A+\tan B+\tan C-\tan A\tan B\tan C=0$$

56 $$\sin(2\alpha+x)+\sin(2\beta+x)=2\sin(\alpha+\beta+x)\cos(\alpha-\beta),$$

$$\sin(2\gamma+x)-\sin(2\alpha+2\beta+2\gamma+3x)=-2\sin(\alpha+\beta+x)\cos(\alpha+\beta+2\gamma+2x),$$

$$2\sin(\alpha+\beta+x)\{\cos(\alpha-\beta)-\cos(\alpha+\beta+2\gamma+2x)\}$$
$$=2\sin(\alpha+\beta+x)\,2\sin(\beta+\gamma+x)\sin(\alpha+\gamma+x)$$
$$=4\sin(\alpha+\beta+x)\sin(\beta+\gamma+x)\sin(\gamma+\alpha+x)$$

57 If $x=0$ we have

$$\sin 2\alpha+\sin 2\beta+\sin 2\gamma-\sin(2\alpha+2\beta+2\gamma)=4\sin(\alpha+\beta)\sin(\beta+\gamma)\sin(\gamma+\alpha)$$

If then $\alpha+\beta+\gamma=\pi$ we have $\sin(2\alpha+2\beta+2\gamma)=0$,

also $$\sin(\alpha+\beta)=\sin\gamma,\quad \sin(\beta+\gamma)=\sin\alpha,\quad \sin(\gamma+\alpha)=\sin\beta,$$

so that $$\sin 2\alpha+\sin 2\beta+\sin 2\gamma=4\sin\gamma\sin\alpha\sin\beta$$

If $x=\frac{\pi}{2}$ we have

$$\cos 2\alpha+\cos 2\beta+\cos 2\gamma+\cos(2\alpha+2\beta+2\gamma)$$
$$=4\cos(\alpha+\beta)\cos(\beta+\gamma)\cos(\gamma+\alpha)$$

If then $\alpha+\beta+\gamma=\frac{\pi}{2}$ we have $\cos(2\alpha+2\beta+2\gamma)=-1$,

also $\cos(\alpha+\beta)=\sin\gamma,\quad \cos(\beta+\gamma)=\sin\alpha,\quad \cos(\gamma+\alpha)=\sin\beta,$

so that $\cos 2\alpha+\cos 2\beta+\cos 2\gamma-1=4\sin\alpha\sin\beta\sin\gamma$

58 $$4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}=2\cos\frac{\gamma}{2}\left\{\cos\frac{1}{2}(\alpha-\beta)+\cos\frac{1}{2}(\alpha+\beta)\right\}$$

$$=\cos\frac{1}{2}(\gamma+\alpha-\beta)+\cos\frac{1}{2}(\gamma+\beta-\alpha)+\cos\frac{1}{2}(\alpha+\beta+\gamma)+\cos\frac{1}{2}(\alpha+\beta-\gamma)$$

Thus the left-hand member of the proposed expression

$$=\sin\alpha+\sin\beta+\sin\gamma-\cos\frac{1}{2}(\alpha+\beta+\gamma)-\cos\frac{1}{2}(\beta+\gamma-\alpha)$$
$$-\cos\frac{1}{2}(\alpha+\gamma-\beta)-\cos\frac{1}{2}(\alpha+\beta-\gamma)$$

Again

$$2\sin\frac{\alpha+\beta+\gamma-\pi}{4}\cos\frac{3\alpha-\beta-\gamma+\pi}{4}=\sin\alpha+\sin\frac{\beta+\gamma-\alpha-\pi}{2}$$
$$=\sin\alpha-\cos\frac{\beta+\gamma-\alpha}{2},$$

so also $$2\sin\frac{\alpha+\beta+\gamma-\pi}{4}\cos\frac{3\beta-\alpha-\gamma+\pi}{4}=\sin\beta-\cos\frac{\alpha+\gamma-\beta}{2},$$

$$2\sin\frac{\alpha+\beta+\gamma-\pi}{4}\cos\frac{3\gamma-\alpha-\beta+\pi}{4}=\sin\gamma-\cos\frac{\alpha+\beta-\gamma}{2},$$

and $$2\sin\frac{\alpha+\beta+\gamma-\pi}{4}\cos\frac{\alpha+\beta+\gamma-\pi}{4}=\sin\frac{\alpha+\beta+\gamma-\pi}{2}$$
$$=-\cos\frac{\alpha+\beta+\gamma}{2}$$

Thus the result is established

59. $$\cos 5\theta=\cos(3\theta+2\theta)=\cos 3\theta\cos 2\theta-\sin 3\theta\sin 2\theta$$
$$=(4\cos^3\theta-3\cos\theta)(2\cos^2\theta-1)-(3\sin\theta-4\sin^3\theta)\,2\sin\theta\cos\theta$$
$$=(4\cos^3\theta-3\cos\theta)(2\cos^2\theta-1)-2\sin^2\theta(3-4\sin^2\theta)\cos\theta$$
$$=(4\cos^3\theta-3\cos\theta)(2\cos^2\theta-1)-2(1-\cos^2\theta)(4\cos^2\theta-1)\cos\theta$$
$$=8\cos^5\theta-10\cos^3\theta+3\cos\theta-2(-4\cos^4\theta+5\cos^2\theta-1)\cos\theta$$
$$=16\cos^5\theta-20\cos^3\theta+5\cos\theta$$

60 $\sin 6\theta = 2\sin 3\theta\cos 3\theta = 2(3\sin\theta - 4\sin^3\theta)(4\cos^3\theta - 3\cos\theta)$

$$= 2\sin\theta(3 - 4\sin^2\theta)(4\cos^3\theta - 3\cos\theta)$$

$$= 2\sin\theta(4\cos^2\theta - 1)(4\cos^3\theta - 3\cos\theta)$$

$$= 2\sin\theta(16\cos^5\theta - 16\cos^3\theta + 3\cos\theta)$$

CHAPTER IX. P 91

1 Let $PCB = A$, so that $BPM = \frac{1}{2}A$ and $PAM = \frac{1}{2}A$ Then

$$\frac{MB}{PM} = \tan\frac{1}{2}A, \text{ and } \frac{PM}{AM} = \tan\frac{1}{2}A,$$

so that
$$\tan^2\frac{1}{2}A = \frac{MB}{PM}\cdot\frac{PM}{AM} = \frac{MB}{AM} = \frac{CB - CM}{CA + CM}$$

$$= \frac{CP - CM}{CP + CM} = \frac{1 - \dfrac{CM}{CP}}{1 + \dfrac{CM}{CP}} = \frac{1 - \cos A}{1 + \cos A}$$

2 $\cos\theta = \dfrac{a\cos\phi - b}{a - b\cos\phi}$; therefore

$$\frac{1 - \cos\theta}{1 + \cos\theta} = \frac{a - b\cos\phi - a\cos\phi + b}{a - b\cos\phi + a\cos\phi - b} = \frac{(a+b)(1 - \cos\phi)}{(a-b)(1 + \cos\phi)},$$

therefore
$$\tan^2\frac{\theta}{2} = \frac{a+b}{a-b}\tan^2\frac{\phi}{2},$$

therefore
$$\frac{\tan^2\dfrac{\theta}{2}}{a+b} = \frac{\tan^2\dfrac{\phi}{2}}{a-b}$$

3. $\cos^2\theta = \dfrac{1}{1 + \tan^2\theta} = \dfrac{1}{1 + 2\tan^2\phi + 1} = \dfrac{1}{2(1 + \tan^2\phi)} = \dfrac{1}{2}\cos^2\phi,$

and
$$\cos 2\theta = 2\cos^2\theta - 1 = \cos^2\phi - 1 = -\sin^2\phi,$$
therefore
$$\cos 2\theta + \sin^2\phi = 0.$$

4 $\sec 2\theta = 2\sec\theta\operatorname{cosec}\theta$, therefore $\dfrac{1}{\cos 2\theta} = \dfrac{2}{\cos\theta\sin\theta},$

therefore
$$1 = \frac{2\cos 2\theta}{\cos\theta\sin\theta},$$

therefore $$\frac{1}{\sin 2\theta}=\frac{2\cos 2\theta}{\sin 2\theta\cos\theta\sin\theta}=\frac{\cos 2\theta}{\sin^2\theta\cos^2\theta}$$

$$=\frac{\cos^2\theta-\sin^2\theta}{\sin^2\theta\cos^2\theta}=\frac{1}{\sin^2\theta}-\frac{1}{\cos^2\theta}$$

Thus $$\operatorname{cosec} 2\theta=\operatorname{cosec}^2\theta-\sec^2\theta$$

5 $$\tan(\theta-\phi)=\frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}=\frac{(n-1)\tan\phi}{1+n\tan^2\phi}=\frac{n-1}{\cot\phi+n\tan\phi},$$

therefore $$\tan^2(\theta-\phi)=\frac{(n-1)^2}{\cot^2\phi+2n+n^2\tan^2\phi}=\frac{(n-1)^2}{(n\tan\phi-\cot\phi)^2+4n}$$

The greatest value of this fraction is when the denominator is least, th is when the term $n\tan\phi-\cot\phi$ vanishes

6 $$\sin\theta+\sin\phi-\cos\theta\sin(\theta+\phi)$$

$$=2\sin\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi)-2\cos\theta\sin\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta+\phi)$$

$$=2\sin\frac{1}{2}(\theta+\phi)\left\{\cos\frac{1}{2}(\theta-\phi)-\cos\theta\cos\frac{1}{2}(\theta+\phi)\right\}$$

$$=2\sin\frac{1}{2}(\theta+\phi)\left\{\cos\left(\theta-\frac{\theta+\phi}{2}\right)-\cos\theta\cos\frac{1}{2}(\theta+\phi)\right\}$$

$$=2\sin\frac{1}{2}(\theta+\phi)\sin\theta\sin\frac{1}{2}(\theta+\phi)=2\sin\theta\sin^2\frac{1}{2}(\theta+\phi)$$

7 $$\frac{\sin\beta\cos\alpha(\tan\alpha+\tan\beta)}{1-\cos(\alpha+\beta)}=\frac{\sin\beta\cos\alpha}{2\sin^2\frac{1}{2}(\alpha+\beta)}\left\{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}\right\}$$

$$=\frac{\sin\beta\cos\alpha}{2\sin^2\frac{1}{2}(\alpha+\beta)}\ \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$$

$$=\frac{\sin\beta\,.\,2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha+\beta)}{2\sin^2\frac{1}{2}(\alpha+\beta)\cos\beta}$$

$$=\frac{\sin\beta\cos\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha+\beta)\cos\beta},$$

and
$$\frac{\sin\beta\cos\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha+\beta)\cos\beta}+\frac{\sin\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha+\beta)\cos\beta}$$

$$=\frac{\sin\left(\frac{\alpha+\beta}{2}-\beta\right)+\sin\beta\cos\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha+\beta)\cos\beta}=\frac{\sin\frac{1}{2}(\alpha+\beta)\cos\beta}{\sin\frac{1}{2}(\alpha+\beta)\cos\beta}=1.$$

8 Let x denote the height in yards, then $\frac{x}{1760}=\tan 1'$, therefore $x=1760\tan 1'$ The value of $\tan 1'$ is approximately equal to the circular measure of $1'$, that is to $\frac{\pi}{180\times 60}$, therefore $x=\frac{1760\pi}{180\times 60}$ approximately

9 Let x denote the distance in inches, then $\frac{3}{x}=\tan\frac{1^\circ}{4}$, and taking the tangent as approximately equal to the circular measure we have

$$\frac{3}{x}=\frac{\pi}{180\times 4},\quad\text{therefore } x=\frac{12\times 180}{\pi}$$

10 We have $3\sin A-4\sin^3 A=n\sin A$, as we suppose that A is not zero nor a multiple of two right angles we may divide by $\sin A$, thus $3-4\sin^2 A=n$, therefore $\sin^2 A=\frac{3-n}{4}$, and as this must lie between zero and unity, n must lie between 3 and -1

If $n=2$ we have $\sin^2 A=\frac{1}{4}=\sin^2\frac{\pi}{6}$, therefore $A=m\pi\pm\frac{\pi}{6}$, where m is zero or any integer

11 $$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

$$=\frac{\tan\alpha-\frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}}{1+\tan\alpha\,\frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}}$$

$$=\frac{\sin\alpha(1-n\sin^2\alpha)-n\sin\alpha\cos^2\alpha}{\cos\alpha(1-n\sin^2\alpha)+n\sin^2\alpha\cos\alpha}$$

$$=\frac{\sin\alpha-n\sin\alpha}{\cos\alpha}=\frac{(1-n)\sin\alpha}{\cos\alpha}=(1-n)\tan\alpha$$

12 All the angles which have the same sine as 3θ are included in the formula $n\pi+(-1)^n 3\theta$ Therefore any expression which gives the value of $\tan\theta$ in terms of $\sin 3\theta$ may be expected to give the value of the tangent of every angle included in the formula $\tan\frac{1}{3}\{n\pi+(-1)^n 3\theta\}$

Now n must be of one of the following forms

$$6m,\ 6m+1,\ 6m+2,\ 6m+3,\ 6m+4,\ 6m+5$$

The corresponding values of $\tan\frac{1}{3}\left\{n\pi+(-1)^n 3\theta\right\}$ are, by Art 45,

$$\tan\theta,\ \tan\left(\frac{\pi}{3}-\theta\right),\ \tan\left(\frac{2\pi}{3}+\theta\right),\ \tan(\pi-\theta),$$

$$\tan\left(\pi+\frac{\pi}{3}+\theta\right),\ \tan\left(\pi+\frac{2\pi}{3}-\theta\right)$$

Thus we have six distinct values They may also by Arts 48 and 50 be expressed thus

$$\pm\tan\theta,\ \pm\tan\left(\frac{\pi}{3}+\theta\right),\ \pm\tan\left(\frac{2\pi}{3}+\theta\right)$$

13 $\cos^2 A=\frac{1}{2}(1+\cos 2A)$, therefore

$$\cos^4 A=\frac{1}{4}(1+2\cos 2A+\cos^2 2A)$$

$$=\frac{1}{4}+\frac{1}{2}\cos 2A+\frac{1+\cos 4A}{8}$$

$$=\frac{3}{8}+\frac{1}{2}\cos 2A+\frac{1}{8}\cos 4A$$

Similarly $\sin^2 A=\frac{1}{2}(1-\cos 2A)$;

therefore $\sin^4 A=\frac{1}{4}(1-2\cos 2A+\cos^2 2A)$

$$=\frac{3}{8}-\frac{1}{2}\cos 2A+\frac{1}{8}\cos 4A$$

Therefore $\cos^8 A+\sin^8 A$

$$=\left(\frac{3}{8}+\frac{1}{2}\cos 2A+\frac{1}{8}\cos 4A\right)^2+\left(\frac{3}{8}-\frac{1}{2}\cos 2A+\frac{1}{8}\cos 4A\right)^2$$

$$=2\left\{\left(\frac{3}{8}\right)^2+\left(\frac{1}{2}\cos 2A\right)^2+\left(\frac{1}{8}\cos 4A\right)^2+2\ \frac{3}{8}\ \frac{1}{8}\cos 4A\right\}$$

$$=2\left\{\frac{9}{64}+\frac{1}{4}\cos^2 2A+\frac{1}{64}\cos^2 4A+\frac{3}{32}\cos 4A\right\}$$

$$=\frac{9}{32}+\frac{1}{4}(1+\cos 4A)+\frac{1}{64}(1+\cos 8A)+\frac{3}{16}\cos 4A$$

$$=\frac{1}{64}\{\cos 8A+28\cos 4A+35\}.$$

14 $\cos\theta\cos\phi = -1$

As the cosine of an angle is never numerically greater than unity, we must have $\cos\theta$ and $\cos\phi$ both numerically equal to unity, one being positive and the other negative Hence one of the angles must be zero or an even multiple of π, and the other must be an odd multiple of π

15 $\sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta\cos(\alpha-\beta)$

$$= \sin\alpha\{\sin\alpha - \sin\beta\cos(\alpha-\beta)\} + \sin\beta\{\sin\beta - \sin\alpha\cos(\alpha-\beta)\}$$

$$= \sin\alpha\{\sin(\alpha-\beta+\beta) - \sin\beta\cos(\alpha-\beta)\} + \sin\beta\{\sin(\alpha-\overline{\alpha-\beta}) - \sin\alpha\cos(\alpha-\beta)\}$$

$$= \sin\alpha\sin(\alpha-\beta)\cos\beta - \sin\beta\cos\alpha\sin(\alpha-\beta)$$

$$= \sin(\alpha-\beta)\{\sin\alpha\cos\beta - \sin\beta\cos\alpha\} = \sin^2(\alpha-\beta)$$

Thus $\sin^2(\alpha-\beta) = n^2\sin^2(\alpha+\beta)$,

therefore $\sin(\alpha-\beta) = \pm n\sin(\alpha+\beta)$,

therefore $\sin\alpha\cos\beta - \cos\alpha\sin\beta = \pm n(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$;

divide by $\cos\alpha\cos\beta$, thus $\tan\alpha - \tan\beta = \pm n(\tan\alpha + \tan\beta)$;

therefore $(1\mp n)\tan\alpha = (1\pm n)\tan\beta$,

therefore $\tan\alpha = \dfrac{1\pm n}{1\mp n}\tan\beta$

16 $$\frac{\sin 4\theta\cot\theta}{\text{vers } 2\theta\cot^2 2\theta} = \frac{\sin 4\theta\sin^2 2\theta\cos\theta}{(1-\cos 2\theta)\cos^2 2\theta\sin\theta} = \frac{2\sin^3 2\theta\cos 2\theta\cos\theta}{2\sin^3\theta\cos^2 2\theta}$$

$$= \frac{2(2\sin\theta\cos\theta)^3\cos\theta}{2\sin^3\theta\cos 2\theta} = \frac{8\cos^4\theta}{\cos 2\theta}$$

When $\theta = 0$ the value is therefore 8

17 $\sin\theta + \cos\theta = \sqrt{2}$, therefore $\dfrac{\sin\theta}{\sqrt{2}} + \dfrac{\cos\theta}{\sqrt{2}} = 1$,

therefore $\cos\left(\theta - \dfrac{\pi}{4}\right) = 1$; therefore $\theta - \dfrac{\pi}{4} = 2n\pi$

18 $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$, therefore $\dfrac{\sqrt{3}}{2}\sin\theta - \dfrac{1}{2}\cos\theta = \dfrac{1}{\sqrt{2}}$,

therefore $\dfrac{1}{2}\cos\theta - \dfrac{\sqrt{3}}{2}\sin\theta = -\dfrac{1}{\sqrt{2}}$,

therefore $\cos\left(\theta + \dfrac{\pi}{3}\right) = -\dfrac{1}{\sqrt{2}}$,

therefore $\theta + \dfrac{\pi}{3} = 2n\pi \pm \dfrac{3\pi}{4}$.

19. $\sin 2\theta=\cos\theta$, therefore $\cos\left(\frac{\pi}{2}-2\theta\right)=\cos\theta$,

therefore $\frac{\pi}{2}-2\theta$ and θ are angles having the same cosine, therefore all the solutions are contained in $\frac{\pi}{2}-2\theta=2n\pi\pm\theta$.

20. $\cos\theta-\cos 2\theta=\sin 3\theta$, therefore

$$2\sin\frac{3\theta}{2}\sin\frac{\theta}{2}=2\sin\frac{3\theta}{2}\cos\frac{3\theta}{2},$$

therefore either $\sin\frac{3\theta}{2}=0$, or $\sin\frac{\theta}{2}=\cos\frac{3\theta}{2}$

If $\sin\frac{3\theta}{2}=0$, then $\frac{3\theta}{2}=n\pi$

If $\sin\frac{\theta}{2}=\cos\frac{3\theta}{2}$, then $\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)=\cos\frac{3\theta}{2}$,

and therefore $$\frac{\pi}{2}-\frac{\theta}{2}=2n\pi\pm\frac{3\theta}{2}$$

21. $(4-\sqrt{3})(\sec\theta+\operatorname{cosec}\theta)=4(\sin\theta\tan\theta+\cos\theta\cot\theta)$,

therefore $$(4-\sqrt{3})\left(\frac{1}{\cos\theta}+\frac{1}{\sin\theta}\right)=4\left(\frac{\sin^2\theta}{\cos\theta}+\frac{\cos^2\theta}{\sin\theta}\right),$$

therefore $$(4-\sqrt{3})(\sin\theta+\cos\theta)=4(\sin^3\theta+\cos^3\theta)$$
$$=4(\sin\theta+\cos\theta)(\sin^2\theta+\cos^2\theta-\sin\theta\cos\theta),$$

therefore either $$\sin\theta+\cos\theta=0,$$

or $$4-\sqrt{3}=4(1-\sin\theta\cos\theta)$$

If $\sin\theta+\cos\theta=0$, then $\sin\theta=-\cos\theta$, therefore $\tan\theta=-1$,

therefore $$\theta=n\pi+\frac{3\pi}{4}$$

If $4-\sqrt{3}=4(1-\sin\theta\cos\theta)$, then $\sqrt{3}=4\sin\theta\cos\theta=2\sin 2\theta$,

therefore $\sin 2\theta=\frac{\sqrt{3}}{2}$, therefore $2\theta=n\pi+(-1)^n\frac{\pi}{3}$

22. $\cot\theta-\tan\theta=\cos\theta+\sin\theta$, therefore $\frac{\cos\theta}{\sin\theta}-\frac{\sin\theta}{\cos\theta}=\cos\theta+\sin\theta$,

therefore $$\cos^2\theta-\sin^2\theta=\sin\theta\cos\theta(\cos\theta+\sin\theta),$$

therefore either $\cos\theta+\sin\theta=0$, or $\cos\theta-\sin\theta=\sin\theta\cos\theta$

If $\sin\theta+\cos\theta=0$, then $\sin\theta=-\cos\theta$, therefore $\tan\theta=-1$,

therefore $$\theta=n\pi+\frac{3\pi}{4}.$$

If $\cos\theta-\sin\theta=\sin\theta\cos\theta$, then by squaring

$$1-2\sin\theta\cos\theta=\sin^2\theta\cos^2\theta,$$

therefore $$1-\sin 2\theta=\frac{\sin^2 2\theta}{4}.$$

By solving this quadratic in the usual way we obtain $\sin 2\theta=-2\pm 2\sqrt{2}$, the upper sign must be taken, for the lower sign would make $\sin 2\theta$ numerically greater than unity.

23 $2\sin^2\theta+\sin^2 2\theta=2$, therefore $\sin^2 2\theta=2-2\sin^2\theta=2(1-\sin^2\theta)$,

therefore $$4\sin^2\theta\cos^2\theta=2\cos^2\theta,$$

therefore either $\cos^2\theta=0$, or $\sin^2\theta=\frac{1}{2}$

If $\cos^2\theta=0$, then $$\theta=n\pi+\frac{\pi}{2}$$

If $\sin^2\theta=\frac{1}{2}$, then $$\sin^2\theta=\sin^2\frac{\pi}{4},$$

therefore $$\theta=n\pi\pm\frac{\pi}{4}$$

24 $\tan\theta+2\cot 2\theta=\sin\theta\left(1+\tan\theta\tan\frac{\theta}{2}\right)$, therefore

$$\frac{\sin\theta}{\cos\theta}+\frac{2\cos 2\theta}{\sin 2\theta}=\sin\theta\left(1+\frac{\sin\theta\sin\frac{\theta}{2}}{\cos\theta\cos\frac{\theta}{2}}\right),$$

therefore $$\frac{\sin^2\theta+\cos 2\theta}{\sin\theta\cos\theta}=\sin\theta\;\frac{\cos\left(\theta-\frac{\theta}{2}\right)}{\cos\theta\cos\frac{\theta}{2}}=\frac{\sin\theta}{\cos\theta},$$

therefore $\sin^2\theta+\cos 2\theta=\sin^2\theta$, therefore $\cos 2\theta=0$,

therefore $$2\theta=n\pi+\frac{\pi}{2}$$

25 $\sin^2 2\theta-\sin^2\theta=\sin^2\frac{\pi}{6}=\frac{1}{4}$; therefore

$$4\sin^2\theta\cos^2\theta-\sin^2\theta=\frac{1}{4},$$

therefore $$4\sin^2\theta\,(1-\sin^2\theta)-\sin^2\theta=\frac{1}{4},$$

therefore $$4\sin^4\theta-3\sin^2\theta+\frac{1}{4}=0$$

By solving this quadratic in the usual way we obtain

$$\sin^2\theta=\frac{3\pm\sqrt{5}}{8}$$

Taking the upper sign we have $\sin^2\theta=\sin^2\frac{3\pi}{10}$, and therefore

$$\theta=n\pi\pm\frac{3\pi}{10}$$

Taking the lower sign we have $\sin^2\theta=\sin^2\frac{\pi}{10}$, and therefore

$$\theta=n\pi\pm\frac{\pi}{10}$$

26. $\operatorname{cosec}\theta=\operatorname{cosec}\frac{\theta}{2}$, therefore $\dfrac{1}{\sin\theta}=\dfrac{1}{\sin\frac{\theta}{2}}$,

therefore $\sin\frac{\theta}{2}=\sin\theta$, therefore $\sin\frac{\theta}{2}=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$,

therefore either $\sin\frac{\theta}{2}=0$, or $\cos\frac{\theta}{2}=\frac{1}{2}$.

If $\sin\frac{\theta}{2}=0$, then $\frac{\theta}{2}=n\pi$

If $\cos\frac{\theta}{2}=\frac{1}{2}$, then $\frac{\theta}{2}=2m\pi\pm\frac{\pi}{3}$.

27 $\cos\theta\cos3\theta=\cos5\theta\cos7\theta$, therefore

$$\cos4\theta+\cos2\theta=\cos12\theta+\cos2\theta,$$

therefore $\cos4\theta=\cos12\theta$, therefore $12\theta=2n\pi\pm4\theta$,

taking the upper sign we obtain $\theta=\frac{2n\pi}{8}=\frac{n\pi}{4}$,

and taking the lower sign we obtain $\theta=\frac{2n\pi}{16}=\frac{n\pi}{8}$

It is obvious however that the second expression includes the first

28 $\sin\theta\sin3\theta=\frac{1}{2}$, therefore $\sin\theta\,(3\sin\theta-4\sin^3\theta)=\frac{1}{2}$,

therefore $$4\sin^4\theta-3\sin^2\theta+\frac{1}{2}=0$$

By solving this quadratic in the usual way we obtain

$$\sin^2\theta=\frac{3\pm 1}{8}=\frac{1}{2} \text{ or } \frac{1}{4}$$

If $\sin^2\theta=\frac{1}{2}$, then $\sin^2\theta=\sin^2\frac{\pi}{4}$, and $\theta=n\pi\pm\frac{\pi}{4}$

If $\sin^2\theta=\frac{1}{4}$, then $\sin^2\theta=\sin^2\frac{\pi}{6}$, and $\theta=n\pi\pm\frac{\pi}{6}$

See Example 5 of Chapter V

29 $4\sin^2\theta+\sin^2 2\theta=3$, therefore $4\sin^2\theta+4\sin^2\theta(1-\sin^2\theta)=3$,

therefore $$4\sin^4\theta-8\sin^2\theta+3=0$$

By solving this quadratic in the usual way we obtain $\sin^2\theta=\frac{1}{2}$ or $\frac{3}{2}$, and only the former value is admissible Thus $\sin^2\theta=\sin^2\frac{\pi}{4}$, therefore $\theta=n\pi\pm\frac{\pi}{4}$

30 $(1-\tan\theta)(1+\sin 2\theta)=1+\tan\theta$,

therefore $$\left(1-\frac{\sin\theta}{\cos\theta}\right)(\sin\theta+\cos\theta)^2=1+\frac{\sin\theta}{\cos\theta},$$

therefore $$(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)^2=\cos\theta+\sin\theta;$$

therefore either $\cos\theta+\sin\theta=0$, or $(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)=1$.

If $\cos\theta+\sin\theta=0$, then $\sin\theta=-\cos\theta$,

therefore $$\tan\theta=-1,$$

therefore $$\theta=n\pi+\frac{3\pi}{4}$$

If $(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)=1$, then $\cos^2\theta-\sin^2\theta=1$,

therefore $$\cos 2\theta=1,$$

therefore $$2\theta=2n\pi.$$

31 $\sin\theta+\sin 2\theta+\sin 3\theta+\sin 4\theta=0$;

therefore $$\sin\theta+\sin 4\theta+\sin 2\theta+\sin 3\theta=0,$$

therefore $$2\sin\frac{5\theta}{2}\cos\frac{3\theta}{2}+2\sin\frac{5\theta}{2}\cos\frac{\theta}{2}=0,$$

therefore $$2\sin\frac{5\theta}{2}\left(\cos\frac{3\theta}{2}+\cos\frac{\theta}{2}\right)=0,$$

therefore $$4\sin\frac{5\theta}{2}\cos\frac{\theta}{2}\cos\theta=0$$

Thus there are three cases

If $\sin\frac{5\theta}{2}=0$, then $\frac{5\theta}{2}=n\pi$,

If $\cos\frac{\theta}{2}=0$, then $\frac{\theta}{2}=n\pi+\frac{\pi}{2}$,

If $\cos\theta=0$, then $\theta=n\pi+\frac{\pi}{2}$

32 $\sin\theta-\cos\theta=4\sin\theta\cos^2\theta$,

therefore $$\sin\theta-4\sin\theta(1-\sin^2\theta)=\cos\theta;$$

therefore $$4\sin^3\theta-3\sin\theta=\cos\theta,$$

therefore $$\cos\theta=-\sin 3\theta=\cos\left(3\theta+\frac{\pi}{2}\right),$$

therefore $$3\theta+\frac{\pi}{2}=2n\pi\pm\theta$$

33 $(\cot\theta-\tan\theta)^2(2-\sqrt{3})=4(2+\sqrt{3})$;

therefore $$\left(\frac{\cos\theta}{\sin\theta}-\frac{\sin\theta}{\cos\theta}\right)^2=\frac{4(2+\sqrt{3})}{2-\sqrt{3}},$$

therefore $$\left(\frac{\cos^2\theta-\sin^2\theta}{2\sin\theta\cos\theta}\right)^2=\frac{2+\sqrt{3}}{2-\sqrt{3}}=\frac{(2+\sqrt{3})^2}{(2-\sqrt{3})(2+\sqrt{3})};$$

therefore $$\left(\frac{\cos 2\theta}{\sin 2\theta}\right)^2=(2+\sqrt{3})^2,$$

therefore $$\cot^2 2\theta=\cot^2\frac{\pi}{12},$$

therefore $$2\theta=n\pi\pm\frac{\pi}{12}.$$

34. $2\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)(1+\sin\theta)=1+\cos 2\theta$,

therefore $$2\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)(1+\sin\theta)=2\cos^2\theta=2(1-\sin^2\theta);$$

therefore either $1+\sin\theta=0$, or $\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)=1-\sin\theta$

If $1+\sin\theta=0$, then $\sin\theta=-1$, therefore $\theta=n\pi+(-1)^n\frac{3\pi}{2}$, which may be expressed more simply as $(4m+3)\frac{\pi}{2}$

If $\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)=1-\sin\theta$, then $\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right)=1-\sin\theta$,

therefore $$2 \sin \theta = 1 - \cos \theta ;$$

therefore $$4 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = 2 \sin^2 \frac{\theta}{2},$$

therefore either $\sin \frac{\theta}{2} = 0$, or $\tan \frac{\theta}{2} = 2$

If $\sin \frac{\theta}{2} = 0$, then $\frac{\theta}{2} = n\pi$

If $\tan \frac{\theta}{2} = 2$, then $\frac{\theta}{2} = n\pi + \alpha$, where α is such that $\tan \alpha = 2$

35 $\sin 9\theta + \sin 5\theta + 2 \sin^2 \theta = 1$, therefore

$$2 \sin 7\theta \cos 2\theta = 1 - 2 \sin^2 \theta = \cos 2\theta,$$

therefore either $\cos 2\theta = 0$, or $\sin 7\theta = \frac{1}{2}$

If $\cos 2\theta = 0$, then $2\theta = n\pi + \frac{\pi}{2}$.

If $\sin 7\theta = \frac{1}{2}$, then $7\theta = n\pi + (-1)^n \frac{\pi}{6}$

CHAPTER X P 104

1 Let x denote the required logarithm, then

$$128 = (\sqrt[3]{4})^x, \text{ that is } 2^7 = 4^{\frac{x}{3}} = 2^{\frac{2x}{3}},$$

therefore $\frac{2x}{3} = 7$, therefore $x = \frac{21}{2}$

2 Let x denote the required logarithm, then

$$243 \sqrt[3]{9} = (\sqrt{3})^x, \text{ that is } 3^5 \sqrt[3]{9} = 3^{\frac{x}{2}}, \text{ that is } 3^{5+\frac{2}{3}} = 3^{\frac{x}{2}},$$

therefore $\frac{x}{2} = \frac{17}{3}$, therefore $x = \frac{34}{3}$

3 Let x denote the logarithm of 2187 to the base 3, then $2187 = 3^x$, that is $3^7 = 3^x$, therefore $x = 7$

Let x denote the logarithm of 0001 to the base 10, then $0001 = 10^x$, that is $\frac{1}{10^4} = 10^x$, that is $10^{-4} = 10^x$, therefore $x = -4$

Let x denote the logarithm of $\cos 45^0$ to the base 2, then $\cos 45^0 = 2^x$, that is $\frac{1}{\sqrt{2}} = 2^x$, that is $2^{-\frac{1}{2}} = 2^x$, therefore $x = -\frac{1}{2}$

4. $5^{6-4x}=2^{x+3}$; therefore $(6-4x)\log 5=(x+3)\log 2$,

therefore $$(6-4x)\log\frac{10}{2}=(x+3)\log 2,$$

therefore $$(6-4x)(1-\log 2)=(x+3)\log 2,$$

therefore $$x(4-3\log 2)=6-9\log 2,$$

therefore $$x=\frac{6-9\log 2}{4-3\log 2}=\frac{3{\cdot}29073}{3{\cdot}09691}=1{\cdot}06$$

5. Here $a=\log{\cdot}224=\log\frac{224}{1000}=\log\frac{7\times 32}{1000}=\log 7+5\log 2-3$,

$$b=\log 125=\log\frac{1000}{8}=3-3\log 2$$

From the second equation we have $\log 2=\frac{1}{3}(3-b)$, and then substituting in the first equation we have $\log 7=a+3-\frac{5}{3}(3-b)$

6. 725 lies between 6^3 and 6^4, and therefore the characteristic of the logarithm of 725 to the base 6 is 3

Then $$\log\sqrt[5]{({\cdot}0725)}=\frac{1}{5}\log{\cdot}0725=\frac{1}{5}\log\frac{725}{10000},$$

and $\frac{725}{10000}$ lies between $\frac{1}{6}$ and $\frac{1}{36}$, that is between 6^{-1} and 6^{-2}. Hence $\frac{1}{5}\log\frac{725}{10000}$ to the base 6 lies between $-\frac{1}{5}$ and $-\frac{2}{5}$, and thus the characteristic will be -1, since by supposition the decimal part of a logarithm is positive

7. $\operatorname{Log} 405=\log(81\times 5)=\log\left(81\times\frac{10}{2}\right)=\log\frac{3^4\times 10}{2}=4\log 3+1-\log 2$,

therefore $$4\log 3=\log 405+\log 2-1=8{\cdot}908485;$$

therefore $$\log 3={\cdot}477121$$

8. $\operatorname{Log} 98=\log(2\times 7^2)=\log 2+2\log 7={\cdot}301030+1{\cdot}690196=1{\cdot}991226$,

$$\log\left(\frac{4}{343}\right)^{\frac{1}{2}}=\frac{1}{2}\log\frac{4}{343}=\frac{1}{2}\log\frac{2^2}{7^3}=\frac{1}{2}(2\log 2-3\log 7)$$

$$=-{\cdot}966617=\bar{1}{\cdot}033383$$

9. $$\operatorname{Log}({\cdot}0020736)^{\frac{1}{3}}=\frac{1}{3}\log{\cdot}0020736=\frac{1}{3}\log\frac{20736}{10^7}$$

$$=\frac{1}{3}\log\frac{3^4\times 2^8}{10^7}=\frac{1}{3}\{4\log 3+8\log 2-7\}$$

$$=-{\cdot}89443=\bar{1}{\cdot}10557$$

10. $\frac{2}{\underline{|3}}=\frac{1}{\underline{|2}}-\frac{1}{\underline{|3}}$, $\frac{4}{\underline{|5}}=\frac{1}{\underline{|4}}-\frac{1}{\underline{|5}}$, $\frac{6}{\underline{|7}}=\frac{1}{\underline{|6}}-\frac{1}{\underline{|7}}$, .

thus we see that the series $=\frac{1}{\underline{|2}}-\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}-\frac{1}{\underline{|5}}+ \quad =e^{-1}$.

11.
$$\frac{1}{\underline{|2}}=\frac{1}{2}\;\frac{1\ 2}{\underline{|2}}=\frac{1}{2},$$
$$\frac{1+2}{\underline{|3}}=\frac{1}{2}\;\frac{2\ 3}{\underline{|3}}=\frac{1}{2}\cdot\frac{1}{1},$$
$$\frac{1+2+3}{\underline{|4}}=\frac{1}{2}\cdot\frac{3\ 4}{\underline{|4}}=\frac{1}{2}\cdot\frac{1}{\underline{|2}},$$
$$\frac{1+2+3+4}{\underline{|5}}=\frac{1}{2}\cdot\frac{4\ 5}{\underline{|5}}=\frac{1}{2}\;\frac{1}{\underline{|3}},$$

and generally $\frac{1+2+3+ \quad +n}{\underline{|n+1}}=\frac{1}{2}\cdot\frac{n(n+1)}{\underline{|n+1}}=\frac{1}{2}\cdot\frac{1}{\underline{|n-1}}$.

Thus we see that the series $=\frac{1}{2}\left\{1+\frac{1}{1}+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+ \quad \right\}=\frac{e}{2}$.

12 $4\sin x\sin(x-a)=2\cos a-1$,

therefore $$2\{\cos a-\cos(2x-a)\}=2\cos a-1,$$

therefore $$\cos(2x-a)=\frac{1}{2},$$

therefore $$2x-a=2n\pi\pm\frac{\pi}{3}.$$

13 $\cos\beta\sqrt{(a^2-x^2)}=x\sin\beta-a\sin\alpha$,

therefore $\cos^2\beta(a^2-x^2)=x^2\sin^2\beta-2xa\sin\beta\sin\alpha+a^2\sin^2\alpha$,

therefore $x^2-2xa\sin\beta\sin\alpha=a^2\cos^2\beta-a^2\sin^2\alpha$,

therefore $(x-a\sin\beta\sin\alpha)^2=a^2\cos^2\beta-a^2\sin^2\alpha+a^2\sin^2\beta\sin^2\alpha$
$$=a^2\cos^2\beta-a^2\sin^2\alpha\cos^2\beta=a^2\cos^2\beta\cos^2\alpha;$$

therefore $x-a\sin\beta\sin\alpha=\pm a\cos\beta\cos\alpha$,

therefore $x=a(\sin\beta\sin\alpha\pm\cos\beta\cos\alpha)=a\cos(\beta-\alpha)$ or $-a\cos(\beta+\alpha)$

14 $\sin a+\sin(x-a)+\sin(2x+a)=\sin(x+a)+\sin(2x-a)$,

therefore $\sin a=\sin(x+a)-\sin(x-a)+\sin(2x-a)-\sin(2x+a)$
$$=2\sin a\cos x-2\sin a\cos 2x,$$

therefore $$1=2\cos x-2\cos 2x=2\cos x-2(2\cos^2 x-1),$$

therefore $$4\cos^2 x-2\cos x-1=0$$

By solving this quadratic in the usual way we obtain $\cos x=\frac{1\pm\sqrt{5}}{4}$

Taking the upper sign we have $\cos x=\cos\frac{\pi}{5}$, and therefore $x=2n\pi\pm\frac{\pi}{5}$

Taking the lower sign we have $\cos x=\cos\frac{3\pi}{5}$, and therefore $x=2n\pi\pm\frac{3\pi}{5}$

15 $$\cos\left(x+\frac{3}{2}\right)a+\cos\left(x+\frac{1}{2}\right)a=\sin a,$$

therefore $$2\cos(x+1)a\cos\frac{a}{2}=\sin a=2\sin\frac{a}{2}\cos\frac{a}{2},$$

therefore $$\cos(x+1)a=\sin\frac{a}{2}=\cos\left(\frac{\pi}{2}-\frac{a}{2}\right)$$

Hence all the solutions are contained in

$$(x+1)a=2n\pi\pm\left(\frac{\pi}{2}-\frac{a}{2}\right)$$

16 $$x^2\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)+x\cos(\alpha-\beta)=2\cos\frac{\beta}{2},$$

therefore $$x^2+\frac{x\cos(\alpha-\beta)}{\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)}=\frac{2\cos\frac{\beta}{2}}{\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)},$$

therefore $$\left\{x+\frac{\cos(\alpha-\beta)}{2\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)}\right\}^2=\frac{2\cos\frac{\beta}{2}}{\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)}+\frac{\cos^2(\alpha-\beta)}{4\cos^2\alpha\cos^2\left(\alpha-\frac{\beta}{2}\right)}$$

$$=\frac{\cos^2(\alpha-\beta)+8\cos\alpha\cos\frac{\beta}{2}\cos\left(\alpha-\frac{\beta}{2}\right)}{4\cos^2\alpha\cos^2\left(\alpha-\frac{\beta}{2}\right)}$$

$$=\frac{\cos^2(\alpha-\beta)+4\cos\alpha\{\cos\alpha+\cos(\alpha-\beta)\}}{4\cos^2\alpha\cos^2\left(\alpha-\frac{\beta}{2}\right)}$$

$$=\frac{\{\cos(\alpha-\beta)+2\cos\alpha\}^2}{4\cos^2\alpha\cos^2\left(\alpha-\frac{\beta}{2}\right)},$$

therefore $$x+\frac{\cos(\alpha-\beta)}{2\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)}=\pm\frac{\cos(\alpha-\beta)+2\cos\alpha}{2\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)}$$

Taking the upper sign we have

$$x = \frac{2\cos\alpha}{2\cos\alpha\cos\left(\alpha - \frac{\beta}{2}\right)} = \sec\left(\alpha - \frac{\beta}{2}\right)$$

Taking the lower sign we have

$$x = -\frac{\cos\alpha + \cos(\alpha-\beta)}{\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)}$$

$$= -\frac{2\cos\left(\alpha-\frac{\beta}{2}\right)\cos\frac{\beta}{2}}{\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)} = -2\cos\frac{\beta}{2}\sec\alpha$$

Or we may write the proposed equation in this form

$$x\cos\alpha\left\{x\cos\left(\alpha-\frac{\beta}{2}\right)-1\right\}+2\left\{x\cos\left(\alpha-\frac{\beta}{2}\right)-1\right\}\cos\frac{\beta}{2}=0;$$

and then the two values of x which satisfy it are obvious

17 $\cot 2^{x-1}\alpha - \cot 2^x\alpha = \operatorname{cosec} 3\alpha$,

put y for $2^{x-1}\alpha$, thus $\cot y - \cot 2y = \operatorname{cosec} 3\alpha$,

therefore $$\frac{\cos y}{\sin y} - \frac{\cos 2y}{\sin 2y} = \operatorname{cosec} 3\alpha;$$

therefore $$\frac{\sin(2y-y)}{\sin y\sin 2y} = \operatorname{cosec} 3\alpha = \frac{1}{\sin 3\alpha},$$

therefore $\sin 2y = \sin 3\alpha$, that is $\sin 2^x\alpha = \sin 3\alpha$

Thus the general solution is $2^x\alpha = n\pi + (-1)^n 3\alpha$

18 $m \operatorname{vers}\theta = n\operatorname{vers}(\alpha-\theta)$,

therefore $$m(1-\cos\theta) = n\{1-\cos(\alpha-\theta)\},$$

therefore $$2m\sin^2\frac{\theta}{2} = 2n\sin^2\frac{\alpha-\theta}{2},$$

therefore $$\sin\frac{\alpha-\theta}{2} = \left(\frac{m}{n}\right)^{\frac{1}{2}}\sin\frac{\theta}{2},$$

therefore $$\sin\frac{\alpha}{2}\cos\frac{\theta}{2} - \cos\frac{\alpha}{2}\sin\frac{\theta}{2} = \left(\frac{m}{n}\right)^{\frac{1}{2}}\sin\frac{\theta}{2}$$

Divide by $\cos\frac{\theta}{2}$, thus we obtain a simple equation for finding $\tan\frac{\theta}{2}$

19 $\cos n\theta+\cos(n-2)\theta=\cos\theta$,

therefore $$2\cos(n-1)\theta\cos\theta=\cos\theta;$$

therefore either $\cos\theta=0$, or $\cos(n-1)\theta=\frac{1}{2}$

If $\cos\theta=0$, then $\theta=m\pi+\frac{\pi}{2}$

If $\cos(n-1)\theta=\frac{1}{2}$, then $(n-1)\theta=2m\pi\pm\frac{\pi}{3}$

20 $\sin\theta+\sin 3\theta=\sin 2\theta+\sin 4\theta$,

therefore $$2\sin 2\theta\cos\theta=2\sin 3\theta\cos\theta,$$

therefore either $\cos\theta=0$, or $\sin 2\theta=\sin 3\theta$

If $\cos\theta=0$, then $\theta=n\pi+\frac{\pi}{2}$

If $\sin 2\theta=\sin 3\theta$, then $\sin 2\theta-\sin 3\theta=0$, therefore $2\sin\frac{\theta}{2}\cos\frac{5\theta}{2}=0$, therefore either $\sin\frac{\theta}{2}=0$, or $\cos\frac{5\theta}{2}=0$ taking $\sin\frac{\theta}{2}=0$ we have $\frac{\theta}{2}=n\pi$, and taking $\cos\frac{5\theta}{2}=0$ we have $\frac{5\theta}{2}=n\pi+\frac{\pi}{2}$

The seven values greater than 0 and less than 2π are

$$\frac{\pi}{5},\ \frac{3\pi}{5},\ \frac{5\pi}{5},\ \frac{7\pi}{5},\ \frac{9\pi}{5},\ \frac{\pi}{2}\text{ and }\frac{3\pi}{2}$$

21. $\tan x=\tan\beta\tan(x+\alpha)=\dfrac{\tan\beta(\tan x+\tan\alpha)}{1-\tan x\tan\alpha}$,

therefore $$\tan x(1-\tan x\tan\alpha)=\tan\beta(\tan x+\tan\alpha),$$

therefore $$\tan^2 x\tan\alpha+(\tan\beta-1)\tan x+\tan\alpha\tan\beta=0$$

By solving this quadratic in the usual way we obtain the values of $\tan x$ It is known by the theory of quadratic equations that for the values to be real we must have $(\tan\beta-1)^2-4\tan^2\alpha\tan\beta$ positive or zero

And $(\tan\beta-1)^2-4\tan^2\alpha\tan\beta$

$$=\tan^2\beta-2\tan\beta-4\tan^2\alpha\tan\beta+1$$

$$=\{\tan\beta-(1+2\tan^2\alpha)\}^2+1-(1+2\tan^2\alpha)^2$$

$$=\left\{\tan\beta-\frac{1+\sin^2\alpha}{\cos^2\alpha}\right\}^2-\frac{4\sin^2\alpha}{\cos^4\alpha}$$

$$=\left\{\tan\beta-\frac{1+\sin^2\alpha}{\cos^2\alpha}-\frac{2\sin\alpha}{\cos^2\alpha}\right\}\left\{\tan\beta-\frac{1+\sin^2\alpha}{\cos^2\alpha}+\frac{2\sin\alpha}{\cos^2\alpha}\right\}$$

$$=\left\{\tan\beta-\left(\frac{1+\sin\alpha}{\cos\alpha}\right)^2\right\}\left\{\tan\beta-\left(\frac{1-\sin\alpha}{\cos\alpha}\right)^2\right\}$$

This expression then must be positive or zero, and therefore $\tan\beta$ must not lie between $\left(\frac{1-\sin\alpha}{\cos\alpha}\right)^2$ and $\left(\frac{1+\sin\alpha}{\cos\alpha}\right)^2$.

22. $$\tan\left(\frac{\pi}{4}-\theta\right)+\tan\left(\frac{\pi}{4}+\theta\right)$$

$$=\frac{\sin\left(\frac{\pi}{4}-\theta\right)}{\cos\left(\frac{\pi}{4}-\theta\right)}+\frac{\sin\left(\frac{\pi}{4}+\theta\right)}{\cos\left(\frac{\pi}{4}+\theta\right)}$$

$$=\frac{\sin\left(\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}+\theta\right)+\sin\left(\frac{\pi}{4}+\theta\right)\cos\left(\frac{\pi}{4}-\theta\right)}{\cos\left(\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}+\theta\right)}$$

$$=\frac{\sin\frac{\pi}{2}}{\cos\left(\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}+\theta\right)}=\frac{1}{\sin\left(\frac{\pi}{4}+\theta\right)\cos\left(\frac{\pi}{4}+\theta\right)}$$

$$=\frac{2}{\sin\left(\frac{\pi}{2}+2\theta\right)}=\frac{2}{\cos 2\theta}.$$

Thus $$\frac{2}{\cos 2\theta}=\left(\frac{8\sqrt{2}}{1+\sqrt{2}}\right)^{\frac{1}{2}};$$

therefore $$\frac{\cos 2\theta}{2}=\left(\frac{1+\sqrt{2}}{8\sqrt{2}}\right)^{\frac{1}{2}};$$

therefore $$\cos^2 2\theta=\frac{1+\sqrt{2}}{2\sqrt{2}};$$

therefore $$2\cos^2 2\theta-1=\frac{1+\sqrt{2}}{\sqrt{2}}-1=\frac{1}{\sqrt{2}};$$

therefore $$\cos 4\theta=\frac{1}{\sqrt{2}}=\cos\frac{\pi}{4},$$

therefore the least value of θ is given by $4\theta=\frac{\pi}{4}$.

23. $$\sin^2(n+1)\theta=\sin^2 n\theta+\sin^2(n-1)\theta,$$

therefore $$\sin^2(n+1)\theta-\sin^2(n-1)\theta=\sin^2 n\theta,$$

therefore $$\sin 2n\theta\sin 2\theta=\sin^2 n\theta \quad \text{(Art. 83.)}$$

But $$(n+1)\,\theta+(n-1)\,\theta+n\theta=\pi,$$

therefore $3n\theta=\pi$, therefore $n\theta=\dfrac{\pi}{3}$;

therefore $$\sin 2\theta \sin\frac{2\pi}{3}=\sin^2\frac{\pi}{3},$$

therefore $$\sin 2\theta=\sin\frac{\pi}{3},$$

thus $2\theta=\dfrac{\pi}{3}$, therefore $\theta=\dfrac{\pi}{6}$. But $n\theta=\dfrac{\pi}{3}$, and therefore $n=2$

24 $\cos^2\theta-\cos^2\alpha=2\cos^3\theta\,(\cos\theta-\cos\alpha)-2\sin^3\theta\,(\sin\theta-\sin\alpha)$,

therefore $$\cos^2\theta-\cos^2\alpha=\frac{\cos 3\theta+3\cos\theta}{2}(\cos\theta-\cos\alpha)$$
$$-\frac{3\sin\theta-\sin 3\theta}{2}(\sin\theta-\sin\alpha),$$

therefore $2\,(\cos^2\theta-\cos^2\alpha)=\cos 3\theta\cos\theta+\sin 3\theta\sin\theta-\cos 3\theta\cos\alpha-\sin 3\theta\sin\alpha$
$$+3\cos^2\theta-3\sin^2\theta-3\cos\theta\cos\alpha+3\sin\theta\sin\alpha,$$

therefore $\cos(3\theta-\theta)-\cos(3\theta-\alpha)-3\cos(\theta+\alpha)=3\sin^2\theta-\cos^2\theta-2\cos^2\alpha$,

therefore $\cos 2\theta-\cos(3\theta-\alpha)-3\cos(\theta+\alpha)=3-4\cos^2\theta-2\cos^2\alpha$
$$=3-2\,(1+\cos 2\theta)-(1+\cos 2\alpha)$$
$$=-2\cos 2\theta-\cos 2\alpha,$$

therefore $$3\cos 2\theta-3\cos(\theta+\alpha)-\cos(3\theta-\alpha)+\cos 2\alpha=0,$$

therefore $$3\sin\frac{3\theta+\alpha}{2}\sin\frac{\alpha-\theta}{2}+\sin\frac{(3\theta+\alpha)}{2}\sin\frac{3\theta-3\alpha}{2}=0,$$

therefore $$\sin\frac{3\theta+\alpha}{2}\left\{\sin\frac{3\,(\theta-\alpha)}{2}-3\sin\frac{\theta-\alpha}{2}\right\}=0,$$

therefore $$4\sin\frac{3\theta+\alpha}{2}\sin^3\frac{\theta-\alpha}{2}=0$$

Hence either $\sin\dfrac{3\theta+\alpha}{2}=0$, or $\sin\dfrac{\theta-\alpha}{2}=0$, the former gives $\dfrac{3\theta+\alpha}{2}=n\pi$, and the latter gives $\dfrac{\theta-\alpha}{2}=n\pi$

25 Let θ denote an angle having the same sine as α, so that $\sin\theta=\sin\alpha$, thus $\cos\left(\theta-\dfrac{\pi}{2}\right)=\cos\left(\dfrac{\pi}{2}-\alpha\right)$, therefore all the solutions are comprised in $\theta-\dfrac{\pi}{2}=2n\pi\pm\left(\dfrac{\pi}{2}-\alpha\right)$.

26 Let θ denote an angle having the same cosine as α, so that $\cos\theta = \cos\alpha$; thus $\sin\left(\theta - \frac{\pi}{2}\right) = \sin\left(\alpha - \frac{\pi}{2}\right)$; therefore all the solutions are comprised in $\theta - \frac{\pi}{2} = n\pi + (-1)^n\left(\alpha - \frac{\pi}{2}\right)$.

27. By Art. 101 it follows that the upper sign ought to be taken if $\frac{A}{2}$ lies between $n\,360^\circ + 225^\circ$ and $n\,360^\circ + 405^\circ$; in this case A lies between $2n\,360^\circ + 450^\circ$ and $2n\,360^\circ + 810^\circ$, and $A + 270^\circ$ lies between $2n\,360^\circ + 720^\circ$ and $2n\,360^\circ + 1080^\circ$, and therefore $\frac{A + 270^\circ}{360^\circ}$ lies between $2n + 2$ and $2n + 3$ thus the integral part of this fraction is an *even* number, so that denoting it by m we have $(-1)^m$ positive

In precisely the same manner we find that the present example agrees with Art 101 for the case in which m is *odd*

28 First suppose the number of degrees in A to lie between $n\,360$ and $n\,360 + 90$; then $\tan A$ and $\tan\frac{A}{2}$ are both positive, and therefore the upper sign must be taken in the ambiguity. Also in this case $\frac{A + 90}{180}$ lies between $\frac{n\,360 + 90}{180}$ and $\frac{n\,360 + 180}{180}$, that is between $2n + \frac{1}{2}$ and $2n + 1$; so that m is *even*.

Next suppose the number of degrees in A to lie between $n\,360 + 90$ and $n\,360 + 180$; then $\tan A$ is negative, and $\tan\frac{A}{2}$ is positive; and therefore the lower sign must be taken in the ambiguity. Also in this case $\frac{A + 90}{180}$ lies between $2n + 1$ and $2n + 2$, so that m is *odd*

Similarly we may proceed if the number of degrees in A lies between $n\,360 + 180$ and $n\,360 + 270$, or between $n\,360 + 270$ and $n\,360 + 360$

It will be observed that in this and the preceding example the *greatest integer* in a certain expression means that integer which with a *positive* proper fraction constitutes the whole expression.

Or we might treat the example thus.

$$\pm\sqrt{(1 + \tan^2 A)} = \pm\sqrt{\frac{1}{\cos^2 A}} = \pm\frac{1}{\cos A};$$

but
$$\tan\frac{A}{2} = \frac{1 - \cos A}{\sin A} = \frac{\frac{1}{\cos A} - 1}{\tan A}:$$

hence the ambiguity in $\pm\sqrt{(1 + \tan^2 A)}$ must be so taken as to ensure that the *sign is the same as the sign* of $\cos A$, and it is easy to shew that $(-1)^m$ is of the same sign as $\cos A$ when m has the prescribed value

5—2

29. $$\tan(\cot x) = \cot(\tan x),$$

therefore $$\tan(\cot x) = \tan\left\{\frac{\pi}{2} - \tan x\right\},$$

therefore, by Art 68, all the possible solutions are comprised in

$$\cot x = n\pi + \frac{\pi}{2} - \tan x;$$

therefore $$\cot x + \tan x = n\pi + \frac{\pi}{2},$$

therefore $$\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = n\pi + \frac{\pi}{2},$$

therefore $$\frac{1}{\sin x \cos x} = \frac{(2n+1)\pi}{2},$$

therefore $$\sin x \cos x = \frac{2}{(2n+1)\pi},$$

therefore $$\sin 2x = \frac{4}{(2n+1)\pi}$$

The value $n = -1$ would make $\sin 2x$ greater than unity.

30 $$2\cos^2\frac{A}{2} = 1 + \cos A,$$

therefore $$4\cos^2\frac{A}{2} = 2 + 2\cos A,$$

therefore $$2\cos\frac{A}{2} = \sqrt{(2 + 2\cos A)}$$

Again $$2\cos^2\frac{A}{4} = 1 + \cos\frac{A}{2},$$

therefore $$4\cos^2\frac{A}{4} = 2 + 2\cos\frac{A}{2};$$

therefore $$2\cos\frac{A}{4} = \sqrt{\left(2 + 2\cos\frac{A}{2}\right)} = \sqrt{\{2 + \sqrt{(2 + 2\cos A)}\}}.$$

Similarly $$2\cos\frac{A}{8} = \sqrt{[2 + \sqrt{\{2 + \sqrt{(2 + 2\cos A)}\}}]},$$

and this process may be continued to any extent

31 Change x successively to $\frac{\pi}{4} - x$ and $\frac{\pi}{4} + x$, thus

$$\cos\left(x - \frac{\pi}{4}\right) = \pm\sqrt{\frac{1}{2}\left\{1 + \cos\left(2x - \frac{\pi}{2}\right)\right\}} = \pm\sqrt{\frac{1 + \sin 2x}{2}},$$

and $\cos\left(\frac{\pi}{4}+x\right)=\pm\sqrt{\frac{1}{2}\left\{1+\cos\left(\frac{\pi}{2}+2x\right)\right\}}=\pm\sqrt{\frac{1-\sin 2x}{2}}$.

Then putting for $\cos\left(\frac{\pi}{4}-x\right)$ and $\cos\left(\frac{\pi}{4}+x\right)$ their values we have

$$\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x=\pm\sqrt{\frac{1+\sin 2x}{2}} \qquad (1),$$

and

$$\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x=\pm\sqrt{\frac{1-\sin 2x}{2}} \qquad (2).$$

Hence by subtraction we find the required expression for $\sin x$. In (1) the upper or lower sign must be taken according as $\cos\left(x-\frac{\pi}{4}\right)$ is positive or negative, that is according as $x-\frac{\pi}{4}$ lies between $2n\pi-\frac{1}{2}\pi$ and $2n\pi+\frac{1}{2}\pi$, or between $2n\pi+\frac{1}{2}\pi$ and $2n\pi+\frac{3\pi}{2}$. Similarly we can determine the sign to be taken in (2).

32. Let k denote the value which the expression retains for all values of θ, so that

$$\frac{A\cos(\theta+\alpha)+B\sin(\theta+\beta)}{A'\sin(\theta+\alpha)+B'\cos(\theta+\beta)}=k,$$

then $\quad A\cos(\theta+\alpha)+B\sin(\theta+\beta)=k\{A'\sin(\theta+\alpha)+B'\cos(\theta+\beta)\}$;

therefore $\quad \cos\theta(A\cos\alpha+B\sin\beta)+\sin\theta(B\cos\beta-A\sin\alpha)$

$$=k\cos\theta(A'\sin\alpha+B'\cos\beta)+k\sin\theta(A'\cos\alpha-B'\sin\beta),$$

therefore $\quad \cos\theta\{A\cos\alpha+B\sin\beta-k(A'\sin\alpha+B'\cos\beta)\}$

$$+\sin\theta\{B\cos\beta-A\sin\alpha-k(A'\cos\alpha-B'\sin\beta)\}=0$$

Now this is to be true for *all values* of θ Put for θ in succession 0 and $\frac{\pi}{2}$, thus we obtain the following two results

$$A\cos\alpha+B\sin\beta=k(A'\sin\alpha+B'\cos\beta),$$
$$B\cos\beta-A\sin\alpha=k(A'\cos\alpha-B'\sin\beta),$$

and it is obvious that if these hold the original expression does always retain the same value

By cross multiplication we obtain

$(A\cos\alpha+B\sin\beta)(A'\cos\alpha-B'\sin\beta)=(A'\sin\alpha+B'\cos\beta)(B\cos\beta-A\sin\alpha)$,

therefore $\quad AA'\cos^2\alpha-BB'\sin^2\beta+(A'B-AB')\cos\alpha\sin\beta$

$$=BB'\cos^2\beta-AA'\sin^2\alpha+(A'B-AB')\sin\alpha\cos\beta,$$

therefore $\quad AA'-BB'=(A'B-AB')\sin(\alpha-\beta)$

33 Let A denote the sum of the two angles x and y Then

$$\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2}=2\sin\frac{A}{2}\cos\frac{x-y}{2},$$

and the numerically greatest value of this expression is when $\cos\frac{x-y}{2}$ is greatest, that is when $x-y=0$, that is when $x=y$

Again $$\tan x+\tan y=\frac{\sin x}{\cos x}+\frac{\sin y}{\cos y}=\frac{\sin(x+y)}{\cos x\cos y}$$

$$=\frac{\sin A}{\cos x\cos y}=\frac{2\sin A}{2\cos x\cos y}$$

$$=\frac{2\sin A}{\cos(x-y)+\cos(x+y)}=\frac{2\sin A}{\cos(x-y)+\cos A},$$

and if $\cos A$ is positive the numerically least value of this is when

$$\cos(x-y)=1,\ \text{that is when}\ x=y$$

34 *By Art 114 we have*

$$\tan A\tan B+\tan B\tan C+\tan C\tan A=1,$$

therefore $$\tan^2 A+\tan^2 B+\tan^2 C=1+\frac{1}{2}(\tan A-\tan B)^2$$

$$+\frac{1}{2}(\tan B-\tan C)^2+\frac{1}{2}(\tan C-\tan A)^2.$$

Hence the least value of the expression is when $\tan A-\tan B$, $\tan B-\tan C$, and $\tan C-\tan A$ all vanish, and the value is then unity

35 By Art 114 we have

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C,$$

therefore $$\frac{1}{\cot A}+\frac{1}{\cot B}+\frac{1}{\cot C}=\frac{1}{\cot A\cot B\cot C},$$

therefore $$\cot B\cot C+\cot A\cot C+\cot A\cot B=1,$$

therefore $$\cot^2 A+\cot^2 B+\cot^2 C$$

$$=1+\frac{1}{2}(\cot A-\cot B)^2+\frac{1}{2}(\cot B-\cot C)^2+\frac{1}{2}(\cot C-\cot A)^2$$

Hence the least value of the expression is when $\cot A-\cot B$, $\cot B-\cot C$, and $\cot C-\cot A$ all vanish, and the value is then unity

36 $\cot B+\cot C-\operatorname{cosec} A=\frac{\cos B}{\sin B}+\frac{\cos C}{\sin C}-\frac{1}{\sin A}$

$$=\frac{\sin(B+C)}{\sin B\sin C}-\frac{1}{\sin A}=\frac{\sin A}{\sin B\sin C}-\frac{1}{\sin A}=\frac{\sin^2 A-\sin B\sin C}{\sin A\sin B\sin C}.$$

Proceeding in this way we find that the difference of the two given expressions is equivalent to a fraction with the denominator $\sin A\sin B\sin C$, while the numerator is

$$\sin^2 A+\sin^2 B+\sin^2 C-\sin B\sin C-\sin C\sin A-\sin A\sin B,$$

that is $\qquad \frac{1}{2}(\sin A-\sin B)^2+\frac{1}{2}(\sin B-\sin C)^2+\frac{1}{2}(\sin C-\sin A)^2.$

This expression is never negative

37. Suppose A, B, C to be three acute angles such that

$$\cos^2 A+\cos^2 B+\cos^2 C=1,$$

then $\qquad \cos^2 A=1-\cos^2 C-\cos^2 B=\sin^2 C-\cos^2 B$

$$=-\cos(C-B)\cos(C+B).$$

This shews that $C+B$ must be greater than a right angle Now if we take $A'=180^\circ-C-B$ we shall have $\cos^2 A'$ numerically equal to $\cos^2(B+C)$, and therefore numerically *less* than $\cos(C-B)\cos(C+B)$, for we may suppose C not less than B, and then $C-B$ is less than $180^\circ-C-B$. Hence $\cos^2 A$ is greater than $\cos^2 A'$, and A is less than A', and therefore $A+B+C$ is less than 180°.

38 By Art 113 we have $\sin A+\sin B+\sin C-\sin(A+B+C)$

$$=\sin A(1-\cos B\cos C)+\sin B(1-\cos C\cos A)+\sin C(1-\cos A\cos B)$$
$$+\sin A\sin B\sin C,$$

and as A, B, and C are acute this expression is necessarily positive

39 Let $u=\left(\cos\frac{a}{n}\right)^{n^2}$,

therefore $\qquad \log u=n^2\log\cos\frac{a}{n}=\frac{n^2}{2}\log\left(1-\sin^2\frac{a}{n}\right)$

$$=-\frac{n^2}{2}\left\{\sin^2\frac{a}{n}+\frac{1}{2}\sin^4\frac{a}{n}+\frac{1}{3}\sin^6\frac{a}{n}+\quad\right\}.$$

Now $n\sin\frac{a}{n}=a\frac{\sin\frac{a}{n}}{\frac{a}{n}}$, and this is equal to a when n is indefinitely increased; and therefore $n^2\sin^2\frac{a}{n}$ is equal to a^2.

Then $n^2 \sin^4\frac{a}{n} = n^2 \sin^2\frac{a}{n} \times \sin^2\frac{a}{n}$; and this vanishes when n is indefinitely increased Similarly the other terms in $\log u$ vanish, and as in Art 150 their sum vanishes also; and thus $\log u = -\frac{a^2}{2}$ ultimately. Therefore $u = e^{-\frac{a^2}{2}}$

40. Let $u = \left(\cos\frac{a}{n}\right)^{n^3}$; therefore

$$\log u = n^3 \log\cos\frac{a}{n} = \frac{n^3}{2}\log\left(1 - \sin^2\frac{a}{n}\right)$$

$$= -\frac{n^3}{2}\left\{\sin^2\frac{a}{n} + \frac{1}{2}\sin^4\frac{a}{n} + \frac{1}{3}\sin^6\frac{a}{n} + \quad\right\}.$$

Now we have shewn in solving the preceding Example that $n^2 \sin^2\frac{a}{n} = a^2$ ultimately; hence $n^3 \sin^2\frac{a}{n} = na^2$, and so becomes infinite. Thus the logarithm of u is negative infinity, and therefore u vanishes ultimately

41. $\sin\theta - (\tan\theta - \frac{1}{2}\tan^3\theta) = \sin\theta - \tan\theta + \frac{1}{2}\tan^3\theta$

$$= \sin\theta - \frac{\sin\theta}{\cos\theta} + \frac{1}{2}\frac{\sin^3\theta}{\cos^3\theta} = \frac{\sin\theta}{\cos^3\theta}\{\cos^3\theta - \cos^2\theta + \frac{1}{2}\sin^2\theta\}$$

$$= \frac{\sin\theta}{2\cos^3\theta}\{2\cos^3\theta - 2\cos^2\theta + 1 - \cos^2\theta\}$$

$$= \frac{\sin\theta\,(1-\cos\theta)}{2\cos^3\theta}\{1 + \cos\theta - 2\cos^2\theta\}$$

$$= \frac{\sin\theta\,(1-\cos\theta)\,(1-\cos\theta)\,(1+2\cos\theta)}{2\cos^3\theta}$$

$$= \frac{\sin\theta\,(1-\cos\theta)^2\,(1+2\cos\theta)}{2\cos^3\theta},$$ which is positive.

42 Let $u = \left(\frac{x-1}{x}\right)^x$; then

$$\log u = x\log\frac{x-1}{x} = x\log\left(1 - \frac{1}{x}\right)$$

$$= -x\left\{\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \quad\right\}$$

$$= -\left\{1 + \frac{1}{2x} + \frac{1}{3x^2} + \quad\right\}.$$

Thus the logarithm is always negative, and as x increases the logarithm diminishes numerically, and so u increases, when x is infinite $\log u = -1$; and therefore $u = e^{-1}$.

CHAPTER XI P 120

1.
$$\begin{array}{r} 4\ 0948553 \\ 4\ 0948204 \\ \hline 0000349 \end{array} \qquad 1 \cdot 35 \ .\cdot\ 0000349 : x;$$

this gives $\qquad x = 0000122,$

therefore $\qquad \log 12440\ 35 = 4\ 0948326$

2
$$\begin{array}{r} 0288558 \\ 0288152 \\ \hline 0000406 \end{array} \qquad \begin{array}{r} 0288355 \\ \cdot 0288152 \\ \hline 0000203 \end{array} \qquad 0000406 : 0000203 \ .:\ \cdot 0001 \ .\ x;$$

this gives $\qquad x = 00005,$

therefore $\qquad \log 1\ 06865 = 0288355$

3
$$\begin{array}{r} 4\ 3702725 \\ 4\ 3702540 \\ \hline 0000185 \end{array}$$

1	185
2	370
3	555
4	740
5	925
6	1110
7	1295
8	1480
9	1665

$$\begin{array}{lr} \log 23456 \cdot = 4\ 3702540 & \\ \text{add for} \quad 3 & 555 \\ \qquad\qquad 8 & 1480 \\ \hline & 4\ 370261030 \end{array}$$

therefore retaining 7 places of decimals

$$\log 23456\ 38 = 4\ 3702610, \text{ and } \log\ 2345638 = \bar{1}\ 3702610$$

4 $\qquad -(1\ 8753145) = \bar{2}\ 1246855$

$$\begin{array}{r} 1246998 \\ 1246672 \\ \hline 0000326 \end{array} \qquad \begin{array}{r} \cdot 1246855 \\ \cdot 1246672 \\ \hline 0000183 \end{array} \qquad 0000326 \quad 0000183 \ ..\ 0001 : x;$$

this gives $\qquad x = 000056,$

therefore $\qquad \log 1\ 332556 = \cdot 1246855,$

therefore $\qquad \log\ 01332556 = \bar{2}\ 1246855$

5
$$\begin{array}{r} 5860356 \\ 5860244 \\ \hline 0000112 \end{array} \qquad 0001\ .\ 00004 \cdot\ 0000112 \cdot x;$$

this gives $\qquad x = 0000045,$

therefore $\qquad \log 3\ 85504 = 5860289,$

therefore $\qquad \log\ 00385504 = \bar{3}\ 5860289,$

therefore $\log(\,00385504)^{\frac{1}{4}} = \frac{1}{4}(\bar{3}\ 5860289) = \frac{1}{4}(-4+1\ 5860289) = \bar{1}\ 3965072.$

6 $$\log (24)^{\frac{1}{2}} = \frac{1}{2}\log 24 = 6901056$$

$$\begin{array}{r} 6901074 \\ 6900986 \\ \hline 0000088 \end{array} \qquad \begin{array}{r} 6901056 \\ 6900986 \\ \hline 0000070 \end{array} \qquad 0000088 \quad 0000070 \quad 0001 \quad x,$$

this gives $$x = 000079,$$

therefore $$\log 4\ 898979 = \cdot 6901056,$$

therefore $$(24)^{\frac{1}{2}} = 4\ 898979$$

7 $$\log (142\ 71)^{\frac{1}{7}} = \frac{1}{7} \times 2\ 1544544 = 3077792$$

$$\begin{array}{r} 3077954 \\ 3077741 \\ \hline 0000213 \end{array} \qquad \begin{array}{r} 3077792 \\ 3077741 \\ \hline 0000051 \end{array} \qquad \cdot 0000213 \quad 0000051 \quad 1 \quad x,$$

this gives $$x = 24,$$

therefore $$\log 20313\ 24 = 4\ 3077792,$$

therefore $$\log 2\ 031324 = 3077792,$$

therefore $$(142\ 71)^{\frac{1}{7}} = 2\ 031324$$

8 $$\log (\ 07)^{\frac{1}{5}} = \frac{1}{5}\log\ 07 = \frac{1}{5}(\bar{2}\ 8450980) = \frac{1}{5}(-5+3\ 8450980) = \bar{1}\ 7690196$$

$$\begin{array}{r} 7690227 \\ 7690153 \\ \hline 0000074 \end{array} \qquad \begin{array}{r} 7690196 \\ 7690153 \\ \hline 0000043 \end{array} \qquad 0000074 \quad 0000043 \quad 1 \quad x,$$

this gives $$x = 58,$$

therefore $$\log 58751\ 58 = 4\ 7690196,$$

therefore $$\log\ 5875158 = \bar{1}\ 7690196,$$

therefore $$(\ 07)^{\frac{1}{5}} = 5875158$$

9 $$\log (\ 0625)^{\frac{1}{5}} = \log \left(\frac{625}{10000}\right)^{\frac{1}{5}} = \log \left(\frac{125}{2000}\right)^{\frac{1}{5}} = \log \left(\frac{25}{400}\right)^{\frac{1}{5}}$$

$$= \log \left(\frac{1}{16}\right)^{\frac{1}{5}} = -\frac{1}{5}\log 16 = -\frac{4}{5}\log 2 = -\ 2408240$$

$$= \bar{1}\ 7591760 = \log\ 5743491,$$

therefore $$(\ 0625)^{\frac{1}{5}} = 5743491.$$

10 $$\log (27)^{-\frac{1}{5}} = -\frac{1}{5}\log 27 = -\frac{1}{5}(1\ 4313638) = -\ 2862728$$

$$= \bar{1}\ 7137272 = \log\ 5172818,$$

therefore $$(27)^{-\frac{1}{5}} = 5172818$$

11. $\log 71968 \cdot 6 = 1 \cdot 8571391 + \frac{6}{10}$ of $\cdot 0000060 = 1 \cdot 8571430$,

$$\log (\cdot 0719686)^{\frac{1}{8}} = \frac{1}{8} (\bar{2} \cdot 8571430) = \frac{1}{8} (-8 + 6 \cdot 8571430) = \bar{1} \cdot 8571429$$

But $\log \cdot 719686 = \bar{1} \cdot 8571430$, therefore $(\cdot 0719686)^{\frac{1}{8}} = \cdot 719686$

12. $\log (1 \cdot 03)^{-10} = -10 \times \cdot 0128372 = -\cdot 128372 = \bar{1} \cdot 871628 = \log \cdot 7440942$,

therefore $$(1 \cdot 03)^{-10} = \cdot 7440942$$

13. $\log (1 \cdot 05)^{-20} = -20 \times \cdot 0211893 = -\cdot 423786 = \bar{1} \cdot 576214 = \log \cdot 37689$,

therefore $$(1 \cdot 05)^{-20} = \cdot 37689,$$

therefore $$64 \{1 - (1 \cdot 05)^{-20}\} = 64 \{1 - \cdot 37689\}$$
$$= 64 \times \cdot 62311 = 39 \cdot 87904$$

14. Denote it by u, then $\log u = \sqrt{5} \log 5 = 2 \sqrt{5} \log \sqrt{5}$;

therefore $$\log (\log u) = \log 2 + \log \sqrt{5} + \log (\log \sqrt{5})$$

Now $$\log \sqrt{5} = \frac{1}{2} \log 5 = \frac{1}{2} \log \frac{10}{2} = \frac{1}{2} (1 - \log 2)$$
$$= \frac{1}{2} (1 - \cdot 301030) = \frac{1}{2} (\cdot 698970) = \cdot 349485,$$

$$\log (\log \sqrt{5}) = \log \cdot 349485 = \bar{1} \cdot 543428$$

Therefore $\log (\log u) = \cdot 301030 + \cdot 349485 + \bar{1} \cdot 543428 = \cdot 193943$

Therefore $$\log u = 1 \cdot 562944$$

$$\begin{array}{ll} 563006 & 562944 \\ 562887 & 562887 \\ \hline 000119 & 000057 \end{array} \qquad \cdot 000119 : \cdot 000057 :: \cdot 001 : x;$$

this gives $x = \cdot 00048$, therefore $u = 36 \cdot 5548$

15. $\log 144 = \log 12^2 = 2 \log 12 = 2 \cdot 1583624$,

$$\log (1 \cdot 44)^{-6} = -6 \log 1 \cdot 44 = -6 (\cdot 1583624) = -\cdot 9501744$$
$$= \bar{1} \cdot 0498256 = \log \cdot 1121568;$$

therefore $$(1 \cdot 44)^{-6} = \cdot 1121568$$

$$\log (1 \cdot 44)^{-12} = -12 \log 1 \cdot 44 = -12 (\cdot 1583624) = -1 \cdot 9003488$$
$$= \bar{2} \cdot 0996512 = \log \cdot 01257916,$$

therefore $$(1 \cdot 44)^{-12} = \cdot 01257916,$$

therefore $(1 \cdot 44)^{-6} - (1 \cdot 44)^{-12} = \cdot 1121568 - \cdot 01257916 = \cdot 09957765$

16. $\log \frac{1}{(1·05)^{13}} = -13 \log 1·05 = -13\,(·0211893) = -·2754609$

$= \bar{1}·7245391 = \log ·5303214,$

therefore $\frac{1}{(1·05)^{13}} = ·5303214,$

$\log \frac{1}{(1·05)^{20}} = -20 \log 1·05 = -20\,(·0211893) = -·423786$

$= \bar{1}·576214 = \log ·3768894,$

therefore $\frac{1}{(1·05)^{20}} = ·3768894;$

therefore $\frac{1}{·05}\left\{\frac{1}{(1·05)^{13}} - \frac{1}{(1·05)^{20}}\right\} = 20\,\{·5303214 - ·3768894\}$

$= 20 \times ·153432 = 3·06864$

17. $\begin{array}{r} ·7431448 \\ ·7313537 \\ \hline ·0117911 \end{array}$ $60' \quad 1' \quad ·0117911 \quad x,$

this gives $x = ·0001965,$

therefore $\sin 47° 1' = ·7313537 + ·0001965 = ·7315502$

18. $\begin{array}{r} ·1270646 \\ ·1267761 \\ \hline ·0002885 \end{array}$ $60'' \quad 25'' \quad ·0002885 : x,$

this gives $x = ·0001202;$

therefore $\sin 7° 17' 25'' = ·1267761 + ·0001202 = ·1268963$

19. $\begin{array}{r} 9·4663483 \\ 9·4659353 \\ \hline ·0004130 \end{array}$ $60'' \quad 12'' \quad ·0004130 \quad x,$

this gives $x = ·0000826,$

therefore $L \sin 17° 0' 12'' = 9·4659353 + ·0000826 = 9·4660179.$

20. $\begin{array}{r} 9·6482582 \\ 9·6480038 \\ \hline ·0002544 \end{array}$ $60'' \quad 12'' \quad ·0002544 \cdot x,$

this gives $x = ·0000509,$

therefore $L \sin 26° 24' 12'' = 9·6480038 + ·0000509 = 9·6480547.$

21. $\begin{array}{r} 9·5052891 \\ 9·5048538 \\ \hline ·0004353 \end{array}$ $60'' \quad 35'' \quad ·0004353 \quad x,$

this gives $x = ·0002539,$

therefore $L \cot 72° 15' 35'' = 9·5052891 - ·0002539 = 9·5050352$

22.

$$\begin{array}{r} 9\ 1604569 \\ 9\ 1603493 \\ \hline 0001076 \end{array} \qquad 0001486 \quad \cdot 0001076 \quad 10 \quad x,$$

this gives $x=7$, therefore the required angle is $81^\circ\,46'\,7''$

23

$$\begin{array}{r} 9\cdot 9713383 \\ 9\text{-}9713351 \\ \hline 0000032 \end{array} \qquad 0000079\ .\ 0000032 \quad 10 \quad x,$$

this gives $x=4$, therefore the required angle is $20^\circ\,35'\,20''-4''$, that is $20^\circ\,35'\,16''$. For as the L cosine increases the angle diminishes

24 $\qquad 60'' \quad 26'' \quad \cdot 0000865 \quad x,$

this gives $\qquad x=\cdot 0000375;$

therefore $\qquad L\cos 34^\circ\,24'\,26''=9\ 9165137-\ 0000375=9\ 9164762$

Again

$$\begin{array}{r} 9\ 9165646 \\ 9\ 9165137 \\ \hline 0000509 \end{array} \qquad 0000865\ .\ \cdot 0000509 \quad 60 \quad x,$$

this gives $x=35$, therefore the required angle is $34^\circ\,24'-35''$, that is $34^\circ\,23'\,25''$

25 Since $\sec\theta\times\cos\theta=1$, we have $\log\sec\theta+\log\cos\theta=0$,

therefore $L\sec\theta+L\cos\theta-20=0$, therefore $L\sec\theta=20-L\cos\theta$.

We shall first find $L\cos 37^\circ\,19'\,47''$

$$60'' \quad 47'' \quad 0000963\ \cdot\ x,$$

this gives $\qquad x=\ 0000754,$

therefore $\qquad L\cos 37^\circ\,19'\,47''=9\ 9005294-\cdot 0000754=9\ 9004540.$

Then $\qquad L\sec 37^\circ\,19'\,47''=20-9\ 9004540=10\ 0995460.$

Next find $L\sin 37^\circ\,19'\,47''$

$$60'' \quad 47''\ \cdot \quad \cdot 0001657 \quad x,$$

this gives $\qquad x=\cdot 0001298;$

therefore $\qquad L\sin 37^\circ\,19'\,47''=9\cdot 7826301+\ 0001298=9\ 7827599.$

Then $\quad \tan\theta=\dfrac{\sin\theta}{\cos\theta}$; therefore $\log\tan\theta=\log\sin\theta-\log\cos\theta$,

therefore $L\tan\theta-10=L\sin\theta-10-(L\cos\theta-10)=L\sin\theta-L\cos\theta$,

therefore $\qquad L\tan\theta=10+L\sin\theta-L\cos\theta.$

Thus $L\tan 37^\circ\,19'\,47''=10+9\cdot 7827599-9\ 9004540=9\ 8823059$

26 $\qquad 60'' \quad 24''\,6 \quad 0001998 \quad x,$

this gives $\qquad x = 0000819;$

therefore $\quad L \sin 32^\circ\,18'\,24''\,6 = 9\,7278277 + 0000819 = 9\,7279096$

$\qquad 60'' \cdot 24''\,6 \quad 0000799 \quad x,$

this gives $\qquad x = 0000328;$

therefore $\quad L \cos 32^\circ\,18'\,24''\,6 = 9\,9269913 - \cdot 0000328 = 9\,9269585$

And $\quad L \tan 32^\circ\,18'\,24''\,6 = 10 + L \sin 32^\circ\,18'\,24''\,6 - L \cos 32^\circ\,18'\,24''\,6$

$$= 9\,8009511$$

CHAPTER XII

1 Let $ABCD$ denote the rectangle From A draw AP perpendicular to the diagonal BD, and from P draw PM perpendicular to BC, and PN perpendicular to CD

Let the angle DBA be denoted by α, then

$$AB = c\cos\alpha, \quad BP = AB\cos\alpha = c\cos^2\alpha,$$

$$PM = BP\cos BPM = BP\cos\alpha = c\cos^3\alpha.$$

Thus denoting PM by p we have $p = c\cos^3\alpha$.

Similarly $AD = c\sin\alpha, \quad PD = AD\sin PAD = AD\sin\alpha = c\sin^2\alpha,$

$$PN = PD\sin PDN = PD\sin\alpha = c\sin^3\alpha.$$

Thus $q = c\sin^3\alpha$

Therefore $p^{\frac{2}{3}} + q^{\frac{2}{3}} = (c\cos^3\alpha)^{\frac{2}{3}} + (c\sin^3\alpha)^{\frac{2}{3}} = c^{\frac{2}{3}}(\cos^2\alpha + \sin^2\alpha) = c^{\frac{2}{3}}$

2 Let a denote the radius of the larger circle, and b the radius of the smaller circle. Let x denote the distance of the point of intersection of the two common tangents from the centre of the larger circle, therefore $x-a-b$ denotes the distance of this point from the centre of the smaller circle

Then $\qquad \sin\frac{\theta}{2} = \frac{a}{x}$, and also $\sin\frac{\theta}{2} = \frac{b}{x-a-b}$,

therefore $\qquad x = \dfrac{a}{\sin\frac{\theta}{2}}$, and $x - a - b = \dfrac{b}{\sin\frac{\theta}{2}}$,

therefore, by subtraction, $\qquad a + b = \dfrac{a-b}{\sin\frac{\theta}{2}}$,

therefore $\qquad \sin\frac{\theta}{2} = \frac{a-b}{a+b}, \qquad$ therefore $\cos\frac{\theta}{2} = \frac{2\sqrt{(ab)}}{a+b};$

and $\qquad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{4(a-b)\sqrt{(ab)}}{(a+b)^2}$

3 $$\sec\alpha\sec\theta+\tan\alpha\tan\theta=\sec\beta,$$

therefore $$\sec\alpha\sec\theta=\sec\beta-\tan\alpha\tan\theta,$$

therefore $$\sec^2\alpha\sec^2\theta=(\sec\beta-\tan\alpha\tan\theta)^2;$$

therefore $$\sec^2\alpha(1+\tan^2\theta)=\sec^2\beta-2\sec\beta\tan\alpha\tan\theta+\tan^2\alpha\tan^2\theta;$$

therefore $$(\sec^2\alpha-\tan^2\alpha)\tan^2\theta+2\sec\beta\tan\alpha\tan\theta=\sec^2\beta-\sec^2\alpha,$$

therefore $$\tan^2\theta+2\sec\beta\tan\alpha\tan\theta=\sec^2\beta-\sec^2\alpha,$$

therefore $$(\tan\theta+\tan\alpha\sec\beta)^2=\sec^2\beta-\sec^2\alpha+\tan^2\alpha\sec^2\beta$$
$$=\sec^2\beta\sec^2\alpha-\sec^2\alpha=\tan^2\beta\sec^2\alpha,$$

therefore $$\tan\theta+\tan\alpha\sec\beta=\pm\tan\beta\sec\alpha,$$

therefore $$\tan\theta=-\tan\alpha\sec\beta\pm\tan\beta\sec\alpha$$
$$=-\frac{\sin\alpha}{\cos\alpha\cos\beta}\pm\frac{\sin\beta}{\cos\beta\cos\alpha}=\frac{-\sin\alpha\pm\sin\beta}{\cos\alpha\cos\beta}.$$

4 $$\frac{\sin\frac{\theta}{2}\cos 2\theta}{\operatorname{vers}\theta\cot\theta}=\frac{\sin\frac{\theta}{2}\cos 2\theta\sin\theta}{\operatorname{vers}\theta\cos\theta}=\frac{2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\cos 2\theta}{(1-\cos\theta)\cos\theta}$$
$$=\frac{2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\cos 2\theta}{2\sin^2\frac{\theta}{2}\cos\theta}=\frac{\cos\frac{\theta}{2}\cos 2\theta}{\cos\theta};$$

and the value of this is unity when $\theta=0$

$$\frac{\tan^2\theta}{\sec 2\theta-1}=\frac{\sin^2\theta}{\cos^2\theta(\sec 2\theta-1)}=\frac{\sin^2\theta\cos 2\theta}{\cos^2\theta(1-\cos 2\theta)}$$
$$=\frac{\sin^2\theta\cos 2\theta}{2\cos^2\theta\sin^2\theta}=\frac{\cos 2\theta}{2\cos^2\theta},$$

and the value of this is $\frac{1}{2}$ when $\theta=0$

5 $$\cot\frac{\theta}{2}-(1+\cot\theta)=\cot\frac{\theta}{2}-\cot\theta-1=\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}-\frac{\cos\theta}{\sin\theta}-1$$
$$=\frac{\sin\theta\cos\frac{\theta}{2}-\cos\theta\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}\sin\theta}-1=\frac{\sin\left(\theta-\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}\sin\theta}-1=\frac{1}{\sin\theta}-1,$$

now this is always positive as θ changes from 0 to π, except when $\theta=\frac{\pi}{2}$, and then it is zero

6
$$\tan\frac{\theta}{2}=\frac{\tan\theta+c-1}{\tan\theta+c+1},$$

therefore
$$\tan\frac{\theta}{2}(\tan\theta+c+1)=\tan\theta+c-1,$$

therefore
$$\tan\frac{\theta}{2}\left(\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}+c+1\right)=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}+c-1;$$

therefore
$$2\tan^2\frac{\theta}{2}+(c+1)\left(1-\tan^2\frac{\theta}{2}\right)\tan\frac{\theta}{2}$$
$$=2\tan\frac{\theta}{2}+(c-1)\left(1-\tan^2\frac{\theta}{2}\right),$$

therefore $(c+1)\tan^3\frac{\theta}{2}-(1+c)\tan^2\frac{\theta}{2}+(1-c)\tan\frac{\theta}{2}+(c-1)=0,$

therefore
$$(c+1)\tan^2\frac{\theta}{2}\left(\tan\frac{\theta}{2}-1\right)=(c-1)\left(\tan\frac{\theta}{2}-1\right),$$

therefore either $\tan\frac{\theta}{2}-1=0$, or $(c+1)\tan^2\frac{\theta}{2}=c-1.$

Thus
$$\tan\frac{\theta}{2}=1, \text{ or } \pm\sqrt{\frac{c-1}{c+1}}$$

7. $a\sec^2\theta-b\cos\theta=2a$, therefore $a-b\cos^3\theta=2a\cos^2\theta$,

therefore
$$b\cos^3\theta=a-2a\cos^2\theta,$$

again $b\cos^2\theta-a\sec\theta=2b$, therefore $b\cos^3\theta=2b\cos\theta+a$,

thus
$$a-2a\cos^2\theta=2b\cos\theta+a,$$

therefore $-a\cos\theta=b$, therefore $\cos\theta=-\frac{b}{a}$

Substitute this value of $\cos\theta$ in either of the given equations, for instance the first, thus $\frac{a^3}{b^2}+\frac{b^2}{a}=2a$, therefore $a^4+b^4-2a^2b^2=0$, therefore $a^2=b^2$

8. $a^2+b^2=(\sin\alpha\cos\beta\sin\theta+\cos\alpha\cos\theta)^2+(\sin\alpha\cos\beta\cos\theta-\cos\alpha\sin\theta)^2$
$$=\sin^2\alpha\cos^2\beta+\cos^2\alpha,$$

$$\frac{c^2}{\sin^2\theta}=\sin^2\alpha\sin^2\beta,$$

therefore $$a^2+b^2+\frac{c^2}{\sin^2\theta}=\sin^2\alpha\cos^2\beta+\sin^2\alpha\sin^2\beta+\cos^2\alpha$$
$$=\sin^2\alpha+\cos^2\alpha=1$$

9 $$b+c\cos\alpha=u\cos(\alpha-\theta),\quad b+c\cos\beta=u\cos(\beta-\theta),$$

therefore $$2b+c(\cos\alpha+\cos\beta)=u\cos(\alpha-\theta)+u\cos(\beta-\theta),$$

therefore $$b+c\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}=u\cos\left(\frac{\alpha+\beta}{2}-\theta\right)\cos\frac{\alpha-\beta}{2},$$

therefore $$b\sec\frac{\alpha-\beta}{2}=u\cos\left(\frac{\alpha+\beta}{2}-\theta\right)-c\cos\frac{\alpha+\beta}{2} \qquad (1)$$

Again from the first two equations, by subtraction,

$$c(\cos\alpha-\cos\beta)=u\cos(\alpha-\theta)-u\cos(\beta-\theta),$$

therefore $$c\sin\frac{\beta-\alpha}{2}\sin\frac{\alpha+\beta}{2}=u\sin\frac{\beta-\alpha}{2}\sin\left(\frac{\alpha+\beta}{2}-\theta\right),$$

therefore $$0=u\sin\left(\frac{\alpha+\beta}{2}-\theta\right)-c\sin\frac{\alpha+\beta}{2} \qquad (2)$$

Square and add (1) and (2), thus

$$b^2\sec^2\delta=u^2+c^2-2uc\left\{\cos\left(\frac{\alpha+\beta}{2}-\theta\right)\cos\frac{\alpha+\beta}{2}+\sin\left(\frac{\alpha+\beta}{2}-\theta\right)\sin\frac{\alpha+\beta}{2}\right\}$$
$$=u^2+c^2-2uc\cos\theta$$

10 $$a\tan^2\theta-x=\frac{2a\tan\theta\tan2\alpha\tan2\alpha'}{\tan2\alpha+\tan2\alpha'},$$

$$a-x=\frac{2a\tan\theta}{\tan2\alpha+\tan2\alpha'},$$

therefore, by subtraction,

$$a(1-\tan^2\theta)=\frac{2a\tan\theta(1-\tan2\alpha\tan2\alpha')}{\tan2\alpha+\tan2\alpha'},$$

therefore $$\frac{\tan2\alpha+\tan2\alpha'}{1-\tan2\alpha\tan2\alpha'}=\frac{2\tan\theta}{1-\tan^2\theta},$$

therefore $$\tan(2\alpha+2\alpha')=\tan2\theta$$

11 $$\sin\theta+\sin\phi=a,\quad \cos\theta+\cos\phi=b,$$

square and add, thus

$$2+2(\cos\theta\cos\phi+\sin\theta\sin\phi)=a^2+b^2,$$

therefore $$2+2\cos(\theta-\phi)=a^2+b^2,$$

therefore $$2+2c=a^2+b^2$$

12 $\quad x\cos\theta+y\sin\theta=a,\quad x\cos(\theta+2\phi)-y\sin(\theta+2\phi)=a,$
therefore, by subtraction,

$$x\{\cos\theta-\cos(\theta+2\phi)\}+y\{\sin\theta+\sin(\theta+2\phi)\}=0,$$

therefore $\quad x\sin(\theta+\phi)\sin\phi+y\sin(\theta+\phi)\cos\phi=0,$

therefore $$x\sin\phi+y\cos\phi=0 \tag{1}$$

Again, by addition,

$$x\{\cos\theta+\cos(\theta+2\phi)\}+y\{\sin\theta-\sin(\theta+2\phi)\}=2a,$$

therefore $\quad x\cos(\theta+\phi)\cos\phi-y\cos(\theta+\phi)\sin\phi=a,$

therefore $$x\cos\phi-y\sin\phi=\frac{a}{\cos(\theta+\phi)} \tag{2}$$

Square and add (1) and (2) thus

$$x^2+y^2=\frac{a^2}{\cos^2(\theta+\phi)}=\frac{a^2}{1-\sin^2(\theta+\phi)}=\frac{a^2}{1-\frac{a^2}{b^2}\sin^2\phi},$$

therefore $$(x^2+y^2)\left(1-\frac{a^2}{b^2}\sin^2\phi\right)=a^2$$

But from (1) $\quad x^2\sin^2\phi=y^2\cos^2\phi=y^2(1-\sin^2\phi),$

therefore $$\sin^2\phi=\frac{y^2}{x^2+y^2}$$

Therefore $$(x^2+y^2)\left\{1-\frac{a^2y^2}{b^2(x^2+y^2)}\right\}=a^2,$$

therefore $$x^2+y^2=a^2+\frac{a^2y^2}{b^2}$$

13 $$\tan c=\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}.$$

Now $\quad \tan x+\tan y=a,$ and $\cot x+\cot y=b,$

therefore $$\frac{1}{\tan x}+\frac{1}{\tan y}=b,$$

therefore $$\tan x+\tan y=b\tan x\tan y,$$

therefore $\quad a=b\tan x\tan y,\quad$ therefore $\quad \tan x\tan y=\frac{a}{b},$

therefore $$\tan c=\frac{a}{1-\frac{a}{b}}=\frac{ab}{b-a},$$

therefore $$\cot c=\frac{b-a}{ab}=\frac{1}{a}-\frac{1}{b}$$

14 $$\frac{x}{a}=\frac{\sec^2\theta-\cos^2\theta}{\sec^2\theta+\cos^2\theta}=\frac{1-\cos^4\theta}{1+\cos^4\theta},$$

$$\frac{2b}{y}=\sec^2\theta+\cos^2\theta=\frac{1+\cos^4\theta}{\cos^2\theta};$$

therefore $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{1-\cos^4\theta}{1+\cos^4\theta}\right)^2+\frac{4\cos^4\theta}{(1+\cos^4\theta)^2}=\left(\frac{1+\cos^4\theta}{1+\cos^4\theta}\right)^2=1$$

15 $$(a+b)\tan(\theta-\phi)=(a-b)\tan(\theta+\phi),$$

therefore $$(a+b)\sin(\theta-\phi)\cos(\theta+\phi)=(a-b)\sin(\theta+\phi)\cos(\theta-\phi),$$

therefore $$b\{\sin(\theta+\phi)\cos(\theta-\phi)+\sin(\theta-\phi)\cos(\theta+\phi)\}$$
$$=a\{\sin(\theta+\phi)\cos(\theta-\phi)-\sin(\theta-\phi)\cos(\theta+\phi)\},$$

therefore $$b\sin 2\theta=a\sin 2\phi,$$

and $$b\cos 2\theta=c-a\cos 2\phi,$$

square and add, thus $$b^2=c^2+a^2-2ac\cos 2\phi$$

16 $$x=\frac{z\sin(\theta+\theta')}{\sin 2\theta},\quad y=\frac{z\sin(\theta-\theta')}{\sin 2\theta}$$

Square and substitute in the first given equation, thus

$$\frac{z^2\sin^2(\theta+\theta')}{a^2\sin^2 2\theta}\cos\theta=\frac{z^2\sin^2(\theta-\theta')}{a^2\sin^2 2\theta}\cos\theta+\frac{z^2}{b^2}\cos\theta',$$

therefore $$\frac{\sin^2(\theta+\theta')\cos\theta-\sin^2(\theta-\theta')\cos\theta}{a^2\sin^2 2\theta}=\frac{\cos\theta'}{b^2},$$

therefore $$\frac{(\sin\theta\cos\theta'+\cos\theta\sin\theta')^2-(\sin\theta\cos\theta'-\cos\theta\sin\theta')^2}{4a^2\sin^2\theta\cos^2\theta}\cos\theta=\frac{\cos\theta'}{b^2},$$

therefore $$\frac{4\sin\theta\cos^2\theta\sin\theta'\cos\theta'}{4a^2\sin^2\theta\cos^2\theta}=\frac{\cos\theta'}{b^2},$$

therefore $$\frac{\sin\theta'}{\sin\theta}=\frac{a^2}{b^2}$$

17. $$y\cos\phi-x\sin\phi=a\cos 2\phi \qquad (1),$$
$$y\sin\phi+x\cos\phi=2a\sin 2\phi \qquad (2)$$

Multiply (1) by $\cos\phi$, and (2) by $\sin\phi$, and add, thus

$$y=a\cos 2\phi\cos\phi+2a\sin 2\phi\sin\phi$$
$$=a\cos\phi(\cos 2\phi+4\sin^2\phi)=a\cos\phi(\cos^2\phi+3\sin^2\phi)$$

Again multiply (2) by $\cos\phi$, and (1) by $\sin\phi$, and subtract thus

$$x=2a\sin 2\phi\cos\phi-a\cos 2\phi\sin\phi$$
$$=a\sin\phi(4\cos^2\phi-\cos 2\phi)=a\sin\phi(3\cos^2\phi+\sin^2\phi)$$

Thus $\quad x+y=a(\sin^3\phi+\cos^3\phi+3\sin^2\phi\cos\phi+3\cos^2\phi\sin\phi)$

$\qquad =a(\sin\phi+\cos\phi)^3,$

and $\quad x-y=a(\sin^3\phi-3\sin^2\phi\cos\phi+3\cos^2\phi\sin\phi-\cos^3\phi)$

$\qquad =a(\sin\phi-\cos\phi)^3$

Therefore $(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=a^{\frac{2}{3}}\{(\sin\phi+\cos\phi)^2+(\sin\phi-\cos\phi)^2\}=2a^{\frac{2}{3}}$

18 $\quad \cos\theta\cos\phi+\sin\theta\sin\phi=\sin\beta\sin\gamma,$

therefore $\quad \sin^2\theta\sin^2\phi=(\sin\beta\sin\gamma-\cos\theta\cos\phi)^2,$

therefore $\quad (1-\cos^2\theta)(1-\cos^2\phi)=(\sin\beta\sin\gamma-\cos\theta\cos\phi)^2,$

therefore $$\left(1-\frac{\sin^2\beta}{\sin^2\alpha}\right)\left(1-\frac{\sin^2\gamma}{\sin^2\alpha}\right)=\left(\sin\beta\sin\gamma-\frac{\sin\beta\sin\gamma}{\sin^2\alpha}\right)^2,$$

therefore $\quad (\sin^2\alpha-\sin^2\beta)(\sin^2\alpha-\sin^2\gamma)=\sin^2\beta\sin^2\gamma(\sin^2\alpha-1)^2,$

therefore $\quad \sin^4\alpha-\sin^2\alpha(\sin^2\beta+\sin^2\gamma)=\sin^2\beta\sin^2\gamma(\sin^4\alpha-2\sin^2\alpha),$

therefore $\quad \sin^2\alpha-\sin^2\beta-\sin^2\gamma=\sin^2\beta\sin^2\gamma(\sin^2\alpha-2),$

therefore $\quad \sin^2\alpha(1-\sin^2\beta\sin^2\gamma)=\sin^2\beta\cos^2\gamma+\cos^2\beta\sin^2\gamma,$

therefore $$\sin^2\alpha=\frac{\sin^2\beta\cos^2\gamma+\cos^2\beta\sin^2\gamma}{1-\sin^2\beta\sin^2\gamma},$$

therefore $$\cos^2\alpha=\frac{1-\sin^2\beta\sin^2\gamma-\sin^2\beta\cos^2\gamma-\cos^2\beta\sin^2\gamma}{1-\sin^2\beta\sin^2\gamma}$$

$$=\frac{(\sin^2\beta+\cos^2\beta)(\sin^2\gamma+\cos^2\gamma)-\sin^2\beta\sin^2\gamma-\sin^2\beta\cos^2\gamma-\cos^2\beta\sin^2\gamma}{1-\sin^2\beta\sin^2\gamma}$$

$$=\frac{\cos^2\beta\cos^2\gamma}{1-\sin^2\beta\sin^2\gamma}$$

Therefore $$\tan^2\alpha=\frac{\sin^2\beta\cos^2\gamma+\cos^2\beta\sin^2\gamma}{\cos^2\beta\cos^2\gamma}$$

$$=\frac{\sin^2\beta}{\cos^2\beta}+\frac{\sin^2\gamma}{\cos^2\gamma}=\tan^2\beta+\tan^2\gamma$$

19 $$m=\operatorname{cosec}\theta-\sin\theta=\frac{1}{\sin\theta}-\sin\theta=\frac{1-\sin^2\theta}{\sin\theta}=\frac{\cos^2\theta}{\sin\theta},$$

$$n=\sec\theta-\cos\theta=\frac{1}{\cos\theta}-\cos\theta=\frac{1-\cos^2\theta}{\cos\theta}=\frac{\sin^2\theta}{\cos\theta},$$

therefore $$mn=\frac{\cos^2\theta\sin^2\theta}{\sin\theta\cos\theta}=\cos\theta\sin\theta,$$

therefore $$\sin\theta=\frac{mn}{\cos\theta},\ \text{and}\ \cos\theta=\frac{mn}{\sin\theta},$$

therefore $$m=\frac{\cos^3\theta}{mn}, \text{ and } n=\frac{\sin^3\theta}{mn},$$

therefore $$\cos\theta=(m^2n)^{\frac{1}{3}}, \text{ and } \sin\theta=(mn^2)^{\frac{1}{3}},$$

therefore $$\cos^2\theta+\sin^2\theta=(m^2n)^{\frac{2}{3}}+(mn^2)^{\frac{2}{3}},$$

therefore $$1=(mn)^{\frac{2}{3}}\{m^{\frac{2}{3}}+n^{\frac{2}{3}}\}$$

20 $$(x\sin\theta-y\cos\theta)^2=x^2+y^2,$$

therefore $$x^2+y^2-(x\sin\theta-y\cos\theta)^2=0,$$

therefore $$x^2\cos^2\theta+2xy\sin\theta\cos\theta+y^2\sin^2\theta=0,$$

therefore $$(x\cos\theta+y\sin\theta)^2=0,$$

therefore $$x\cos\theta+y\sin\theta=0,$$

therefore $$\tan\theta=-\frac{x}{y}$$

Hence we obtain $\cos^2\theta=\dfrac{y^2}{x^2+y^2}$ and $\sin^2\theta=\dfrac{x^2}{x^2+y^2}$.

Substitute in the second given equation thus

$$\frac{1}{x^2+y^2}\left(\frac{y^2}{a^2}+\frac{x^2}{b^2}\right)=\frac{1}{x^2+y^2},$$

therefore $$\frac{y^2}{a^2}+\frac{x^2}{b^2}=1$$

21. $$a\sin^2\theta+a'\cos^2\theta=b,$$

therefore $$a\sin^2\theta+a'(1-\sin^2\theta)=b,$$

therefore $$\sin^2\theta=\frac{b-a'}{a-a'},$$

therefore $$\cos^2\theta=\frac{a-b}{a-a'},$$

therefore $$\tan^2\theta=\frac{b-a'}{a-b}$$

Similarly we find $$\tan^2\theta'=\frac{b'-a}{a'-b'}$$

But $$a^2\tan^2\theta=a'^2\tan^2\theta';$$

therefore $$a^2\frac{b-a'}{a-b}=a'^2\frac{b'-a}{a'-b'},$$

therefore $$a^2(b-a')(b'-a')=a'^2(b'-a)(b-a),$$

therefore $$a^2\{bb'-a'(b+b')\}=a'^2\{bb'-a(b+b')\},$$

therefore $$bb'(a^2-a'^2)=aa'(a-a')(b+b');$$

therefore $$bb'(a+a')=aa'(b+b')$$

Divide by $aa'bb'$, thus $\frac{1}{a'}+\frac{1}{a}=\frac{1}{b'}+\frac{1}{b}$.

22 $$y^2\cos^2\theta+x^2\sin^2\theta=\frac{x^2y^2}{a^2}=\frac{a^2b^2\sin^2\alpha}{a^2}=b^2\sin^2\alpha,$$

therefore $$\frac{y^2}{2}(1+\cos 2\theta)+\frac{x^2}{2}(1-\cos 2\theta)=b^2\sin^2\alpha,$$

therefore $$x^2+y^2+(y^2-x^2)\cos 2\theta=2b^2\sin^2\alpha,$$

therefore $$a^2+b^2+\{(y^2+x^2)^2-4y^2x^2\}^{\frac{1}{2}}\cos 2\theta=2b^2\sin^2\alpha,$$

therefore $$a^2+b^2+\{(a^2+b^2)^2-4a^2b^2\sin^2\alpha\}^{\frac{1}{2}}\cos 2\theta=2b^2\sin^2\alpha,$$

therefore $$\cos 2\theta=\frac{2b^2\sin^2\alpha-b^2-a^2}{\{(a^2+b^2)^2-4a^2b^2\sin^2\alpha\}^{\frac{1}{2}}},$$

therefore $$\sin^2 2\theta=\frac{-4a^2b^2\sin^2\alpha+4b^2\sin^2\alpha(b^2+a^2)-4b^4\sin^4\alpha}{(a^2+b^2)^2-4a^2b^2\sin^2\alpha}$$

$$=\frac{4b^4\sin^2\alpha(1-\sin^2\alpha)}{(a^2+b^2)^2-4a^2b^2\sin^2\alpha},$$

therefore $$\pm\sin 2\theta=\frac{2b^2\sin\alpha\cos\alpha}{\{(a^2+b^2)^2-4a^2b^2\sin^2\alpha\}^{\frac{1}{2}}}$$

Hence by division

$$\pm\cot 2\theta=\frac{2b^2\sin^2\alpha-b^2-a^2}{2b^2\sin\alpha\cos\alpha}=-\frac{a^2+b^2\cos 2\alpha}{b^2\sin 2\alpha}$$

$$=-\cot 2\alpha-\frac{a^2}{b^2}\operatorname{cosec} 2\alpha,$$

which we may also express thus

$$\pm\cot 2\theta=\cot 2\alpha+\frac{a^2}{b^2}\operatorname{cosec} 2\alpha$$

23 Let $\frac{\cos x}{a_1}$, $\frac{\cos 2x}{a_2}$, and $\frac{\cos 3x}{a_3}$ each be equal to $\frac{1}{\lambda}$, then

$$a_1=\lambda\cos x,\quad a_2=\lambda\cos 2x,\quad\text{and}\quad a_3=\lambda\cos 3x.$$

Therefore $$\frac{2a_2-a_1-a_3}{4a_2}=\frac{2\cos 2x-\cos x-\cos 3x}{4\cos 2x}$$

$$=\frac{2\cos 2x-2\cos 2x\cos x}{4\cos 2x}=\frac{1-\cos x}{2}=\sin^2\frac{x}{2}$$

24 Let $\frac{\sin x}{a_1}$, $\frac{\sin 3x}{a_3}$, and $\frac{\sin 5x}{a_5}$ each be equal to $\frac{1}{\lambda}$, then

$$a_1=\lambda \sin x, \quad a_3=\lambda \sin 3x, \text{ and } a_5=\lambda \sin 5x$$

Therefore $\frac{a_1-2a_3+a_5}{a_3}=\frac{\sin x-2\sin 3x+\sin 5x}{\sin 3x}=\frac{2\sin 3x\cos 2x-2\sin 3x}{\sin 3x}$

$$=2(\cos 2x-1)=-4\sin^2 x,$$

and $\frac{a_3-3a_1}{a_1}=\frac{\sin 3x-3\sin x}{\sin x}=\frac{3\sin x-4\sin^3 x-3\sin x}{\sin x}=-4\sin^2 x$

Thus $$\frac{a_1-2a_3+a_5}{a_3}=\frac{a_3-3a_1}{a_1}$$

25 Let $\frac{1}{\lambda}$ denote the value of the fractions which are given equal, thus

$$a_1=\lambda\cos x, \quad a_2=\lambda\cos(x+\theta), \quad a_3=\lambda\cos(x+2\theta), \quad a_4=\lambda\cos(x+3\theta),$$

therefore $\frac{a_1+a_3}{a_2}=\frac{\cos x+\cos(x+2\theta)}{\cos(x+\theta)}=\frac{2\cos(x+\theta)\cos\theta}{\cos(x+\theta)}=2\cos\theta,$

and $\frac{a_2+a_4}{a_3}=\frac{\cos(x+\theta)+\cos(x+3\theta)}{\cos(x+2\theta)}=\frac{2\cos(x+2\theta)\cos\theta}{\cos(x+2\theta)}=2\cos\theta,$

thus the required result is established

26 $$\sin^2\phi=\frac{\cos 2\alpha\cos 2\alpha'}{\cos^2(\alpha+\alpha')},$$

therefore $\cos^2\phi=\frac{\cos^2(\alpha+\alpha')-\cos 2\alpha\cos 2\alpha'}{\cos^2(\alpha+\alpha')}$

$$=\frac{1+\cos 2(\alpha+\alpha')-\cos 2(\alpha+\alpha')-\cos 2(\alpha-\alpha')}{2\cos^2(\alpha+\alpha')}=\frac{\sin^2(\alpha-\alpha')}{\cos^2(\alpha+\alpha')},$$

therefore $\cos\phi=\pm\frac{\sin(\alpha-\alpha')}{\cos(\alpha+\alpha')}$

Take the upper sign, then $\cos\phi=\frac{\sin(\alpha-\alpha')}{\cos(\alpha+\alpha')}$, therefore

$$\frac{1-\cos\phi}{1+\cos\phi}=\frac{\cos(\alpha+\alpha')-\sin(\alpha-\alpha')}{\cos(\alpha+\alpha')+\sin(\alpha-\alpha')}=\frac{\sin\left(\frac{\pi}{2}-\alpha-\alpha'\right)-\sin(\alpha-\alpha')}{\sin\left(\frac{\pi}{2}-\alpha-\alpha'\right)+\sin(\alpha-\alpha')}$$

$$=\frac{2\sin\left(\frac{\pi}{4}-\alpha\right)\cos\left(\frac{\pi}{4}-\alpha'\right)}{2\sin\left(\frac{\pi}{4}-\alpha'\right)\cos\left(\frac{\pi}{4}-\alpha\right)}=\frac{\tan\left(\frac{\pi}{4}-\alpha\right)}{\tan\left(\frac{\pi}{4}-\alpha'\right)},$$

therefore $$\tan^2\frac{\phi}{2}=\frac{\tan\left(\frac{\pi}{4}-\alpha\right)}{\tan\left(\frac{\pi}{4}-\alpha'\right)}.$$

Take the lower sign; then $\cos\phi=-\frac{\sin(\alpha-\alpha')}{\cos(\alpha+\alpha')}$, therefore

$$\frac{1-\cos\phi}{1+\cos\phi}=\frac{\cos(\alpha+\alpha')+\sin(\alpha-\alpha')}{\cos(\alpha+\alpha')-\sin(\alpha-\alpha')}=\frac{\sin\left(\frac{\pi}{2}-\alpha-\alpha'\right)+\sin(\alpha-\alpha')}{\sin\left(\frac{\pi}{2}-\alpha-\alpha'\right)-\sin(\alpha-\alpha')}$$

$$=\frac{2\sin\left(\frac{\pi}{4}-\alpha'\right)\cos\left(\frac{\pi}{4}-\alpha\right)}{2\sin\left(\frac{\pi}{4}-\alpha\right)\cos\left(\frac{\pi}{4}-\alpha'\right)}=\frac{\cot\left(\frac{\pi}{4}-\alpha\right)}{\cot\left(\frac{\pi}{4}-\alpha'\right)}=\frac{\tan\left(\frac{\pi}{4}+\alpha\right)}{\tan\left(\frac{\pi}{4}+\alpha'\right)},$$

therefore $$\tan^2\frac{\phi}{2}=\frac{\tan\left(\frac{\pi}{4}+\alpha\right)}{\tan\left(\frac{\pi}{4}+\alpha'\right)}.$$

27 $$\frac{\sin(\theta-\beta)\cos\alpha}{\sin(\phi-\alpha)\cos\beta}+\frac{\cos(\alpha+\theta)\sin\beta}{\cos(\phi-\beta)\sin\alpha}=0,$$

therefore $$\frac{\sin(\theta-\beta)\cos\alpha}{\cos(\alpha+\theta)\cos\beta}+\frac{\sin(\phi-\alpha)\sin\beta}{\cos(\phi-\beta)\sin\alpha}=0,$$

therefore $$\frac{(\sin\theta\cos\beta-\cos\theta\sin\beta)\cos\alpha}{(\cos\alpha\cos\theta-\sin\alpha\sin\theta)\cos\beta}+\frac{(\sin\phi\cos\alpha-\cos\phi\sin\alpha)\sin\beta}{(\cos\phi\cos\beta+\sin\phi\sin\beta)\sin\alpha}=0,$$

therefore $$\frac{(\tan\theta\cos\beta-\sin\beta)\cos\alpha}{(\cos\alpha-\sin\alpha\tan\theta)\cos\beta}+\frac{(\tan\phi\cos\alpha-\sin\alpha)\sin\beta}{(\cos\beta+\tan\phi\sin\beta)\sin\alpha}=0,$$

therefore $$\frac{\tan\theta-\tan\beta}{1-\tan\alpha\tan\theta}+\frac{\tan\phi\cot\alpha-1}{\cot\beta+\tan\phi}=0,$$

therefore $(\tan\theta-\tan\beta)(\cot\beta+\tan\phi)+(\tan\phi\cot\alpha-1)(1-\tan\alpha\tan\theta)=0$,

therefore $$\tan\theta(\cot\beta+\tan\alpha)+\tan\phi(\cot\alpha-\tan\beta)=2$$

But $\tan\theta=-\tan\phi\,\frac{\tan\beta}{\tan\alpha}\,\frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)}$, therefore

$$-\tan\phi\;(\cot\beta+\tan\alpha)\frac{\tan\beta}{\tan\alpha}\,\frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)}+\tan\phi(\cot\alpha-\tan\beta)=2,$$

therefore $-\tan\phi(\cot\alpha+\tan\beta)\cos(\alpha-\beta)+\tan\phi(\cot\alpha-\tan\beta)\cos(\alpha+\beta)$

$$=2\cos(\alpha+\beta),$$

therefore $\tan\phi\,\{\cot\alpha\,[\cos(\alpha+\beta)-\cos(\alpha-\beta)]-\tan\beta\,[\cos(\alpha+\beta)+\cos(\alpha-\beta)]\}$
$$=2\cos(\alpha+\beta),$$

therefore $\tan\phi\,\{\cot\alpha\sin\alpha\sin\beta+\tan\beta\cos\alpha\cos\beta\}=-\cos(\alpha+\beta)$,

therefore $$\tan\phi=-\frac{\cos(\alpha+\beta)}{2\cos\alpha\sin\beta}=\frac{1}{2}(\tan\alpha-\cot\beta),$$

and $$\tan\theta=-\tan\phi\,\frac{\tan\beta}{\tan\alpha}\,\frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)}$$
$$=\frac{\cos(\alpha-\beta)}{2\sin\alpha\cos\beta}=\frac{1}{2}(\cot\alpha+\tan\beta)$$

28 $$\frac{2}{1+x}=\frac{\sin\beta\sin\theta}{\cos(\beta-\theta)}=\frac{\sin\beta\sin\theta}{\cos\beta\cos\theta+\sin\beta\sin\theta}=\frac{1}{\cot\beta\cot\theta+1},$$

therefore $$\cot\beta\cot\theta+1=\frac{1+x}{2},$$

therefore $$\cot\beta\cot\theta=\frac{1+x}{2}-1=\frac{x-1}{2} \qquad (1)$$

Again $$\frac{2}{1+x}=\frac{\tan(\theta-\alpha)}{\cot\beta}=\frac{(\tan\theta-\tan\alpha)\tan\beta}{1+\tan\theta\tan\alpha},$$

therefore $$2(1+\tan\theta\tan\alpha)=(1+x)(\tan\theta-\tan\alpha)\tan\beta,$$

therefore $$\tan\theta=\frac{2+(1+x)\tan\alpha\tan\beta}{(1+x)\tan\beta-2\tan\alpha} \qquad (2)$$

From (1) and (2) by multiplication
$$\cot\beta=\frac{2+(1+x)\tan\alpha\tan\beta}{(1+x)\tan\beta-2\tan\alpha}\;\frac{x-1}{2},$$

therefore $2\cot\beta\,\{(1+x)\tan\beta-2\tan\alpha\}=2(x-1)+(x^2-1)\tan\alpha\tan\beta$,

therefore $$2(1+x)-4\cot\beta\tan\alpha=2(x-1)+(x^2-1)\tan\alpha\tan\beta,$$

therefore $$x^2\tan\alpha\tan\beta=4-4\cot\beta\tan\alpha+\tan\alpha\tan\beta,$$

therefore $$x^2=4\cot\alpha\cot\beta-4\cot^2\beta+1$$
$$=2\left(\cot\frac{\alpha}{2}-\tan\frac{\alpha}{2}\right)\cot\beta-4\cot^2\beta+1$$
$$=\left(\cot\frac{\alpha}{2}-2\cot\beta\right)\left(\tan\frac{\alpha}{2}+2\cot\beta\right)$$

29 $\sin\theta\sin\phi=\sin\alpha\sin\beta$, therefore $2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=\frac{\sin\alpha\sin\beta}{\sin\phi}$,

therefore $$4\sin^2\frac{\theta}{2}-4\sin^4\frac{\theta}{2}=\frac{\sin^2\alpha\sin^2\beta}{\sin^2\phi},$$

therefore $$4\sin^4\frac{\theta}{2}-4\sin^2\frac{\theta}{2}+1=1-\frac{\sin^2\alpha\sin^2\beta}{\sin^2\phi},$$

but $$\sin^2\phi=\frac{\cot^2\frac{\alpha}{2}}{\cot^2\frac{\alpha}{2}+\cos^2\beta},$$

therefore $$4\sin^4\frac{\theta}{2}-4\sin^2\frac{\theta}{2}+1=1-\frac{\sin^2\alpha\left(\cot^2\frac{\alpha}{2}+\cos^2\beta\right)\sin^2\beta}{\cot^2\frac{\alpha}{2}}$$

$$=1-4\sin^4\frac{\alpha}{2}\left(\cot^2\frac{\alpha}{2}+\cos^2\beta\right)\sin^2\beta$$

$$=1-4\sin^4\frac{\alpha}{2}\left(\cot^2\frac{\alpha}{2}+1-\sin^2\beta\right)\sin^2\beta$$

$$=1-4\sin^2\frac{\alpha}{2}\sin^2\beta+4\sin^4\frac{\alpha}{2}\sin^4\beta,$$

therefore $$2\sin^2\frac{\theta}{2}-1=\pm\left(1-2\sin^2\frac{\alpha}{2}\sin^2\beta\right)$$

Taking the lower sign we have $\sin^2\frac{\theta}{2}=\sin^2\frac{\alpha}{2}\sin^2\beta$

30 $\sin\phi=n\sin\theta$, therefore $\cos\phi=\sqrt{(1-n^2\sin^2\theta)}$,

therefore $$\tan\phi=\frac{n\sin\theta}{\sqrt{(1-n^2\sin^2\theta)}}$$

Hence we have $$\frac{n\sin\theta}{\sqrt{(1-n^2\sin^2\theta)}}=2\tan\theta=\frac{2\sin\theta}{\cos\theta},$$

therefore $$n\cos\theta=2\sqrt{(1-n^2\sin^2\theta)},$$

therefore $$n^2(1-\sin^2\theta)=4(1-n^2\sin^2\theta),$$

therefore $3n^2\sin^2\theta=4-n^2$, therefore $\sin^2\theta=\frac{4-n^2}{3n^2}$

This must lie between 0 and 1, so that $4-n^2$ must lie between 0 and $3n^2$, therefore 4 must lie between n^2 and $4n^2$, therefore n^2 must lie between 1 and 4

31 Assume $x=\tan A$ and $y=\tan B$; then by Art 114 we have $z=\tan C$, where $A+B+C=180^0$

Therefore $2A+2B+2C=360^0$, therefore $\tan(2A+2B+2C)=0$, and therefore, as in Art 114,

$$\tan 2A+\tan 2B+\tan 2C=\tan 2A\tan 2B\tan 2C;$$

therefore $$\frac{2\tan A}{1-\tan^2 A}+\frac{2\tan B}{1-\tan^2 B}+\frac{2\tan C}{1-\tan^2 C}$$
$$=\frac{2\tan A}{1-\tan^2 A}\cdot\frac{2\tan B}{1-\tan^2 B}\cdot\frac{2\tan C}{1-\tan^2 C}$$

32 $$v\sin c=\sin z=\sin(2\pi-x-y)=-\sin(x+y)$$
$$=-\sin x\cos y-\cos x\sin y=-v\sin a\cos y-v\sin b\cos x$$

Therefore either $v=0$, or $\sin c=-\sin a\cos y-\sin b\cos x$

Take the latter, thus $\sin a\cos y=-\sin c-\sin b\cos x$,

but $\sin a\sin y=\sin b\sin x$,

square and add, thus $\sin^2 a=\sin^2 b+\sin^2 c+2\sin b\sin c\cos x$,

therefore $$\cos x=\frac{\sin^2 a-\sin^2 b-\sin^2 c}{2\sin b\sin c}$$

Similarly $\cos y$ and $\cos z$ may be found, and then v

If $v=0$, we have $\sin x=0$, $\sin y=0$, and $\sin z=0$ This will give us three solutions; $x=0, y=\pi, z=\pi$, $x=\pi, y=0, z=\pi$, $x=\pi, y=\pi, z=0$ · and also three solutions, $x=0, y=0, z=2\pi$, $x=0, y=2\pi, z=0$, $x=2\pi, y=0, z=0$

33 Let $u=(\cos ax)^{\operatorname{cosec}^2\beta x}$, therefore

$$\log u=\operatorname{cosec}^2\beta x\log\cos ax=\frac{1}{2}\operatorname{cosec}^2\beta x\log(1-\sin^2 ax)$$
$$=-\frac{1}{2\sin^2\beta x}\left\{\sin^2 ax+\frac{1}{2}\sin^4 ax+\frac{1}{3}\sin^6 ax+\quad\right\}$$

Now $$\frac{\sin ax}{\sin\beta x}=\frac{a}{\beta}\cdot\frac{\sin ax}{ax}\cdot\frac{\beta x}{\sin\beta x},$$

when x is zero the value of $\frac{\sin ax}{ax}$ is unity, and so also is the value of $\frac{\beta x}{\sin\beta x}$, thus $\frac{\sin ax}{\sin\beta x}=\frac{a}{\beta}$; therefore $\frac{\sin^2 ax}{\sin^2\beta x}=\frac{a^2}{\beta^2}$

The limit of $\frac{\sin^4 ax}{\sin^2\beta x}$ is zero, and so also the other terms in $\log u$ vanish, and as in Art 150 their sum vanishes also Hence $\log u=-\frac{a^2}{2\beta^2}$, and therefore $u=e^{-\frac{a^2}{2\beta^2}}$.

34. By Art 183 if h is very small we have $\tan(\theta+h)-\tan\theta=h\sec^2\theta$, thus if θ be nearly equivalent to 60° we have approximately

$$\tan(\theta+h)-\tan\theta=4h$$

Since the tables extend to 7 places of decimals it follows that we can discriminate angles which are near 60°, by means of their tangents, when the circular measure h of the difference is such that $4h = 0000001$ Thus $h = \frac{1}{4}$ of $\frac{1}{10^7}$; the corresponding value in seconds is $\frac{1}{4} \times \frac{1}{10^7} \times \frac{180}{\pi} \times 60 \times 60$, that is $\frac{18 \times 9}{10000\pi}$, that is about $\frac{1}{200}$.

35 By Art 196 if h is very small we have

$$L \sin(\theta + h) - L \sin\theta = \mu h \cot\theta = \frac{h}{(\log_e 10) \tan\theta},$$

thus if θ be nearly equivalent to $64^\circ 36'$ we have approximately

$$L \sin(\theta + h) - L \sin\theta = \frac{h}{4\ 8492}$$

Since the tables extend to 7 places of decimals it follows that we can discriminate angles which are near $64^\circ 36'$ by means of their L sines, when the circular measure of the difference is such that $\frac{h}{4\ 8492} = 0000001$ Thus $h = \frac{4\ 8492}{10^7}$, the corresponding value in seconds is $\frac{4\ 8492}{10^7} \times \frac{180}{\pi} \times 60 \times 60$ this will be found to be about $\frac{1}{10}$.

36.
$$1 - \tan^2\frac{\alpha}{2} = \frac{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}} = \frac{\cos\alpha}{\cos^2\frac{\alpha}{2}},$$

$$1 - \tan^2\frac{\alpha}{4} = \frac{\cos^2\frac{\alpha}{4} - \sin^2\frac{\alpha}{4}}{\cos^2\frac{\alpha}{4}} = \frac{\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{4}},$$

and so on

In this way we find that the proposed expression

$$= \frac{\cos\alpha \cos\frac{\alpha}{2} \cos\frac{\alpha}{2^2} \cos\frac{\alpha}{2^3}}{\cos^2\frac{\alpha}{2} \cos^2\frac{\alpha}{2^2} \cos^2\frac{\alpha}{2^3}}.$$

$$= \frac{\cos\alpha}{\cos\frac{\alpha}{2} \cos\frac{\alpha}{2^2} \cos\frac{\alpha}{2^3}}$$

$$= \cos\alpha \frac{\sin\alpha}{\alpha} = \frac{\alpha}{\tan\alpha}.$$ See Art 129.

37 We have universally

$$\sin^2(A+B)=(\sin A\cos B+\cos A\sin B)^2$$

$$=\sin^2 A\cos^2 B+\cos^2 A\sin^2 B+2\sin A\cos A\sin B\cos B$$

$$=\sin^2 A(1-\sin^2 B)+\sin^2 B(1-\sin^2 A)+2\sin A\cos A\sin B\cos B$$

$$=\sin^2 A+\sin^2 B+2\sin A\sin B\{\cos A\cos B-\sin A\sin B\}$$

$$=\sin^2 A+\sin^2 B+2\sin A\sin B\cos(A+B) \qquad (1)$$

Also in the present case

$$\sin^2 A+\sin^2 B=1-\sin^2 C=\cos^2 C \qquad (2)$$

If $A+B$ is greater than 90^0, then *a fortiori* $A+B+C$ is greater than 90^0

If $A+B$ is less than 90^0, then $\sin^2(A+B)$ is greater than $\sin^2 A+\sin^2 B$ by (1), and therefore greater than $\cos^2 C$ by (2), and therefore $A+B$ is greater than 90^0-C, so that $A+B+C$ is greater than 90^0

38 Take the diagram of Art 71 Let α be the angle PAB Suppose a circle having its centre O within the space bounded by PB, BT, and TP; let it touch the arc PB, the tangent BT, and the secant APT Let ρ denote the radius of this circle, and r the radius of the original circle

OT will bisect the angle ATB, and OA will pass through the point of contact of the circles Let N be the point of contact of the secant APT and the circle with centre O Then

$$NT=\rho\cot\frac{1}{2}\left(\frac{\pi}{2}-\alpha\right), \quad OA=r+\rho,$$

therefore $$AN=\sqrt{(r+\rho)^2-\rho^2}=\sqrt{(r^2+2r\rho)}.$$

Hence $$\sqrt{(r^2+2r\rho)}+\rho\cot\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=AT=r\sec\alpha;$$

therefore $$\sqrt{(r^2+2r\rho)}=r\sec\alpha-\rho\cot\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$$

By squaring we obtain a quadratic equation for determining ρ The reason why we have a quadratic equation is that another circle can also be drawn, which may be said to fulfil the conditions For produce PA through A to meet the original circle again at p, then we may have a circle outside the arc Bp, touching this arc, touching TB produced through B, and touching Tp produced through p The corresponding equation would be

$$\rho\cot\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)-\sqrt{(r^2+2r\rho)}=r\sec\alpha,$$

this differs from the former only in the sign of the radical, and therefore leads to the same quadratic equation

Suppose $\rho = r$, then $\pm\sqrt{3} = \dfrac{1}{\cos\alpha} - \dfrac{\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)}$

$$= \frac{1}{\cos\alpha} - \frac{\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}$$

$$= \frac{1}{\cos\alpha} - \frac{\left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right)^2}{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}$$

$$= \frac{1}{\cos\alpha} - \frac{1 + \sin\alpha}{\cos\alpha} = -\tan\alpha$$

Hence taking $\sqrt{3} = \tan\alpha$ we have $\alpha = \dfrac{\pi}{3}$

39 Let x denote the value of $l\sin(\theta-\beta) - m\sin(\theta-\alpha)$, so that $l\cos(\theta-\beta) - m\cos(\theta-\alpha) = n$, $l\sin(\theta-\beta) - m\sin(\theta-\alpha) = x$

Square and add, thus

$$l^2 + m^2 - 2lm\{\cos(\theta-\beta)\cos(\theta-\alpha) + \sin(\theta-\beta)\sin(\theta-\alpha)\} = n^2 + x^2,$$

that is $\qquad l^2 + m^2 - 2lm\cos(\alpha-\beta) = n^2 + x^2,$

therefore $\qquad x = \sqrt{l^2 + m^2 - n^2 - 2lm\cos(\alpha-\beta)}$

40 $\theta - \sin\theta$ is less than $\tan\theta - \theta$ if 2θ is less than $\sin\theta + \tan\theta$, that is if 2θ is less than $\tan\theta(1+\cos\theta)$, that is if 2θ is less than $\dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} \times \dfrac{2}{1+\tan^2\frac{\theta}{2}}$, that is if $\dfrac{\theta}{2}$ is less than $\dfrac{\tan\frac{\theta}{2}}{1-\tan^4\frac{\theta}{2}}$ and this is obviously the case, because $\dfrac{\theta}{2}$ is less than $\tan\dfrac{\theta}{2}$

XIII [illegible]

1 The greatest angle is opposite to the greatest side, thus the cosine

$$= \frac{(x^2-1)^2 + (2x+1)^2 - (x^2+x+1)^2}{2(x^2-1)(2x+1)}$$

$$= \frac{x^4 - 2x^2 + 1 + 4x^2 + 4x + 1 - (x^4 + x^2 + 1 + 2x^3 + 2x^2 + 2x)}{2(x^2-1)(2x+1)}$$

$$= \frac{-2x^3 - x^2 + 2x + 1}{2(2x^3 + x^2 - 2x - 1)} = -\frac{1}{2}$$

Therefore the angle is 120^0

2 $2 \sin C \cos B = \sin A = \sin(B+C) = \sin B \cos C + \cos B \sin C$,

therefore $\sin C \cos B = \sin B \cos C$,

therefore $\sin(C-B) = 0$; therefore $B = C$

3 We have $\cos A = \dfrac{b}{c}$, $\sin A = \dfrac{a}{c}$,

therefore $\dfrac{1+\cos A}{\sin A} = \dfrac{c+b}{a}$, therefore $\cot\dfrac{A}{2} = \dfrac{c+b}{a}$

4 $a \tan A + b \tan B = (a+b)\tan\dfrac{A+B}{2}$,

therefore $$a\left(\tan A - \tan\frac{A+B}{2}\right) = b\left(\tan\frac{A+B}{2} - \tan B\right),$$

therefore $$\frac{a\left(\sin A \cos\frac{A+B}{2} - \cos A \sin\frac{A+B}{2}\right)}{\cos A \cos\frac{A+B}{2}}$$

$$= \frac{b\left(\sin\frac{A+B}{2}\cos B - \cos\frac{A+B}{2}\sin B\right)}{\cos B \cos\frac{A+B}{2}},$$

therefore $$\frac{a \sin\frac{A-B}{2}}{\cos A} = \frac{b \sin\frac{A-B}{2}}{\cos B}, \text{ therefore } \frac{a}{b} = \frac{\cos A}{\cos B}.$$

But $\dfrac{a}{b} = \dfrac{\sin A}{\sin B}$, therefore $\dfrac{\sin A}{\sin B} = \dfrac{\cos A}{\cos B}$,

therefore $\tan A = \tan B$, therefore $A = B$

5 Let $2a$ denote the least angle, then the other angles are $4a$ and $8a$ respectively therefore $2a + 4a + 8a = \pi$, therefore $a = \dfrac{\pi}{11}$

Then by Art 211 the ratio of the greatest side to the perimeter

$$= \frac{\sin 8a}{\sin 2a + \sin 4a + \sin 8a}$$

$$= \frac{\sin 8a}{\sin 2a + \sin 4a + \sin 6a} = \frac{2 \sin 4a \cos 4a}{2 \sin 3a \cos a + 2 \sin 3a \cos 3a},$$

but $4a + 3a = \dfrac{\pi}{2}$, therefore $\cos 4a = \sin 3a$, hence this expression

$$= \frac{\sin 4a}{\cos a + \cos 3a} = \frac{2 \sin 2a \cos 2a}{2 \cos a \cos 2a} = \frac{\sin 2a}{\cos a} = 2 \sin a$$

6 $2bc \operatorname{vers} A' + 2ca \operatorname{vers} B' + 2ab \operatorname{vers} C'$

$$=2bc\,(1-\cos A') + 2ca\,(1-\cos B') + 2ab\,(1-\cos C')$$

$$=2bc\,(1+\cos A) + 2ca\,(1+\cos B) + 2ab\,(1+\cos C)$$

$$=4bc\cos^2\frac{A}{2} + 4ca\cos^2\frac{B}{2} + 4ab\cos^2\frac{C}{2}$$

$$=4s\,(s-a) + 4s\,(s-b) + 4s\,(s-c)$$

$$=4s\,(3s-a-b-c) = 4s^2 = (2s)^2 = (a+b+c)^2$$

7 Let $AD=p$ Suppose the angles B and C to be acute, as in the left-hand diagram of Art 214 Then

$$AE = p\cos(90^\circ - B) = p\sin B,$$

$$DE = p\sin(90^\circ - B) = p\cos B,$$

$$EB = DE\cot B = p\cos B\cot B$$

therefore $$AE\ \ EB = p^2\cos^2 B$$

Similarly $$AF\ \ FC = p^2\cos^2 C$$

Therefore $$AE\ \ EB\cos^2 C = AF\ \ FC\cos^2 B$$

Next suppose one of the angles B and C to be obtuse, say the angle C, as in the right-hand diagram of Art 214

Then $$AE\ \ EB = p^2\cos^2 B \text{ as before,}$$

$$AF = p\cos(C-90^\circ) = p\sin C,$$

$$DF = p\sin(C-90^\circ) = -p\cos C,$$

$$FC = DF\cot(180^\circ - C) = -DF\cot C = p\cos C\cot C,$$

therefore $$AF\ \ FC = p^2\cos^2 C, \text{ as before}$$

8 $\dfrac{\sin 2\theta + \sin 4\theta}{\sin 3\theta} = \dfrac{a+c}{b}$, therefore $2\cos\theta = \dfrac{a+c}{b}$; therefore $\cos\theta = \dfrac{a+c}{2b}$, therefore $\tan^2\theta = \dfrac{1}{\cos^2\theta} - 1 = \left(\dfrac{2b}{a+c}\right)^2 - 1$

9 Since C is obtuse, $A+B$ is less than 90°, therefore $\cos(A+B)$ is positive, therefore $\cos A\cos B - \sin A\sin B$ is positive, therefore $\sin A\sin B$ is less than $\cos A\cos B$, therefore $\dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}$ is less than unity, that is $\tan A\tan B$ is less than unity.

10 Since a, b, c are in Arithmetical Progression, so are $\sin A$, $\sin B$, $\sin C$, hence $\sin A + \sin C = 2\sin B$,

therefore $\sin\frac{A+C}{2}\cos\frac{A-C}{2}=2\sin\frac{B}{2}\cos\frac{B}{2}=2\sin\frac{B}{2}\sin\frac{A+C}{2}$,

therefore $$\cos\frac{A-C}{2}=2\sin\frac{B}{2}$$

Again $$a\cos^2\frac{C}{2}+c\cos^2\frac{A}{2}=\frac{a}{2}(1+\cos C)+\frac{c}{2}(1+\cos A)$$

$$=\frac{1}{2}(a+c)+\frac{1}{2}(a\cos C+c\cos A)=\frac{1}{2}(a+c)+\frac{b}{2},$$ by Art. 216,

$$=b+\frac{b}{2},$$ by hypothesis, $=\frac{3b}{2}$

11 From the triangle ABD we have

$$\frac{\sin ADB}{\sin BAD}=\frac{AB}{BD}=\frac{2c}{a}$$

Put ϕ for BAD, thus

$$\frac{\sin(\phi+B)}{\sin\phi}=\frac{2c}{a}=\frac{2\sin C}{\sin A},$$

therefore $$\frac{\sin\phi\cos B+\cos\phi\sin B}{\sin\phi}=\frac{2\sin C}{\sin A},$$

therefore $$\cot B+\cot\phi=\frac{2\sin C}{\sin A\sin B}=\frac{2\sin(A+B)}{\sin A\sin B}$$

$$=2\cot A+2\cot B,$$

therefore $$\cot\phi-\cot B=2\cot A.$$

12 Let the angle A of a triangle be divided into two parts by a straight line AD, denote BAD by ϕ and CAD by ψ, and suppose that $\frac{\sin\phi}{\sin\psi}=\frac{c}{b}$

Thus $$\frac{\sin(A-\psi)}{\sin\psi}=\frac{c}{b}=\frac{\sin C}{\sin B},$$

therefore $$\sin A\cot\psi-\cos A=\frac{\sin C}{\sin B},$$

therefore $$\cot\psi=\cot A+\frac{\sin(A+B)}{\sin A\sin B}=2\cot A+\cot B$$

Similarly $$\cot\phi=2\cot A+\cot C.$$

Therefore $$\cot\psi-\cot\phi=\cot B-\cot C$$

13 Suppose $\cot A+\cot C=2\cot B$,

thus $$\frac{\cos A}{\sin A}+\frac{\cos C}{\sin C}=\frac{2\cos B}{\sin B};$$

therefore $$\frac{\sin(A+C)}{\sin A \sin C}=\frac{2\cos B}{\sin B},$$

therefore $$\frac{\sin^2 B}{\sin A \sin C}=2\cos B,$$

therefore $$\frac{b^2}{ac}=\frac{a^2+c^2-b^2}{ac};$$

therefore $$2b^2=a^2+c^2$$

Thus a^2, b^2, c^2 are in Arithmetical Progression

14 Let a perpendicular AD be drawn from the angle A of a triangle on the base BC Let $BAD=\phi$, and $CAD=\psi$ Let m denote the ratio of the base BC to the perpendicular AD

Then in the case of the left-hand diagram of Art 214 we have

$$\tan\phi=\frac{BD}{AD}, \quad \tan\psi=\frac{CD}{AD},$$

therefore $$\tan\phi+\tan\psi=\frac{BD+CD}{AD}=\frac{BC}{AD}=m \qquad (1)$$

Also $\phi+\psi=A$, thus

$$\tan A=\tan(\phi+\psi)=\frac{\tan\phi+\tan\psi}{1-\tan\phi\tan\psi} \qquad (2)$$

Hence from (1) and (2) we can find $\tan\phi$ and $\tan\psi$

Similarly in the case of the right-hand diagram of Art 214 we have

$$\tan\phi-\tan\psi=m,$$

and $$\tan A=\tan(\phi-\psi)=\frac{\tan\phi-\tan\psi}{1+\tan\phi\tan\psi}$$

15 Let the base BC of a triangle be divided at D and E, so that $BD=DE=EC$ Let the angle BAD be denoted by ϕ_1, the angle DAE by ϕ_2, and the angle EAC by ϕ_3

Then from the triangle AEB we have $\dfrac{\sin(\phi_1+\phi_2)}{\sin AEB}=\dfrac{BE}{AB}=\dfrac{2}{3}\,\dfrac{a}{c}$, and from the triangle AEC we have $\dfrac{\sin\phi_3}{\sin AEC}=\dfrac{EC}{AC}=\dfrac{1}{3}\,\dfrac{a}{b}$,

therefore by division $$\frac{\sin(\phi_1+\phi_2)}{\sin\phi_3}=\frac{2b}{c}$$

In the same manner we see that

$$\frac{\sin(\phi_3+\phi_2)}{\sin\phi_1}=\frac{2c}{b}.$$

Therefore $$\frac{\sin(\phi_1+\phi_2)\sin(\phi_3+\phi_2)}{\sin\phi_1\sin\phi_3}=4=4(\sin^2\phi_2+\cos^2\phi_2),$$

therefore $(\cos\phi_2+\sin\phi_2\cot\phi_1)(\cos\phi_2+\sin\phi_2\cot\phi_3)=4(\sin^2\phi_2+\cos^2\phi_2)$,

therefore $$(\cot\phi_2+\cot\phi_1)(\cot\phi_2+\cot\phi_3)=4(1+\cot^2\phi_2)$$

16 Suppose that $\sin A+\sin C=2\sin B$,

then $$2\sin\frac{A+C}{2}\cos\frac{A-C}{2}=4\sin\frac{B}{2}\cos\frac{B}{2}=4\cos\frac{A+C}{2}\sin\frac{A+C}{2},$$

therefore $$\cos\frac{A-C}{2}=2\cos\frac{A+C}{2},$$

therefore $$\cos\frac{A}{2}\cos\frac{C}{2}+\sin\frac{A}{2}\sin\frac{C}{2}=2\cos\frac{A}{2}\cos\frac{C}{2}-2\sin\frac{A}{2}\sin\frac{C}{2},$$

therefore $$3\sin\frac{A}{2}\sin\frac{C}{2}=\cos\frac{A}{2}\cos\frac{C}{2},$$

therefore $$\tan\frac{A}{2}\tan\frac{C}{2}=\frac{1}{3}$$

17 Denote ADB by ϕ From the triangle ABD we have

$$\frac{\sin BAD}{\sin ADB}=\frac{BD}{AB}=\frac{a}{2c},$$

therefore $$\frac{\sin(\phi+B)}{\sin\phi}=\frac{a}{2c},$$

therefore $$\cos B+\sin B\cot\phi=\frac{a}{2c},$$

therefore $$\cot\phi=\frac{\frac{a}{2c}-\cos B}{\sin B},$$

therefore $$\tan\phi=\frac{2c\sin B}{a-2c\cos B}=\frac{2ac\sin B}{a^2-(a^2+c^2-b^2)}$$

$$=\frac{2ac\sin B}{b^2-c^2}=\frac{2bc\sin A}{b^2-c^2}$$

18 Here $\cot\frac{A}{2}+\cot\frac{C}{2}=2\cot\frac{B}{2}$,

therefore $$\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}+\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}=\frac{2\cos\frac{B}{2}}{\sin\frac{B}{2}}=\frac{2\sin\frac{A+C}{2}}{\cos\frac{A+C}{2}},$$

therefore $$\frac{\sin\frac{A+C}{2}}{\sin\frac{A}{2}\sin\frac{C}{2}}=\frac{2\sin\frac{A+C}{2}}{\cos\frac{A+C}{2}},$$

therefore $$\cos\frac{A+C}{2}=2\sin\frac{A}{2}\sin\frac{C}{2},$$

therefore $$\cos\frac{A}{2}\cos\frac{C}{2}-\sin\frac{A}{2}\sin\frac{C}{2}=2\sin\frac{A}{2}\sin\frac{C}{2},$$

therefore $$\cos\frac{A}{2}\cos\frac{C}{2}=3\sin\frac{A}{2}\sin\frac{C}{2},$$

therefore $$\cot\frac{A}{2}\cot\frac{C}{2}=3$$

19 First suppose that $$\frac{\sin DAC}{\sin DAB}=\frac{1}{n},$$

and that $$\frac{\sin DBC}{\sin DBA}=\frac{1}{n}.$$

We have $$\frac{\sin DCB}{\sin DBC}=\frac{BD}{DC},$$

and $$\frac{\sin DBC}{\sin DBA}=\frac{1}{n},$$

therefore $$\frac{\sin DCB}{\sin DBA}=\frac{BD}{DC}\cdot\frac{1}{n}$$

Similarly $$\frac{\sin DCA}{\sin DAB}=\frac{AD}{DC}\cdot\frac{1}{n}.$$

Therefore $$\frac{\sin DCB}{\sin DCA}\cdot\frac{\sin DAB}{\sin DBA}=\frac{BD}{DA},$$

therefore $$\frac{\sin DCB}{\sin DCA}\cdot\frac{DB}{DA}=\frac{BD}{DA},$$

therefore $$\frac{\sin DCB}{\sin DCA}=1$$

In this case the angle C is bisected by DC

Next suppose that $$\frac{\sin DAC}{\sin DAB}=\frac{1}{n},$$

and that $$\frac{\sin DBA}{\sin DBC}=\frac{1}{n},$$

thus the angle B is divided into two parts equal to the two former, but differently situated

Then proceeding as before we have

$$\frac{\sin DCB}{\sin DBC}=\frac{BD}{DC},$$

and $$\frac{\sin DBC}{\sin DBA}=n,$$

therefore
$$\frac{\sin DCB}{\sin DBA} = \frac{n \ BD}{DC}.$$

Also
$$\frac{\sin DCA}{\sin DAB} = \frac{AD}{DC}\frac{1}{n}.$$

Hence we find that
$$\frac{\sin DCB}{\sin DCA} = n^2$$

20 Let the straight line which bisects the angle A of a triangle meet the base at D Then

the angle ADC = the angle B + the angle BAD,

thus
$$\sin\theta = \sin\left(B + \frac{A}{2}\right).$$

Hence
$$s\left(\sin\theta - \sin\frac{A}{2}\right) = s\left\{\sin\left(B + \frac{A}{2}\right) - \sin\frac{A}{2}\right\}$$
$$= 2s\cos\frac{B+A}{2}\sin\frac{B}{2} = 2s\sin\frac{C}{2}\sin\frac{B}{2},$$

put for $\sin\frac{C}{2}$ and $\sin\frac{B}{2}$ their values by Art 217, thus we have

$$2s\sin\frac{C}{2}\sin\frac{B}{2} = \frac{2s}{a}(s-a)\sqrt{\frac{(s-b)(s-c)}{bc}}$$
$$= \frac{2s(s-a)}{a}\sin\frac{A}{2} = \frac{2bc}{a}\cos^2\frac{A}{2}\sin\frac{A}{2}$$
$$= \frac{bc}{a}\cos\frac{A}{2}\sin A.$$

Again
$$l\sin\theta = \text{the perpendicular from } A \text{ on } BC$$
$$= b\sin C.$$

Therefore
$$l\sin\theta\cos\frac{A}{2} = b\sin C\cos\frac{A}{2}$$
$$= \frac{bc}{a}\sin A\cos\frac{A}{2}, \text{ by Art. 214}$$

Therefore
$$s\left(\sin\theta - \sin\frac{A}{2}\right) = l\sin\theta\cos\frac{A}{2}.$$

21 The third angle of the triangle will be $\pi-\theta-\phi$, and as the sines of the angles must be in Arithmetical Progression, we have

$$\sin\theta+\sin\phi=2\sin(\pi-\theta-\phi)=2\sin(\theta+\phi),$$

therefore $$2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}=4\sin\frac{\theta+\phi}{2}\cos\frac{\theta+\phi}{2},$$

therefore $$2\cos\frac{\theta+\phi}{2}=\cos\frac{\theta-\phi}{2},$$

therefore $$2\left(\cos\frac{\theta}{2}\cos\frac{\phi}{2}-\sin\frac{\theta}{2}\sin\frac{\phi}{2}\right)=\cos\frac{\theta}{2}\cos\frac{\phi}{2}+\sin\frac{\theta}{2}\sin\frac{\phi}{2},$$

therefore $$\cos\frac{\theta}{2}\cos\frac{\phi}{2}=3\sin\frac{\theta}{2}\sin\frac{\phi}{2},$$

therefore $$\cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2}=9\sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2}$$

$$=9\left(1-\cos^2\frac{\theta}{2}\right)\left(1-\cos^2\frac{\phi}{2}\right),$$

therefore $$8\left(1-\cos^2\frac{\theta}{2}\right)\left(1-\cos^2\frac{\phi}{2}\right)$$

$$=\cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2}-\left(1-\cos^2\frac{\theta}{2}\right)\left(1-\cos^2\frac{\phi}{2}\right)$$

$$=\cos^2\frac{\theta}{2}+\cos^2\frac{\phi}{2}-1=\cos^2\frac{\theta}{2}-\sin^2\frac{\phi}{2},$$

therefore $$8\sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2}=\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},$$

therefore $$4(1-\cos\theta)(1-\cos\phi)=2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}$$

$$=\cos\theta+\cos\phi$$

Or thus, $\cos\theta=\dfrac{a^2+b^2-c^2}{2ab}$, and $b=\dfrac{a+c}{2}$, therefore

$$\cos\theta=\frac{a-c}{a}+\frac{b}{2a}=\frac{a-c}{a}+\frac{a+c}{4a}=\frac{5a-3c}{4a}$$

Similarly $$\cos\phi=\frac{5c-3a}{4c}.$$

Hence $$4(1-\cos\theta)(1-\cos\phi)=\frac{(3c-a)(3a-c)}{4ac}=\frac{10ac-3a^2-3c^2}{4ac},$$

and $$\cos\theta+\cos\phi=\frac{5a-3c}{4a}+\frac{5c-3a}{4c}=\frac{10ac-3a^2-3c^2}{4ac}.$$

22 Draw from A, B, C respectively straight lines to meet the opposite sides at D, E, F, so that the angle BAD = the angle CBE = the angle $ACF = \alpha$ Let LMN be the triangle formed by the straight lines thus drawn so that A, L, M, D are in one straight line, B, M, N, E on another, and C, N, L, F on a third Then will the triangle LMN be similar to the triangle ABC

For the angle MLN = the angle MAC + the angle $LCA = A - \alpha + \alpha = A$, similarly the angle $NML = B$, and the angle $LNM = C$ Thus the triangle LMN is equiangular to the original triangle, and therefore similar to it

Again $$\frac{BN}{BC} = \frac{\sin BCN}{\sin BNC} = \frac{\sin(C-\alpha)}{\sin(\pi - C)} = \frac{\sin(C-\alpha)}{\sin C},$$

therefore $$BN = \frac{a \sin(C-\alpha)}{\sin C},$$

and $$\frac{BM}{BA} = \frac{\sin BAM}{\sin BMA} = \frac{\sin\alpha}{\sin(\pi - B)} = \frac{\sin\alpha}{\sin B},$$

therefore $$BM = \frac{c\sin\alpha}{\sin B}.$$

Hence $$MN = \frac{a\sin(C-\alpha)}{\sin C} - \frac{c\sin\alpha}{\sin B}$$

$$= a\cos\alpha - a\cot C\sin\alpha - \frac{a\sin C}{\sin A\sin B}\sin\alpha$$

$$= a\cos\alpha - a\cot C\sin\alpha - \frac{a\sin(A+B)}{\sin A\sin B}\sin\alpha$$

$$= a\cos\alpha - a\sin\alpha(\cot C + \cot B + \cot A)$$

The ratio of this to a is the same as the ratio of

$\cos\alpha - \sin\alpha(\cot A + \cot B + \cot C)$ to unity

23 $$ab\cos C - ac\cos B = \frac{a^2+b^2-c^2}{2} - \frac{a^2+c^2-b^2}{2} = b^2 - c^2$$

24 $$a(\cos B\cos C + \cos A) = a\{\cos B\cos C - \cos(B+C)\}$$

$$= a\sin B\sin C = \frac{a}{\sin A}\sin A\sin B\sin C$$

Similarly $$b(\cos A\cos C + \cos B) = \frac{b}{\sin B}\sin A\sin B\sin C,$$

and $$c(\cos A\cos B + \cos C) = \frac{c}{\sin C}\sin A\sin B\sin C$$

Thus the three expressions are equal by Art 214

25 $$(b+c-a)\tan\frac{A}{2} = 2(s-a)\tan\frac{A}{2} = 2(s-a)\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= \frac{2\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{s}}$$

Similarly the other two proposed expressions reduce to the same symmetrical form

26 $$b\cos B+c\cos C=\frac{a\sin B}{\sin A}\cos B+\frac{a\sin C}{\sin A}\cos C$$
$$=\frac{a}{2\sin A}(\sin 2B+\sin 2C)$$
$$=\frac{2a\sin(B+C)\cos(B-C)}{2\sin(B+C)}=a\cos(B-C).$$

27 By Art 216
$$c\cos B+b\cos C=a,\quad a\cos C+c\cos A=b,\quad b\cos A+a\cos B=c,$$
therefore by addition
$$c(\cos B+\cos A)+b(\cos A+\cos C)+a(\cos C+\cos B)=a+b+c$$

28 Let k stand for $\frac{a}{\sin A}$, $\frac{b}{\sin B}$, and $\frac{c}{\sin C}$ which we know are all equal Then
$$(a^2-b^2)\cot C+(b^2-c^2)\cot A+(c^2-a^2)\cot B$$
$$=k^2\{(\sin^2 A-\sin^2 B)\cot C+(\sin^2 B-\sin^2 C)\cot A+(\sin^2 C-\sin^2 A)\cot B\}$$
$$=k^2\{\sin(A+B)\sin(A-B)\cot C+\sin(B+C)\sin(B-C)\cot A+\sin(C+A)\sin(C-A)\cot B\}$$
$$=k^2\{\sin(A-B)\cos C+\sin(B-C)\cos A+\sin(C-A)\cos B\}$$
$$=-k^2\{\sin(A-B)\cos(A+B)+\sin(B-C)\cos(B+C)+\sin(C-A)\cos(C+A)\}$$
$$=-\frac{k^2}{2}\{\sin 2A-\sin 2B+\sin 2B-\sin 2C+\sin 2C-\sin 2A\}$$
$$=0.$$

29 Let k have the same meaning as in the preceding solution; then
$$(a-b)\cot\frac{C}{2}+(c-a)\cot\frac{B}{2}+(b-c)\cot\frac{A}{2}$$
$$=k\left\{(\sin A-\sin B)\cot\frac{C}{2}+(\sin C-\sin A)\cot\frac{B}{2}+(\sin B-\sin C)\cot\frac{A}{2}\right\}$$
$$=2k\left\{\sin\frac{A-B}{2}\sin\frac{A+B}{2}+\sin\frac{C-A}{2}\sin\frac{C+A}{2}+\sin\frac{B-C}{2}\sin\frac{B+C}{2}\right\}$$
$$=2k\left\{\sin^2\frac{A}{2}-\sin^2\frac{B}{2}+\sin^2\frac{C}{2}-\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}\right\}$$
$$=0$$

30 $$1-\tan\frac{A}{2}\tan\frac{B}{2}=1-\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\times\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$
$$=1-\frac{s-c}{s}=1-\frac{a+b-c}{a+b+c}=\frac{2c}{a+b+c}$$

31 $(a+b+c)(\cos A+\cos B+\cos C)$

$$=a\cos A+b\cos B+c\cos C$$
$$+a\cos B+b\cos A+a\cos C+c\cos A+b\cos C+c\cos B$$
$$=a\cos A+b\cos B+c\cos C+c+b+a, \text{ by Art 216,}$$
$$=a(1+\cos A)+b(1+\cos B)+c(1+\cos C)$$
$$=2a\cos^2\frac{A}{2}+2b\cos^2\frac{B}{2}+2c\cos^2\frac{C}{2}$$

32 Let k have the same meaning as in the solution of Example 28, then

$$\frac{\cos A\cos B}{ab}+\frac{\cos A\cos C}{ac}+\frac{\cos B\cos C}{bc}$$
$$=\frac{1}{k^2}\left\{\frac{\cos A\cos B}{\sin A\sin B}+\frac{\cos A\cos C}{\sin A\sin C}+\frac{\cos B\cos C}{\sin B\sin C}\right\}$$
$$=\frac{1}{k^2}\{\cot A\cot B+\cot A\cot C+\cot B\cot C\}$$
$$=\frac{1}{k^2}\frac{\tan A+\tan B+\tan C}{\tan A\tan B\tan C}=\frac{1}{k^2}, \text{ by Art 114,}$$
$$=\frac{\sin^2 A}{a^2}.$$

33 $a\cos A+b\cos B+c\cos C=a\cos A+\dfrac{a\sin B}{\sin A}\cos B+\dfrac{a\sin C}{\sin A}\cos C$

$$=a\cos A+\frac{a(\sin 2B+\sin 2C)}{2\sin A}=a\cos A+\frac{2a\sin(B+C)\cos(B-C)}{2\sin A}$$
$$=a\cos A+a\cos(B-C)=-a\cos(B+C)+a\cos(B-C)$$
$$=2a\sin B\sin C$$

34 $\dfrac{2a\sin B\sin C}{a+b+c}=\dfrac{2\sin B\sin C}{1+\dfrac{b}{a}+\dfrac{c}{a}}=\dfrac{2\sin B\sin C}{1+\dfrac{\sin B}{\sin A}+\dfrac{\sin C}{\sin A}}$

$$=\frac{2\sin A\sin B\sin C}{\sin A+\sin B+\sin C}=\frac{2\sin A\sin B\sin C}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}, \text{ by Example VIII 16,}$$
$$=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\cos A+\cos B+\cos C-1, \text{ by Art. 114,}$$

therefore $\qquad \cos A+\cos B+\cos C=1+\dfrac{2a\sin B\sin C}{a+b+c}.$

35 $$a^2 - 2ab\cos(60^\circ + C) = a^2 - 2ab(\cos 60^\circ \cos C - \sin 60^\circ \sin C)$$
$$= a^2 - ab\cos C + 2ab \sin 60^\circ \sin C$$
$$= a^2 - \frac{a^2+b^2-c^2}{2} + 2cb\sin 60^\circ \sin A$$
$$= c^2 - \frac{c^2+b^2-a^2}{2} + 2bc \sin 60^\circ \sin A$$
$$= c^2 - bc\cos A + 2bc\sin 60^\circ \sin A$$
$$= c^2 - 2bc\cos(60^\circ + A)$$

36 $$\frac{b+c-a}{2a} = \frac{\sin B + \sin C - \sin A}{2\sin A}$$
$$= \frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2} - 2\sin\frac{A}{2}\cos\frac{A}{2}}{4\sin\frac{A}{2}\cos\frac{A}{2}}$$
$$= \frac{\cos\frac{B-C}{2} - \sin\frac{A}{2}}{2\sin\frac{A}{2}} = \frac{\cos\frac{B-C}{2} - \cos\frac{B+C}{2}}{2\sin\frac{A}{2}} = \frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}$$

Again $$\cot\frac{A}{4} - \operatorname{cosec}\frac{A}{2} = \frac{\cos\frac{A}{4}}{\sin\frac{A}{4}} - \frac{1}{\sin\frac{A}{2}} = \frac{2\cos^2\frac{A}{4} - 1}{\sin\frac{A}{2}} = \frac{\cos\frac{A}{2}}{\sin\frac{A}{2}},$$

and $$\cot\frac{B}{2} + \cot\frac{C}{2} = \frac{\cos\frac{B}{2}}{\sin\frac{B}{2}} + \frac{\cos\frac{C}{2}}{\sin\frac{C}{2}} = \frac{\sin\frac{B+C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}},$$

therefore $$\frac{\cot\frac{A}{4} - \operatorname{cosec}\frac{A}{2}}{\cot\frac{B}{2} + \cot\frac{C}{2}} = \frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}} = \frac{b+c-a}{2a}$$

37. $4\Sigma\left(\Sigma - \cos\frac{A}{2}\right)\left(\Sigma - \cos\frac{B}{2}\right)\left(\Sigma - \cos\frac{A}{2}\right) =$ the product of
$$\frac{1}{4}\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right)\left(\cos\frac{B}{2} + \cos\frac{C}{2} - \cos\frac{A}{2}\right)$$

into $$\left(\cos\frac{A}{2} + \cos\frac{C}{2} - \cos\frac{B}{2}\right)\left(\cos\frac{A}{2} + \cos\frac{B}{2} - \cos\frac{C}{2}\right).$$

Now substitute for the trinomial expressions results given by Examples VIII 20 and 21, thus we obtain

$$\left\{8\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi+C}{4}\right\}^2,$$

that is $$\left\{8\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}\right\}^2,$$

that is $$\left\{\sin\frac{\pi-A}{2}\sin\frac{\pi-B}{2}\sin\frac{\pi-C}{2}\right\}^2,$$

that is $$\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}$$

38 The perimeter$=a+b+c=\dfrac{c\sin A}{\sin C}+\dfrac{c\sin B}{\sin C}+c$

$$=\frac{c(\sin A+\sin B+\sin C)}{\sin C}=\frac{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\sin C}, \text{ by Example VIII 16,}$$

$$=\frac{2\cos\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{C}{2}}=\frac{2\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{A+B}{2}}$$

$$=2\cos\frac{A}{2}\cos\frac{B}{2}\sec\frac{A+B}{2}.$$

39 Let $h=y\sin^2 A+x\sin^2 B=z\sin^2 B+y\sin^2 C=x\sin^2 C+z\sin^2 A.$

Thus $h(\sin^2 C-\sin^2 A)=x\sin^2 B\sin^2 C-z\sin^2 A\sin^2 B,$

and $h=x\sin^2 C+z\sin^2 A,$

therefore $h(\sin^2 C-\sin^2 A)+h\sin^2 B=2x\sin^2 B\sin^2 C,$

therefore $h\sin(C-A)\sin(C+A)+h\sin^2 B=2x\sin^2 B\sin^2 C,$

therefore $h\sin(C-A)+h\sin(C+A)=2x\sin B\sin^2 C,$

therefore $x\sin B\sin^2 C=h\sin C\cos A,$

therefore $x=\dfrac{h\cos A}{\sin B\sin C}=\dfrac{h\sin 2A}{2\sin A\sin B\sin C}.$

Similarly $y=\dfrac{h\sin 2B}{2\sin A\sin B\sin C}$, and $z=\dfrac{h\sin 2C}{2\sin A\sin B\sin C}.$

40 Since $A+B+C=\pi$, we may shew that $8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ has its greatest value when A, B, and C are all equal

For $$\sin\frac{A}{2}\sin\frac{B}{2}=\sin\left(\frac{A+B}{4}+\frac{A-B}{4}\right)\sin\left(\frac{A+B}{4}-\frac{A-B}{4}\right)$$
$$=\sin^2\frac{A+B}{4}-\sin^2\frac{A-B}{4},$$

thus, whatever may be the value of C, it follows that $\sin\frac{A}{2}\sin\frac{B}{2}$ has its greatest value when $A=B$, for $\sin\frac{A+B}{4}$ does not change while A and B change in such a manner as to leave C unchanged In this way we see that the greatest value of the expression is when all the angles are equal, and the value then is $8\sin^3\frac{\pi}{6}$, that is 1

41 Let k have the same meaning as in the solution of Example 28, then

$$a\sin(B-C)\cos(B+C-A)=k\sin A\sin(B-C)\cos(180^\circ-2A)$$
$$=-k\sin(B+C)\sin(B-C)\cos 2A=k(\sin^2 B-\sin^2 C)(2\sin^2 A-1)$$
$$=2k\sin^2 A(\sin^2 B-\sin^2 C)-k(\sin^2 B-\sin^2 C)$$

Similarly the other two terms of the proposed expression may be transformed, and then the whole vanishes because

$$\sin^2 A(\sin^2 B-\sin^2 C)+\sin^2 B(\sin^2 C-\sin^2 A)+\sin^2 C(\sin^2 A-\sin^2 B)=0,$$

and $$\sin^2 B-\sin^2 C+\sin^2 C-\sin^2 A+\sin^2 A-\sin^2 B=0$$

42 $$\frac{\sin A}{\cos B}+\frac{\sin B}{\cos A}=\frac{\sin A\cos A+\sin B\cos B}{\cos A\cos B}=\frac{\sin 2A+\sin 2B}{2\cos A\cos B}$$
$$=\frac{2\sin(A+B)\cos(A-B)}{2\cos A\cos B}=\frac{\sin C}{\cos A\cos B}(\cos A\cos B+\sin A\sin B)$$
$$=\sin C+\cos C\tan A\tan B\tan C$$

Similarly $$\frac{\sin B}{\cos C}+\frac{\sin C}{\cos B}=\sin A+\cos A\tan A\tan B\tan C,$$

and $$\frac{\sin C}{\cos A}+\frac{\sin A}{\cos C}=\sin B+\cos B\tan A\tan B\tan C$$

Hence by addition we obtain the required result.

CHAPTER XIV. P 150

1 $\sin A=\frac{a}{b}\sin B=\frac{5}{2\,5}\,25=\frac{1}{2}$, therefore $A=30^\circ$ or 150°.

2 Suppose $c=\frac{1}{2}b$, and $A=60^\circ$, then, by Art 229,

$$\tan\frac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{1-\frac{1}{2}}{1+\frac{1}{2}}\cot 30^\circ=\frac{1}{3}\sqrt{3}=\frac{1}{\sqrt{3}};$$

therefore $\frac{1}{2}(B-C)=30^\circ$, and $\frac{1}{2}(B+C)=60^\circ$.

Hence $B=90^\circ$ and $C=30^\circ$.

3 Let a, b, c denote these sides in order Then

$$\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{6+(1+\sqrt{3})^2-4}{2(1+\sqrt{3})\sqrt{6}}=\frac{6+2\sqrt{3}}{2(1+\sqrt{3})\sqrt{6}}$$

$$=\frac{\sqrt{3}(1+\sqrt{3})}{(1+\sqrt{3})\sqrt{6}}=\frac{1}{\sqrt{2}},\quad \text{therefore } A=45^\circ$$

$$\cos B=\frac{a^2+c^2-b^2}{2ac}=\frac{4+(1+\sqrt{3})^2-6}{4(1+\sqrt{3})}=\frac{2+2\sqrt{3}}{4(1+\sqrt{3})}=\frac{1}{2},$$

therefore $B=60^\circ$

$$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{4+6-(1+\sqrt{3})^2}{4\sqrt{6}}=\frac{6-2\sqrt{3}}{4\sqrt{6}}=\frac{3-\sqrt{3}}{2\sqrt{6}}$$

$$=\frac{\sqrt{3}-1}{2\sqrt{2}},\quad \text{therefore } C=75^\circ$$

4. $\sin B=\frac{b}{a}\sin A=\frac{100}{40}\cdot\frac{1}{2}=\frac{5}{4}$, but this is impossible, for a sine cannot be greater than unity

5 $$\sin B=\frac{b}{a}\sin A=\frac{4+\sqrt{(80)}}{4}\sin 18^\circ=(1+\sqrt{5})\sin 18^\circ$$

$$=\frac{(1+\sqrt{5})(\sqrt{5}-1)}{4}=1,\quad \text{therefore } B=90^\circ.$$

Thus $C=72^\circ$, and $c^2=b^2-a^2=\{4+\sqrt{(80)}\}^2-16$

$$=80+8\sqrt{(80)}=16(5+2\sqrt{5}),$$

therefore $c=4\sqrt{(5+2\sqrt{5})}$.

6 $$\sin B=\frac{b}{a}\sin A=\frac{4+\sqrt{(48)}}{4}\sin 15^{0}=(1+\sqrt{3})\frac{\sqrt{3}-1}{2\sqrt{2}}=\frac{1}{\sqrt{2}};$$

therefore $$B=45^{0} \text{ or } 135^{0}$$

If $B=45^{0}$, then $C=120^{0}$, and $c=\frac{a\sin C}{\sin A}=4\ \frac{2\sqrt{2}}{\sqrt{3}-1}\cdot\frac{\sqrt{3}}{2}$

$$=\frac{4\sqrt{6}}{\sqrt{3}-1}=\frac{4\sqrt{6}(\sqrt{3}+1)}{2}=2\sqrt{6}(\sqrt{3}+1)$$

If $B=135^{0}$, then $C=30^{0}$, and $c=\frac{a\sin C}{\sin A}=4\ \frac{2\sqrt{2}}{\sqrt{3}-1}\ \frac{1}{2}$

$$=\frac{4\sqrt{2}}{\sqrt{3}-1}=\frac{4\sqrt{2}(\sqrt{3}+1)}{2}=2\sqrt{2}(\sqrt{3}+1)$$

7 With the first diagram of Art 234 we may put $c=AB$ and $c'=AB'$; thus

$$c=b\cos A-a\cos CBB', \text{ and } c'=b\cos A+a\cos CBB';$$

therefore $$c+c'=2b\cos A,$$

and $$cc'=b^2\cos^2 A-a^2\cos^2 CBB'=b^2\cos^2 A-a^2\cos^2 B$$
$$=b^2(1-\sin^2 A)-a^2(1-\sin^2 B)=b^2-a^2$$

Hence $$(c+c')^2=4b^2\cos^2 A,$$
$$4cc'\cos^2 A=4(b^2-a^2)\cos^2 A,$$

therefore $$c^2+2cc'+c'^2-4cc'\cos^2 A=4a^2\cos^2 A,$$

that is $$c^2-2cc'\cos 2A+c'^2=4a^2\cos^2 A$$

8 With the notation of the preceding solution the area of the smaller triangle is $\frac{c}{2}b\sin A$, and the area of the larger triangle is $\frac{c'}{2}b\sin A$, hence the sum of the areas $=\frac{1}{2}(c+c')b\sin A=b^2\sin A\cos A$

9 With the notation of the two preceding solutions we have

$$\frac{\sin C_1}{\sin B_1}=\frac{c}{b} \text{ and } \frac{\sin C_2}{\sin B_2}=\frac{c'}{b};$$

therefore $$\frac{\sin C_1}{\sin B_1}+\frac{\sin C_2}{\sin B_2}=\frac{c+c'}{b}=\frac{2b\cos A}{b}=2\cos A$$

10 As in the solution of Example 8, we have

$$\frac{1}{2}c'b\sin A=\frac{n}{2}cb\sin A,$$

therefore $c'=nc$.

And as in the solution of Example 7,

$$\frac{c'+c}{c'-c}=\frac{2b\cos A}{2a\cos CBB'},$$

therefore $$\frac{b}{a}=\frac{n+1}{n-1}\,\frac{\cos CBB'}{\cos A},$$

but the angle CBB' is greater than A, and therefore $\frac{\cos CBB'}{\cos A}$ is less than unity Hence $\frac{b}{a}$ is less than $\frac{n+1}{n-1}$

11 $\sin B=\frac{b}{a}\sin A$, therefore $L\sin B-10=\log b+L\sin A-10-\log a$

Thus if $\log a+10=\log b+L\sin A$ we have $L\sin B-10=0$, therefore $L\sin B=10$, therefore $\log\sin B=0$, therefore $\sin B=1$, therefore $B=90^\circ$, and the triangle is not ambiguous

12 $$\frac{a+b}{c}=\frac{\sin A+\sin B}{\sin C}=\frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\sin\frac{C}{2}\cos\frac{C}{2}}$$

$$=\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C},$$

therefore $$\cos\frac{1}{2}(A-B)=\frac{(a+b)\sin\frac{C}{2}}{c}$$

Now assume $\cos\theta=\frac{a-b}{c}$, therefore

$$\sin^2\theta=\frac{c^2-(a-b)^2}{c^2}=\frac{a^2+b^2-2ab\cos C-(a-b)^2}{c^2}$$

$$=\frac{2ab(1-\cos C)}{c^2}=\frac{4ab}{c^2}\sin^2\frac{C}{2},$$

therefore $$\sin\theta=\frac{2\sqrt{(ab)}}{c}\sin\frac{C}{2},$$

therefore $$\cos\frac{1}{2}(A-B)=\frac{(a+b)\sin\theta}{2\sqrt{(ab)}}$$

And $$\sin\theta=\frac{2\sqrt{(ab)}}{c}\sin\frac{C}{2}=\frac{2\sqrt{(ab)}}{c}\cos\frac{1}{2}(A+B),$$

therefore $$\cos\frac{1}{2}(A+B)=\frac{c\sin\theta}{2\sqrt{(ab)}}$$

13 $$c^2=a^2+b^2-2ab\cos C=a^2+b^2-2ab\left(1-2\sin^2\frac{C}{2}\right)$$
$$=(a-b)^2+4ab\sin^2\frac{C}{2}=(a-b)^2+(a-b)^2\tan^2\phi$$
$$=(a-b)^2\{1+\tan^2\phi\}=(a-b)^2\sec^2\phi$$

14 Here $s=30,\quad s-a=12,\quad s-b=10,\quad s-c=8$
$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\sqrt{\frac{10\times 8}{30\times 12}}=\sqrt{\frac{8}{36}}=\sqrt{\frac{2}{9}},$$

therefore $L\tan\frac{A}{2}=10+\log\sqrt{\frac{2}{9}}=10+\frac{1}{2}\log 2-\log 3=9\ 6733937$

15 The greatest angle is opposite to the side 66, denote this angle by C Then
$$\cot\frac{C}{2}=\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}.$$

Here $s=69,\quad s-a=37,\quad s-b=29,\quad s-c=3,$

therefore $$\cot\frac{C}{2}=\sqrt{\frac{69\times 3}{37\times 29}}=\sqrt{\frac{207}{1073}};$$

therefore $$L\cot\frac{C}{2}=10+\log\sqrt{\frac{207}{1073}}$$
$$=10+\frac{1}{2}(\log 207-\log 1073)=9\ 6426853.$$

$$\begin{array}{r}9\ 6426853\\9\ 6424342\\\hline 0002511\end{array}\qquad 0003431\qquad 0002511\qquad 60'':x'',$$

this gives $x=44$, therefore $\frac{C}{2}=66^\circ\,18'-44''=66^\circ\,17'\,16''$,

therefore $$C=132^\circ\,34'\,32''$$

16. Here $s=\frac{15}{2},\quad s-a=\frac{7}{2},\quad s-b=\frac{5}{2},\quad s-c=\frac{3}{2}$
$$\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ac}}=\sqrt{\frac{15\times 5}{8\times 12}}=\sqrt{\frac{25}{32}}=\sqrt{\frac{100}{2^7}};$$

therefore $L\cos\frac{B}{2}=10+\log\sqrt{\frac{100}{2^7}}=10+\frac{1}{2}(\log 100-\log 2^7)$
$$=10+1-\frac{7}{2}\log 2=9\ 9463950$$

$$\begin{array}{r} 9\ 9464040 \\ 9\ 9463950 \\ \hline 0000090 \end{array} \qquad 0000669 \qquad 0000090 \qquad 60'' \,.\, x'',$$

this gives $x=8$; therefore $\frac{B}{2}=27^0\,53'\,8''$, therefore $B=55^0\,46'\,16''$.

17 Here $a=7$, $s=9$, $s-a=2$, therefore

$$\cos\frac{A}{2}=\sqrt{\frac{9\times 2}{5\times 6}}=\sqrt{\frac{3}{5}}=\sqrt{\frac{6}{10}},$$

therefore $$L\cos\frac{A}{2}=10+\log\sqrt{\frac{6}{10}}=10+\frac{1}{2}(\log 6-\log 10)$$

$$=10+\frac{1}{2}\log 6-\frac{1}{2}=9\ 8890756$$

$$\begin{array}{r} 9\ 8890756 \\ 9\ 8890644 \\ \hline \cdot 0000112 \end{array} \qquad \cdot 0001032 \qquad \cdot 0000112 \qquad 60'' : x'',$$

this gives $x=6\ 5$; therefore $\frac{A}{2}=39^0\,14'-6''\,5=39^0\,13'\,53''\,5$, therefore $A=78^0\,27'\,47''$.

18 As in Art 229 we have

$$\tan\frac{1}{2}(B-C)=\frac{18-2}{18+2}\cot\frac{A}{2}=\frac{8}{10}\cot 27^0\,30',$$

therefore $$L\tan\frac{1}{2}(B-C)=L\cot 27^0\,30'+\log 8-\log 10$$

$$=L\cot 27^0\,30'+3\log 2-1=10\ 1866133$$

$$\begin{array}{r} 10\ 1866133 \\ 10\ 1863769 \\ \hline \cdot 0002364 \end{array} \qquad \cdot 0002763 \qquad 0002364 \qquad 60'' \quad x'',$$

this gives $x=51$, therefore $\frac{1}{2}(B-C)=56^0\,56'\,51''$

And $\frac{1}{2}(B+C)=62^0\,30'$; therefore $B=119^0\,26'\,51''$, $C=5^0\,33'\,9''$.

19 $$\tan\frac{1}{2}(B-C)=\frac{9-7}{9+7}\cot\frac{A}{2}=\frac{1}{8}\cot 32^0\,6',$$

therefore $$L\tan\frac{1}{2}(B-C)=L\cot 32^0\,6'-\log 8$$

$$=L\tan 57^0\,54'-3\log 2=9\ 2994355.$$

$$\begin{array}{r} 9\ 2999804 \\ 9\ 2993216 \\ \hline 0006588 \end{array} \qquad \begin{array}{r} 9\ 2994355 \\ 9\ 2993216 \\ \hline 0001139 \end{array} \qquad 0006588 \,.\, 0001139 \quad 60'' \quad x'',$$

this gives $x=10$, therefore $\frac{1}{2}(B-C)=11^\circ\,16'\,10''$

And $\frac{1}{2}(B+C)=57^\circ\,54'$, therefore $B=69^\circ\,10'\,10''$, $C=46^\circ\,37'\,50''$

20 $\quad \tan\frac{1}{2}(A-B)=\frac{a-b}{a+b}\cot\frac{C}{2}=\frac{70-35}{70+35}\cot\frac{C}{2}=\frac{1}{3}\cot 18^\circ\,26'\,6''$,

therefore $\quad L\tan\frac{1}{2}(A-B)=L\cot 18^\circ\,26'\,6''-\log 3=10$,

therefore $\quad \log\tan\frac{1}{2}(A-B)=0$,

therefore $\quad \tan\frac{1}{2}(A-B)=1$, $\quad$ therefore $\frac{1}{2}(A-B)=45^\circ$

And $\frac{1}{2}(A+B)=71^\circ\,33'\,54''$, therefore $A=116^\circ\,33'\,54''$, $B=26^\circ\,33'\,54''$

21 $\quad \tan\frac{1}{2}(B-C)=\frac{9-7}{9+7}\cot\frac{A}{2}=\frac{1}{8}\cot 23^\circ\,42'\,30''$,

therefore $\quad L\tan\frac{1}{2}(B-C)=L\cot 23^\circ\,42'\,30''-\log 8$

$$=L\tan 66^\circ\,17'\,30''-3\log 2=9\ 4543042$$

$$\begin{array}{r} 9\ 4543042 \\ 9\ 4541479 \\ \hline 0001563 \end{array} \qquad 0004797 \quad 0001563 \quad 60'' \quad x'',$$

this gives $x=20''$, therefore $\frac{1}{2}(B-C)=15^\circ\,53'\,20''$

And $\frac{1}{2}(B+C)=66^\circ\,17'\,30''$, therefore $B=82^\circ\,10'\,50''$, $C=50^\circ\,24'\,10''$

22 $\quad \tan\frac{1}{2}(A-B)=\frac{a-b}{a+b}\cot\frac{C}{2}=\frac{30-20}{30+20}\cot\frac{C}{2}=\frac{2}{10}\cot 11^\circ$,

therefore $\quad L\tan\frac{1}{2}(A-B)=L\cot 11^\circ+\log 2-\log 10$

$$=L\cot 11^\circ+\log 2-1=10\ 0123777$$

$$\begin{array}{r} 10\ 0123821 \\ 10\ 0121294 \\ \hline 0002527 \end{array} \qquad \begin{array}{r} 10\ 0123777 \\ 10\ 0121294 \\ \hline 0002483 \end{array} \qquad 0002527 \quad 0002483 \quad 60'' \,.\, x'',$$

this gives $x=59$, therefore $\frac{1}{2}(A-B)=45^\circ\,48'\,59''$

And $\frac{1}{2}(A+B)=79^\circ$, therefore $A=124^\circ\,48'\,59''$, $B=33^\circ\,11'\,1''$.

23 $$\tan\frac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{3}{25}\cot 30^\circ=\frac{3\sqrt{3}}{25},$$

therefore $$L\tan\frac{1}{2}(B-C)=10+\log\frac{3\sqrt{3}}{25}=10+\frac{3}{2}\log 3-\log 25$$

$$=10+\frac{3}{2}\log 3-\log\frac{100}{4}=10+\frac{3}{2}\log 3-2+2\log 2=9{\cdot}31774,$$

therefore $$\frac{1}{2}(B-C)=11^\circ\,44'\,29''$$

And $\frac{1}{2}(B+C)=60^\circ$, therefore $B=71^\circ\,44'\,29''$

24 Let $a=7$, $b=8$, $c=9$; then $s=12$, $s-a=5$, $s-b=4$, $s-c=3$

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\sqrt{\frac{4\times 3}{12\times 5}}=\sqrt{\frac{1}{5}}=\sqrt{\frac{2}{10}}$$

$$L\tan\frac{A}{2}=10+\log\sqrt{\frac{2}{10}}=10+\frac{1}{2}(\log 2-\log 10)$$

$$=10+\frac{1}{2}(\log 2-1)=9\ 6505150$$

$$\begin{array}{r}9\ 6505634\\ 9\ 6505069\\ \hline 0000565\end{array}\qquad\begin{array}{r}9\ 6505150\\ 9\ 6505069\\ \hline 0000081\end{array}\qquad 0000565 \quad 0000081 \quad 10'' \ . \ x'',$$

this gives $x=1\ 5$, therefore $\frac{A}{2}=24^\circ\,5'\,41''\,5$, therefore $A=48^\circ\,11'\,23''$

$$\tan\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}=\sqrt{\frac{5\times 3}{12\times 4}}=\sqrt{\frac{5}{16}}=\sqrt{\frac{10}{32}}$$

$$L\tan\frac{B}{2}=10+\log\sqrt{\frac{10}{32}}=10+\frac{1}{2}(\log 10-\log 32)$$

$$=10+\frac{1}{2}-\frac{5}{2}\log 2=9\ 7474250$$

$$\begin{array}{r}9{\cdot}7474677\\ 9\ 7474183\\ \hline 0000494\end{array}\qquad\begin{array}{r}9\ 7474250\\ 9\ 7474183\\ \hline 0000067\end{array}\qquad 0000494 \cdot 0000067 \cdot 10'' \quad x'',$$

this gives $x=1\ 5$, therefore $\frac{B}{2}=29^\circ\,12'\,21''\,5$; therefore $B=58^\circ\,24'\,43''$

Hence $$C=180^\circ-48^\circ\,11'\,23''-58^\circ\,24'\,43''=73^\circ\,23'\,54''$$

25 As in Art 238 we have

$$\sin\left(45^0 - \frac{B}{2}\right) = \sqrt{\frac{1-\sin B}{2}} = \sqrt{\frac{1}{2}\left(1-\frac{3}{6953}\right)}$$

$$= \sqrt{\frac{1}{2} \times \frac{6950}{6953}} = \sqrt{\frac{3475}{6953}},$$

therefore $$L\sin\left(45^0 - \frac{B}{2}\right) = 10 + \log\sqrt{\frac{3475}{6953}}$$

$$= 10 + \frac{1}{2}\log(3475 - \log 6953)$$

$$= 10 - \frac{1}{2}(\ 3012174) = 9\ 8493913$$

$$\begin{array}{r} 9\ 8493913 \\ 9\ 8493902 \\ \hline 0000011 \end{array} \qquad 0000021 \cdot 0000011 \quad 1'' \quad x'',$$

this gives $x = 5$, therefore $45^0 - \frac{B}{2} = 44^0\,59'\,15''\,5$, therefore $\frac{B}{2} = 44''\cdot 5$, therefore $B = 1'29''$

26 Let $b=100,\ c=80,$

$$\tan\frac{1}{2}(B-C) = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{1}{9}\cot 30^0 = \frac{\sqrt{3}}{9} = 3^{-\frac{3}{2}},$$

therefore $$L\tan\frac{1}{2}(B-C) = 10 + \log 3^{-\frac{3}{2}} = 10 - \frac{3}{2}\log 3 = 9\ 28432,$$

therefore $$\frac{1}{2}(B-C) = 10^0\,53'\,36''.$$

And $\frac{1}{2}(B+C) = 60^0$, therefore $B = 70^0\,53'\,36''$, $C = 49^0\,6'\,24''$

27 Let $b=5,\ c=3,$

$$\tan\frac{1}{2}(B-C) = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{1}{4}\cot 60^0 = \frac{1}{4\sqrt{3}} = \frac{1}{\sqrt{(48)}},$$

therefore $$L\tan\frac{1}{2}(B-C) = 10 + \log\frac{1}{\sqrt{(48)}} = 10 - \frac{1}{2}\log 48$$

$$= 10 - \frac{1}{2}(1\ 6812412) = 9\ 1593794$$

$$\begin{array}{r} 9\ 1593794 \\ 9\ 1586706 \\ \hline 0007088 \end{array} \qquad 0008940 \quad 0007088 \quad 60'' \quad x'',$$

this gives $x = 48$, therefore $\frac{1}{2}(B-C) = 8^0\,12'\,48''$.

And $\frac{1}{2}(B+C)=30^\circ$, therefore $B=38^\circ\,12'\,48''$ and $C=21^\circ\,47'\,12''$

28 Let $ABCD$ denote the square base, P the vertex From P suppose a perpendicular PQ drawn to the ground, and from Q draw QR perpendicular to AB Let ϕ denote the required inclination, then $\tan\phi=\frac{PQ}{QR}$.

Now $QR=100$ Also $PQ^2+QR^2=PR^2$, and $PR^2+AR^2=AP^2$; thus

$$PQ^2=PR^2-QR^2=AP^2-AR^2-QR^2=(150)^2-(100)^2-(100)^2=2500,$$

therefore $PQ=50$ Therefore $\tan\phi=\frac{50}{100}=\frac{1}{2}$

Hence $$L\tan\phi=10+\log\frac{1}{2}=10-\log 2=9\ 69897$$

$$\begin{array}{r} 9\ 69900 \\ 9\ 69868 \\ \hline 00032 \end{array} \qquad \begin{array}{r} 9\ 69897 \\ 9\ 69868 \\ \hline 00029 \end{array} \qquad 00032 \quad 00029 \quad 60'' \quad x'',$$

this gives $x=54$, therefore $\phi=26^\circ\,33'\,54''$.

29 $\tan\frac{1}{2}(A-B)=\frac{a-b}{a+b}\cot\frac{C}{2}=\frac{1\frac{2}{9}-1}{1\frac{2}{9}+1}\cot 30^\circ=\frac{1}{10}\cot 30^\circ=\frac{\sqrt{3}}{10}$,

therefore $L\tan\frac{1}{2}(A-B)=10+\log\frac{\sqrt{3}}{10}=10+\frac{1}{2}\log 3-1=9\ 2385606$

Now $L\cot 9^\circ\,49'=10\ 7618797$, and as $\tan\theta\times\cot\theta=1$, we have

$$\log\tan\theta+\log\cot\theta=0,$$

therefore $$L\tan\theta-10+L\cot\theta-10=0,$$

therefore $$L\tan\theta=20-L\cot\theta$$

Thus $$L\tan 9^\circ\,49'=9\ 2381203$$

$$\begin{array}{r} 9\ 2385606 \\ 9\ 2381203 \\ \hline 0004403 \end{array} \qquad 0007514 \quad 0004403 \cdot 60'' \quad x'',$$

this gives $x=35$, therefore $\frac{1}{2}(A-B)=9^\circ\,49'\,35''$

And $\frac{1}{2}(A+B)=60^\circ$, therefore $A=69^\circ\,49'\,35''$, $B=50^\circ\,10'\,25''$.

30 $\sin C=\frac{c}{a}\sin A$, $L\sin C=L\sin A+\log c-\log a$

$$=L\sin A+\log 3-\log 2$$

$$=9\ 5228787+\ 4771213-\log 2=10-\log 2;$$

therefore $$\log \sin C = -\log 2 = \log \frac{1}{2},$$

therefore $\sin C = \frac{1}{2}$, therefore $C = 30^\circ$ or 150°

31. Let c be the given base and let h denote the given height. With the left-hand diagram of Art. 214 we have

$$\cot B = \frac{BD}{h} \quad \text{and} \quad \cot C = \frac{CD}{h},$$

therefore $$\cot B + \cot C = \frac{BD + CD}{h} = \frac{c}{h} \qquad (1)$$

Also $B - C$ is supposed given, so that $\cot(B - C)$ is known; call it m; thus

$$\frac{\cot B - \cot C}{1 + \cot B \cot C} = m \qquad (2)$$

From (1) and (2) we can find $\cot A$ and $\cot B$.

32. Let a, b, c denote the sides, and l, m, n the perpendiculars on them respectively from the opposite angles. Then $al = bm = cn$, for each of these expressions denotes twice the area of the triangle. Hence the sides a, b, c are respectively inversely proportional to l, m, n. Thus the *ratios* of the sides are known, and hence the angles of the triangle can be calculated by Art. 217. Then the actual lengths of the sides can be found, for $l = c \sin B$, and l and B are known, so that c can be found, and then a and b can be deduced as the ratios of the sides are already known.

33.
$$\begin{aligned} a &= 43257, & s - a &= 7151, \\ b &= 24692, & s - b &= 25716, \\ c &= 32867, & s - c &= 17541, \\ 2s &= 100816, & s &= 50408 \end{aligned}$$

$$\tan \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

$$L \tan \frac{1}{2}A = 10 + \frac{1}{2}\{\log(s-b) + \log(s-c) - \log(s-a) - \log s\}$$

$$\begin{aligned} \log(s-b) &= 4.4102034 \\ \log(s-c) &= 4.2440543 \\ -\log(s-a) &= -3.8543668 \\ -\log s &= -4.7024995 \\ \hline & .0973914 \end{aligned}$$

$$L \tan \frac{1}{2}A = 10.0486957, \qquad 60'' \times \frac{833}{2542} = 19.6''$$

$$L \tan 48^\circ 12' = 10.0486124,$$

$$\text{difference for } 60'' = .0002542,$$

$$\frac{1}{2}A = 48^\circ 12' 19.6'',$$

$$A = 96^\circ 24' 39.2'',$$

$$L\tan\frac{1}{2}C=10+\frac{1}{2}\{\log(s-b)+\log(s-a)-\log(s-c)-\log s\}$$

$$=10+\frac{1}{2}\{-2+1\ 3180164\}$$

$$=9\ 6590082$$

$L\tan 24^\circ 30'=9\ 6587041$,
difference for $60''=\ \ 0003346$, $\qquad 60''\times\frac{3041}{3346}=54\ 5''$.

$$\frac{1}{2}C=24^\circ 30' 54\ 5'',$$

$$C=49^\circ 1' 49'',$$

$$B=180^\circ-(A+C)=84^\circ 33' 32''$$

34 $\qquad a=279,\quad s-a=200,$
$b=366,\quad s-b=\ 93,$
$c=293,\quad s-c=186,$
$2s=958,\quad s=479$

$$L\tan\frac{1}{2}A=10+\frac{1}{2}\{\log(s-b)+\log(s-c)-\log s-\log(s-a)\}$$

$$\begin{aligned}\log(s-b)&=\ \ 1\ 9684829\\ \log(s-c)&=\ \ 2\ 2695129\\ -\log(s-a)&=-2\ 3010300\\ -\log s\quad&=-2\ 6803355\\ \hline &\ \ -2+1\ 2566303\end{aligned}$$

$L\tan\frac{1}{2}A=9{\cdot}6283152$, $\qquad 60''\times\frac{1121}{3509}=19\ 2''$
$L\tan 23^\circ 1'=9\ 6282031$,
difference for $60''=\ \ 0003509$.

$$\frac{1}{2}A=23^\circ 1' 19\ 2'',$$

$$A=46^\circ 2' 38\ 4''$$

$$L\tan\frac{1}{2}C=10+\frac{1}{2}\{\log(s-a)+\log(s-b)-\log(s-c)-\log s\}$$

$$=10+\frac{1}{2}\{-2+1\ 3196645\}$$

$$=9\ 6598323$$

$L\tan 24^\circ 33'=9\ 6597076$;
difference for $60''=\ \ 0003342$. $\qquad 60''\times\frac{1247}{3342}=22\ 4''$

$$\frac{1}{2}C=24^\circ 33' 22\ 4'',$$

$$C=49^\circ 6' 45'',$$

$$B=180^\circ-(A+C)=84^\circ 50' 37''$$

35

$$a = 49\ 23,\qquad s-a = 22\ 06,$$
$$b = 63\ 75,\qquad s-b = 7\ 54,$$
$$c = 29\ 6,\qquad s-c = 41\ 69,$$
$$2s = 142\ 58,\qquad s = 71\ 29$$

$$L\tan\frac{1}{2}A = 10 + \frac{1}{2}\{\log(s-b) + \log(s-c) - \log s - \log(s-a)\}$$

$$\begin{aligned} \log(s-b) &= 8773713 \\ \log(s-c) &= 1\ 6200319 \\ -\log(s-a) &= -1\ 3436055 \\ -\log s \quad &= -1\ 8530286 \\ \hline &-2+1\ 3007691 \end{aligned}$$

$$L\tan\frac{1}{2}A = 9{\cdot}6503846, \qquad 60''\times\frac{1037}{3390} = 18\ 3''$$

$$L\tan 24^\circ 5' = 9\ 6502809,$$

$$\text{difference for } 60'' = \ 0003390$$

$$\frac{1}{2}A = 24^\circ\, 5'\, 18\ 3'',$$

$$A = 48^\circ\, 10'\, 37''$$

$$L\tan\frac{1}{2}B = 10 + \frac{1}{2}\{\log(s-a) + \log(s-c) - \log s - \log(s-b)\}$$

$$= 10 + \frac{1}{2}(\ 2332375)$$

$$= 10\ 1166188$$

$$L\tan 52^\circ 36' = 10\ 1165897, \qquad 60''\times\frac{291}{2619} = 6\ 6''$$

$$\text{difference for } 60'' = \ 0002619$$

$$\frac{1}{2}B = 52^\circ\, 36'\, 6\ 6'',$$

$$B = 105^\circ\, 12'\, 13'',$$

$$C = 180^\circ - (A+B) = 26^\circ\, 37'\, 10''.$$

36

$$\tan\frac{1}{2}(B-C) = \frac{b-c}{b+c}\cot\frac{1}{2}A$$

$$= \frac{6\ 996}{119\ 562}\cot 23^\circ\, 14'\, 47\ 5''$$

$$\begin{aligned} L\cot\frac{1}{2}A &= \ 10\ 3672499 \\ &\ -\quad 0002758 \\ \log(b-c) &= 8448498 \\ -\log(b+c) &= -\ 2\ 0775932 \\ \hline L\tan\frac{1}{2}(B-C) &= \ 9\ 1342307 \end{aligned}$$

$$L\tan 7^\circ 45' = 9\ 1338391, \qquad 60''\times\frac{3916}{9444} = 24\ 9''$$

$$\text{difference for } 60'' = {\cdot}0009444$$

$$\frac{1}{2}(B-C)=\ 7^\circ\,45'\,24\cdot 9'',$$

$$\frac{1}{2}(B+C)=66^\circ\,45'\,12\cdot 5''$$

$$B=74^\circ\,30'\,37\cdot 4'',$$
$$C=58^\circ\,59'\,47\cdot 6'',$$

$$a=\frac{c\sin A}{\sin C}.$$

$$\log a=\log c+L\sin A-L\sin C;$$

$$\begin{aligned}
\log c &= 1\cdot 7503772\\
L\sin A &= 9\cdot 8601423\\
&\ +\ \ \cdot 0000699\\
-L\sin C &= -9\cdot 9329897\\
&\ -\ \ \cdot 0000602\\
\log a &= \overline{1\cdot 6778395}
\end{aligned}$$

$$a=47\cdot 6255$$

37 $$\tan\frac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\frac{1}{2}A=\frac{89}{565}\cot 54^\circ\,27'\,13'',$$

$$\begin{aligned}
L\cot\frac{1}{2}A &= 9\cdot 8540694\\
&\ -\ \ \cdot 0000579\\
\log(b-c) &= 1\cdot 9493900\\
-\log(b+c) &= -\underline{2\cdot 7520484}\\
L\tan\frac{1}{2}(B-C) &= 9\cdot 0513531
\end{aligned}$$

$$\begin{aligned}
L\tan 6^\circ\,25' &= 9\cdot 0510078,\\
\text{difference for } 60'' &= \ \ \cdot 0011361.
\end{aligned}\qquad 60''\times\frac{3453}{11361}=18\cdot 2''$$

$$\frac{1}{2}(B-C)=\ 6^\circ\,25'\,18\cdot 2'',$$

$$\frac{1}{2}(B+C)=35^\circ\,32'\,47'',$$

$$B=41^\circ\,58'\ \ 5\cdot 2'',$$
$$C=29^\circ\ \ 7'\,28\cdot 8'',$$

$$\log a=\log b+L\sin A-L\sin B,$$
$$L\sin 108^\circ\,54'\,26''=L\sin 71^\circ\,5'\,34''$$

$$\begin{aligned}
L\sin 71^\circ\,5'\,34'' &= 9\cdot 9758870\\
&\ +\ \ \cdot 0000245\\
\log b &= 2\cdot 5145478\\
-L\sin B &= -9\cdot 8252301\\
&\ -\ \ \cdot 0000122\\
\log a &= \overline{2\cdot 6652170}
\end{aligned}$$

$$\therefore\ a=462\cdot 612$$

38. $$\tan\frac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\frac{1}{2}A=\frac{31395}{156357}\cot 18^\circ 21' 37''$$

$$\begin{aligned}
L\cot\frac{1}{2}A &= 10\ 4792718 \\
&\ -\ 0002606 \\
\log(b-c) &= 4\ 4968605 \\
-\log(b+c) &= -\ 5\ 1941174 \\
\hline
L\tan\frac{1}{2}(B-C) &= 9\ 7817543
\end{aligned}$$

$$L\tan 31^\circ 10' = 9\ 7816309,$$

difference for $60'' = \ 0002853,$

$$\frac{1}{2}(B-C) = 31^\circ 10' 26'',$$

$$\frac{1}{2}(B+C) = 71^\circ 38' 23'',$$

$$B = 102^\circ 48' 49'',$$

$$C = 40^\circ 27' 57'',$$

$$\log a = \log c + L\sin A - L\sin C,$$

$$\begin{aligned}
L\sin A &= 9\ 7765983 \\
&\ +\ 0000395 \\
\log c &= 4\ 7957480 \\
-L\sin C &= -9\ 8121003 \\
&\ -\ 0001407 \\
\hline
\log a &= 4\ 7601448
\end{aligned}$$

$$a = 57563\ 2$$

39 $$A = 180^\circ - (B+C) = 67^\circ 59',$$

$$b = \frac{a\sin B}{\sin A},$$

$$\log b = \log a + L\sin B - L\sin A,$$

$$\begin{aligned}
\log a &= 2\ 1931246 \\
L\sin B &= 9\ 8050385 \\
-L\sin A &= -9\ 9671148 \\
\hline
\log b &= 2\ 0310483
\end{aligned}$$

$$b = 107\ 411$$

Again, $$\log c = \log a + L\sin C - L\sin A,$$

$$\begin{aligned}
\log a &= 2\ 1931246 \\
L\sin C &= 9\ 9790594 \\
-L\sin A &= -9\ 9671148 \\
\hline
\log c &= 2\ 2050692
\end{aligned}$$

$$c = 160\ 35$$

40
$$A=180^\circ-(B+C)=70^\circ\ 34'\ 31'',$$
$$\log c=\log a+L\sin C-L\sin A,$$
$$\begin{array}{rr}
\log a= & 6658529\\
L\sin C= & 9\ 9534134\\
+ & 0000576\\
-L\sin A= & -9\ 9745252\\
- & 0000230\\
\hline
\log c= & 6447757
\end{array}$$
$$c=4\ 41342$$
$$\log b=\log a+L\sin B-L\sin A,$$
$$\begin{array}{rr}
\log a= & 6658529\\
L\sin B= & 9\ 8529936\\
+ & 0000684\\
-L\sin A= & -9\ 9745482\\
\hline
\log b= & 5443667
\end{array}$$
$$\cdot\ b=3\ 50241$$

41
$$\sin B=\frac{b\sin A}{a},$$
$$L\sin B=L\sin A+\log b-\log a,$$
$$\begin{array}{rr}
L\sin A= & 9\ 9209393\\
\log b= & 1\ 4351433\\
-\log a= & -1\ 7199938\\
\hline
L\sin B= & 9\ 6360888\\
L\sin 25^\circ\ 38'= & 9\ 6360969,
\end{array}$$
$$\text{difference for } 1'=\quad 0002634; \qquad 60''\times\frac{81}{2634}=1\ 8''$$
$$B=25^\circ\ 37'\ 58\ 2'',$$
$$C=180^\circ-(A+B)=97^\circ\ 54'\ 1\ 8'',$$
$$\log c=\log a+L\sin C-L\sin A,$$
$$L\sin 97^\circ\ 54'\ 1{\cdot}8''=L\cos 7^\circ\ 54'\ 1\ 8'',$$
$$\begin{array}{rr}
L\cos 7^\circ\ 54'\ 1{\cdot}8''= & 9\ 9958586\\
- & 0000005\\
\log a= & 1\ 7199938\\
-L\sin A= & -9\ 9209393\\
\hline
\log c= & 1\ 7949126\\
c= & 62\ 3609
\end{array}$$

42
$$L\sin A=\log a+L\sin B-\log b,$$
$$\begin{array}{rr}
L\sin B= & 9\ 9971559\\
+ & 0000111\\
\log a= & 4\ 6925209\\
-\log b= & -4\ 7736036\\
\hline
L\sin A= & 9\ 9160843
\end{array}$$

$$L \sin 55^\circ\, 31' = \quad 9\,9160805,$$

$$\text{difference for } 1' = \quad 0000868, \qquad 60'' \times \frac{38}{868} = 2\,6''$$

$$A = 55^\circ\, 31'\, 2\,6'',$$

$$C = 180^\circ - (A + B) = 41^\circ\, 1'\, 11\,4''$$

$$\log c = \log b + L \sin C - L \sin B,$$

$$\begin{aligned} \log b &= 4\,7736036 \\ L \sin C &= 9\,8170882 \\ &\ + 0000276 \\ -L \sin B &= -9\,9971670 \\ \hline \log c &= 4{\cdot}5935524 \\ c &= 39224 \end{aligned}$$

43 This is an example of the ambiguous case (Art 233).

$$L \sin A = L \sin B + \log a - \log b,$$

$$\begin{aligned} L \sin B &= 9\,8725466 \\ &\ + 0000470 \\ \log a &= 1\,8656961 \\ -\log b &= -1\,8122447 \\ \hline L \sin A &= 9\,9260450 \end{aligned}$$

$$L \sin 57^\circ\, 30' = \quad 9\,9260292,$$

$$\text{difference for } 60'' = \quad 0000804, \qquad 60'' \times \frac{158}{804} = 11\,8''$$

(i) $$A = 57^\circ\, 30'\, 11\,8'',$$

$$C = 180^\circ - (A + B) = 74^\circ\, 16'\, 23\,2'',$$

or (ii) $$A = 180^\circ - 57^\circ\, 30'\, 11\,8'' = 122^\circ\, 29'\, 48\,2'',$$

$$C = 180^\circ - (A + B) = 9^\circ\, 16'\, 46\,8'',$$

$$\log c = \log b + L \sin C - L \sin B$$

(i)
$$\begin{aligned} \log b &= 1\,8122447 \\ L \sin 74^\circ\, 16'\, 23\,2'' &= 9\,9834161 \\ &\ + 0000138 \\ -L \sin B &= -9\,8725936 \\ \hline \log c &= 1\,9230810 \end{aligned}$$

$$c = 83\,7686$$

(ii)
$$\begin{aligned} \log b &= 1\,8122447 \\ L \sin 9^\circ\, 16'\, 46\,8'' &= 9\,2069059 \\ &\ + 0006034 \\ -L \sin B &= -9\,8725936 \\ \hline \log c &= 1\,1471604 \end{aligned}$$

$$c = 14\,0333.$$

44 This is an example of the ambiguous case

$$L\sin A = L\sin B + \log a - \log b,$$

$$\begin{aligned}
L\sin B &= 9.6251346 \\
&\ + .0001312 \\
\log a &= 3.7293649 \\
-\log b &= -3.4723673 \\
\hline
L\sin A &= 9.8822634
\end{aligned}$$

$$L\sin 49^\circ\, 41' = 9.8822285,$$

$$\text{difference for } 1' = .0001072, \qquad 60''\times\frac{349}{1072} = 19.5''$$

$$\therefore\ A = 49^\circ\, 41'\, 19.5'',$$

$$\text{or } A = 180^\circ - 49^\circ\, 41'\, 19.5'' = 130^\circ\, 18'\, 40.5'',$$

$$\begin{aligned}
C &= 180^\circ - (A+B) \\
&= 105^\circ\, 21'\, 11.5'' \text{ or } 24^\circ\, 13'\, 50.5'',
\end{aligned}$$

$$\log c = \log b + L\sin C - L\sin B.$$

(i) $\sin 105^\circ\, 21'\, 11.5'' = \sin 74^\circ\, 38'\, 48.5''$,

$$\begin{aligned}
\log b &= 3.4723673 \\
L\sin 76^\circ\, 38'\, 48.5'' &= 9.9841895 \\
&\ + .0000280 \\
-L\sin B &= -9.6252658 \\
\hline
\log c &= 3.8313190
\end{aligned}$$

$$c = 6781.4$$

(ii)

$$\begin{aligned}
\log b &= 3.4723673 \\
L\sin 24^\circ\, 13'\, 50.5'' &= 9.6213127 \\
&\ + .0002310 \\
-L\sin B &= -9.6252658 \\
\hline
\log c &= 3.4686452
\end{aligned}$$

$$c = 2942.02$$

45 This is an example of the ambiguous case

$$L\sin B = L\sin A + \log b - \log a,$$

$$\begin{aligned}
L\sin A &= 9.8455332 \\
&\ + .0001136 \\
\log b &= 2.8292523 \\
-\log a &= -2.7732304 \\
\hline
L\sin B &= 9.9016687
\end{aligned}$$

$$L\sin 52^\circ\, 52' = 9.9015852,$$

$$\text{difference for } 1' = .0000956, \qquad 60''\times\frac{835}{956} = 52.4''.$$

$$B = 52^{\circ}\ 52'\ 52\ 4'' \text{ or } 127^{\circ}\ 7'\ 7\ 6'',$$

$$C = 180^{\circ} - (A + B)$$

$$= 82^{\circ}\ 37'\ 14\ 6'' \text{ or } 8^{\circ}\ 22'\ 59\ 4'',$$

$$\log c = \log a + L \sin C - L \sin A$$

(i)

$$\begin{array}{rr}
\log a = & 2\ 7732304 \\
L \sin 82^{\circ}\ 37'\ 14\ 6'' = & 9\ 9963841 \\
& +\ \ 0000040 \\
- L \sin A = & -9\ 8456468 \\
\hline
\log c = & 2\ 9239717
\end{array}$$

$$c = 839\ 405$$

(ii)

$$\begin{array}{rr}
\log a = & 2\ 7732304 \\
L \sin 8^{\circ}\ 22'\ 59\ 4'' = & 9\ 1628853 \\
& +\ \ 0008495 \\
- L \sin A = & -9\ 8456468 \\
\hline
\log c = & 2\ 0913184
\end{array}$$

$$c = 123\ 401.$$

46

$$C = 180^{\circ} - (A + B) = 10^{\circ}\ 4'\ 55'',$$

$$\log c = \log a + L \sin C - L \sin A,$$

$$\begin{array}{rr}
\log a = & 2\ 7629230 \\
L \sin C = & 9\ 2425264 \\
& +\ \ 0006518 \\
- L \sin A = & -9\ 9351715 \\
& -\ \ 0000459 \\
\hline
\log c = & 2\ 0708838
\end{array}$$

$$c = 117\ 729$$

$$180^{\circ} - B = 69^{\circ}\ 33'\ 32'',$$

$$\log b = \log a + L \sin B - L \sin A,$$

$$\begin{array}{rr}
\log a = & 2\ 7629230 \\
L \sin 69^{\circ}\ 33'\ 32'' = & 9\ 9717291 \\
& +\ \ 0000251 \\
- L \sin A = & -9\ 9352174 \\
\hline
\log b = & 2\ 7994598
\end{array}$$

$$b = 630\ 173$$

47 $$\log b = \log a + L\sin B - L\sin A,$$

$$\begin{aligned}
\log a &= 2\ 7196218\\
L\sin B &= 9\ 9661987\\
&\ \ +\ \ 0000418\\
-L\sin A &= -9\ 8794199\\
\hline
\log b &= 2\ 8067414
\end{aligned}$$

$$b = 640\ 828$$

$$C = 180^\circ - (A+B) = 62^\circ\ 57'\ 12'',$$

$$\log c = \log a + L\sin C - L\sin A,$$

$$\begin{aligned}
\log a &= 2\ 7196218\\
L\sin C &= 9\ 9496876\\
&\ \ +\ \ 0000124\\
-L\sin A &= -9\ 8794199\\
\hline
\log c &= 2\ 7899019
\end{aligned}$$

$$c = 616\ 456$$

48 $$\sqrt{(a^2+b^2)} = a\sqrt{\left(1+\frac{b^2}{a^2}\right)} = a\sqrt{(1+\tan^2\theta)}, \text{ where } \tan\theta = \frac{b}{a},$$

$$= a\sec\theta.$$

$$\begin{aligned}
L\tan\theta &= \log b - \log a + 10\\
&= 5\ 1866004 - 5\ 3915685 + 10\\
&= 9\ 7950319
\end{aligned}$$

$$L\tan 31^\circ\ 57' = 9\ 7949455,$$

difference for $1' = \ 0002813$; $\qquad 60''\times\frac{864}{2813} = 18\ 4''$

$$\therefore\ \theta = 31^\circ\ 57'\ 18\ 4''.$$

$$\begin{aligned}
\log\sqrt{(a^2+b^2)} &= \log a + L\sec\theta - 10\\
&= 5\ 3915685 + 10\ 0718671 - 10\\
&= 5\ 4629356,
\end{aligned}$$

$$\sqrt{(a^2+b^2)} = 290359$$

49 As in Example 48,

$$\sqrt{(a^2+b^2)} = a\sec\theta, \text{ where } \tan\theta = \frac{b}{a}$$

$$\begin{aligned}
L\tan\theta &= 10 + \log b - \log a\\
&= 10 + 3\ 7951150 - 3\ 7199111\\
&= 10\ 0752039
\end{aligned}$$

$$L\tan 49^\circ\ 56' = 10\ 0751604,$$

difference for $1' = \ 0002565$; $\qquad 60''\times\frac{435}{2565} = 10\ 18''$

$$\theta = 49^\circ\ 56'\ 10\ 2''$$

$$\log\sqrt{(a^2+b^2)}=\log a+\log\sec\theta$$
$$=3\ 7199111+\ 1913310+\ 0000255$$
$$=3\ 9112676,$$
$$\sqrt{(a^2+b^2)}=8152\ 06$$

50 As in Example 48,

$$\sqrt{(a^2+b^2)}=a\sec\theta,\ \text{where}\tan\theta=\frac{b}{a}.$$
$$L\tan\theta=10+\log b-\log a$$
$$=10+4\ 2873090-4\ 6593742$$
$$=9\ 6279348$$
$$L\tan 23^0\ 0'=9\ 6278519,$$
$$\text{difference for }60''=\ 0003512,\qquad 60''\times\frac{829}{3512}=14\ 1''$$
$$\theta=23^0\ 0'\ 14\ 1''$$
$$\log\sqrt{(a^2+b^2)}=\log a+\log\sec\theta$$
$$=4\ 6593742+\ 0359739+{\cdot}0000126$$
$$=4\ 6953607,$$
$$\sqrt{(a^2+b^2)}=49586\ 2$$

XV 178.

1 Take the diagram of Art 240 The angle $PBC=60^0$, the angle $PAC=30^0$, therefore the angle $APB=30^0$ Also $AB=40$ feet

Since the angle $PAB=$the angle APB, we have $BP=AB=40$ Then

$$PC=BP\sin 60^0=40\frac{\sqrt{3}}{2}=20\sqrt{3},$$

and
$$BC=BP\cos 60^0=40\ \frac{1}{2}=20$$

2. Let AC produced through C meet the horizontal plane which contains B at D Then the angle $ABD=60^0$, and the angle $CBD=30^0$, therefore the angle $ABC=30^0$. The angle $ACB=135^0$ Hence

$$\text{the angle } BAC=180^0-30^0-135^0=15^0,$$
$$\frac{AB}{BC}=\frac{\sin ACB}{\sin BAC}=\frac{\sin 135^0}{\sin 15^0}=\frac{1}{\sqrt{2}}\div\frac{\sqrt{3}-1}{2\sqrt{2}}=\frac{2}{\sqrt{3}-1},$$

therefore
$$AB=\frac{2\times1760}{\sqrt{3}-1}\text{ yards.}$$

The height of the mountain $= AB \sin 60^\circ = AB \dfrac{\sqrt{3}}{2}$

$$= \frac{1760\sqrt{3}}{\sqrt{3}-1} = \frac{1760\sqrt{3}(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$= \frac{1760\sqrt{3}(\sqrt{3}+1)}{2} = 880(3+\sqrt{3})$$

3 Let h denote the height of the tower in yards; then

$$\frac{h}{100} = \tan 30^\circ = \frac{1}{\sqrt{3}}, \quad \text{therefore } h = \frac{100}{\sqrt{3}}$$

4 Let h denote the height of the tower, x the distance of the foot from A, and y the distance of the foot from B. Then

$$x = h \cot 30^\circ, \text{ and } y = h \cot 18^\circ.$$

But $y^2 - x^2 = a^2$, therefore $h^2(\cot^2 18^\circ - \cot^2 30^\circ) = a^2$,

therefore $$h^2\left\{\frac{10+2\sqrt{5}}{(\sqrt{5}-1)^2} - 3\right\} = a^2,$$

therefore $$h^2\left\{\frac{5+\sqrt{5}}{3-\sqrt{5}} - 3\right\} = a^2,$$

therefore $$4h^2(\sqrt{5}-1) = a^2(3-\sqrt{5}),$$

therefore $$h^2 = \frac{3-\sqrt{5}}{4(\sqrt{5}-1)}a^2 = \frac{(3-\sqrt{5})(3+\sqrt{5})}{4(\sqrt{5}-1)(3+\sqrt{5})}$$

$$= \frac{4a^2}{4(2+2\sqrt{5})} = \frac{a^2}{2+2\sqrt{5}}$$

5 Let A denote the eye of the spectator, and B the centre of the balloon. The angle a is formed by straight lines drawn from A in the vertical plane which contains B, so as to touch the balloon Hence

$$\frac{r}{AB} = \sin\frac{a}{2}; \quad \text{therefore } AB = r \operatorname{cosec}\frac{a}{2}$$

And the height of the centre of the balloon $= AB \sin\beta = r \sin\beta \operatorname{cosec}\dfrac{a}{2}$.

6 Let O denote the station which is in the same straight line as A and B, let P be the station which is in the same straight line as A and C, and let Q be the station which is in the same straight line as B and C Then O, P, and Q are in a straight line which is at right angles to AB Let $OP = p$, $OQ = q$; let $APO = a$, and $BQO = \beta$. Then $OA = p \tan a$, and $OB = q \tan\beta$ Thus $AB = q\tan\beta - p\tan a$ And the angles of the triangle ABC are known, for $ABQ = \dfrac{\pi}{2} - \beta$, and $OAP = \dfrac{\pi}{2} - a$ Hence AC and BC can be found

7 The tangent of the angle which AB subtends at E is $\frac{AB}{AE}$, and the tangent of the angle which CD subtends at E is $\frac{CD}{CE}$, therefore $\frac{AB}{AE}=\frac{CD}{CE}$,

therefore $$CE=\frac{AE \cdot CD}{AB}, \qquad \text{therefore} \quad CE^2=\frac{AE^2 \cdot CD^2}{AB^2},$$

therefore $$CA^2+AE^2=\frac{AE^2 \cdot CD^2}{AB^2}$$

but $$CA^2=AB^2, \qquad \text{therefore} \quad AE^2=\frac{AB^4}{CD^2-AB^2}$$

Again $$\cos DEA=\frac{EA}{ED}, \text{ and } \cos BEC=\frac{EB^2+EC^2-BC^2}{2EB \cdot EC}$$

$$=\frac{EA^2+AB^2+EA^2+AC^2-(AB^2+AC^2)}{2EB \cdot EC}=\frac{EA^2}{EB \cdot EC}$$

But by hypothesis the cosine of BEA is equal to the cosine of DEC, that is $\frac{EA}{EB}=\frac{EC}{ED}$; therefore $EA \cdot ED=EB \cdot EC$, therefore $\frac{EA^2}{EB \cdot EC}=\frac{EA}{ED}$

8 Let A be the top of the flag staff, B the top of the tower, C the foot of the tower, E the eye. From E draw a perpendicular ED on the horizontal plane which contains C. Then the angle BEC is to be equal to the angle BEA

Now $$\frac{\sin BEC}{\sin EBC}=\frac{BC}{EC}, \text{ and } \frac{\sin BEA}{\sin EBA}=\frac{AB}{AE},$$

therefore $$\frac{BC}{EC}=\frac{AB}{AE}.$$

This coincides with Euclid VI 3

Let $CD=x$, then $EC=\sqrt{(h^2+x^2)}$, $EA=\sqrt{(a+b-h)^2+x^2}$;

therefore $$\frac{b}{\sqrt{h^2+x^2}}=\frac{a}{\sqrt{(a+b-h)^2+x^2}},$$

therefore $$\{(a+b-h)^2+x^2\}\,b^2=(h^2+x^2)\,a^2,$$

therefore $$x^2=\frac{b^2(a+b-h)^2-h^2a^2}{a^2-b^2},$$

therefore $$EC^2=\frac{h^2(a^2-b^2)+b^2(a+b-h)^2-h^2a^2}{a^2-b^2}$$

$$=\frac{b^2\{(a+b-h)^2-h^2\}}{a^2-b^2}=\frac{b^2(a+b)(a+b-2h)}{a^2-b^2}=\frac{b^2(a+b-2h)}{a-b},$$

therefore $$EC=b\left(\frac{a+b-2h}{a-b}\right)^{\frac{1}{2}}.$$

9. Let P denote the top of the tower, from P draw PQ perpendicular to the ground, then $PQ=h$ Let x denote the distance of Q from the base of the tower, $x+a$ is the distance of Q from one point of observation, and $x+b$ is the distance of Q from the other point of observation

Thus $$\cot\theta=\frac{x}{h},\quad \cot\alpha=\frac{x+a}{h},\quad \cot\beta=\frac{x+b}{h},$$

therefore $$h\cot\alpha=x+a,\quad h\cot\beta=x+b,$$

therefore $$h=\frac{b-a}{\cot\beta-\cot\alpha},$$

and $$x=h\cot\alpha-a=\frac{(b-a)\cot\alpha}{\cot\beta-\cot\alpha}-a=\frac{b\cot\alpha-a\cot\beta}{\cot\beta-\cot\alpha}$$

Thus $$\tan\theta=\frac{h}{x}=\frac{b-a}{b\cot\alpha-a\cot\beta}$$

10 Let x denote the required height, and suppose θ the angle which the tower subtends then

$$x=b\tan\theta,\quad x+a=b\tan(\theta+\gamma),$$

therefore $$x+a=\frac{b(\tan\theta+\tan\gamma)}{1-\tan\theta\tan\gamma}=\frac{x+b\tan\gamma}{1-\frac{x\tan\gamma}{b}},$$

thus we have a quadratic equation for finding x

11. Let x denote the breadth of the river in feet, let α denote the angle subtended by the column, and β the angle subtended by the column and statue.

Thus $$\tan\alpha=\frac{200}{x},\ \text{and}\ \tan\beta=\frac{230}{x},$$

therefore $$\tan(\beta-\alpha)=\frac{\frac{230}{x}-\frac{200}{x}}{1+\frac{200\times 230}{x^2}}=\frac{30x}{x^2+46000}$$

But, by hypothesis, $\tan(\beta-\alpha)=\frac{6}{x}$, therefore

$$\frac{6}{x}=\frac{30x}{x^2+46000},\ \text{therefore}\ x^2+46000=5x^2,$$

$$\text{therefore}\ x^2=11500,\ \text{therefore}\ x=10\sqrt{115}$$

12 The part of the house above the horizontal straight line subtends an angle of 60^0, and thus the height of the top of the house above the window is $30\tan 60^0$ feet The part of the house below the horizontal straight line subtends an angle of 30^0, and thus the depth of the foot of the house below the window is $30\tan 30^0$ feet Hence the distance from the foot of the house to the top of the house in feet

$$=30(\tan 60^0+\tan 30^0)=30\left(\sqrt{3}+\frac{1}{\sqrt{3}}\right)=\frac{4}{\sqrt{3}}30=40\sqrt{3}.$$

13 Let x denote the height of each chimney in feet, and y the distance between them The distance of the first point of observation from the nearer chimney is $x \cot 60^0$, and therefore the distance of the second point of observation is $\sqrt{(80)^2 + x^2 \cot^2 60^0}$ Thus

$$\frac{x}{\sqrt{(80)^2 + x^2 \cot^2 60^0}} = \tan 45^0 = 1,$$

therefore $x^2 = (80)^2 + x^2 \cot^2 60^0 = (80)^2 + \dfrac{x^2}{3}$, therefore $2x^2 = 3\,(80)^2$,

therefore $\qquad x^2 = 6\,(40)^2$, therefore $x = 40\sqrt{6}$

The distance of the first point of observation from the further chimney is $y - x \cot 60^0$, and therefore the distance of the second point of observation is $\sqrt{(80)^2 + (y - x \cot 60^0)^2}$. Thus

$$\frac{x}{\sqrt{(80)^2 + (y - x \cot 60^0)^2}} = \tan 30^0 = \frac{1}{\sqrt{3}},$$

therefore $3x^2 = (80)^2 + (y - x \cot 60^0)^2$, therefore $14\,(40)^2 = (y - x \cot 60^0)^2$,

therefore $\qquad y = x \cot 60^0 + 40\sqrt{14} = 40\,(\sqrt{2} + \sqrt{14})$

14 Let P be the object, PQ the perpendicular from P on the horizontal plane which contains A, B, and C

Let $PQ = x$, $CQ = y$ Suppose θ the angle PAQ, then $PBQ = 2\theta$, and $PCQ = 3\theta$ Thus

$$\tan\theta = \frac{x}{y+a+b}, \qquad \tan 2\theta = \frac{x}{y+b}, \qquad \tan 3\theta = \frac{x}{y},$$

therefore $\qquad y + a + b = x \cot\theta, \quad y + b = x \cot 2\theta, \quad y = x \cot 3\theta,$

therefore $\qquad a = x(\cot\theta - \cot 2\theta), \quad b = x(\cot 2\theta - \cot 3\theta),$

therefore $\qquad a = x\left(\dfrac{\cos\theta}{\sin\theta} - \dfrac{\cos 2\theta}{\sin 2\theta}\right) = \dfrac{x \sin(2\theta - \theta)}{\sin\theta \sin 2\theta} = \dfrac{x}{\sin 2\theta},$

and $\qquad b = x\left(\dfrac{\cos 2\theta}{\sin 2\theta} - \dfrac{\cos 3\theta}{\sin 3\theta}\right) = \dfrac{x \sin(3\theta - 2\theta)}{\sin 2\theta \sin 3\theta} = \dfrac{x \sin\theta}{\sin 2\theta \sin 3\theta}$

$$= \frac{x}{\sin 2\theta\,(3 - 4\sin^2\theta)}$$

Thus $\qquad \sin 2\theta = \dfrac{x}{a}$, and $3 - 4\sin^2\theta = \dfrac{x}{b \sin 2\theta} = \dfrac{a}{b},$

therefore $\qquad 3 - 2(1 - \cos 2\theta) = \dfrac{a}{b}$, therefore $\cos 2\theta = \dfrac{1}{2}\left(\dfrac{a}{b} - 1\right)$

Hence
$$\frac{x^2}{a^2} + \frac{1}{4}\left(\frac{a}{b} - 1\right)^2 = 1,$$

therefore $\dfrac{x^2}{a^2} = 1 - \dfrac{1}{4}\left(\dfrac{a}{b} - 1\right)^2 = \dfrac{4b^2 - (a-b)^2}{4b^2} = \dfrac{3b^2 + 2ab - a^2}{4b^2} = \dfrac{(3b-a)(a+b)}{4b^2},$

therefore $\qquad x = \dfrac{a}{2b}\sqrt{(a+b)(3b-a)}$

If $\tan\theta=\frac{1}{3}$, then $\sin 2\theta=\frac{2\tan\theta}{1+\tan^2\theta}=\frac{\frac{2}{3}}{1+\frac{1}{9}}=\frac{3}{5}$, and $\sin 2\theta=\frac{x}{a}$, thus

$$\frac{3}{5}=\frac{\sqrt{(a+b)(3b-a)}}{2b};$$

therefore $\qquad 36b^2=25(a+b)(3b-a)=25(3b^2+2ab-a^2);$

therefore $\qquad 39b^2+50ab-25a^2=0,$

therefore $\qquad (13b-5a)(3b+5a)=0$, therefore $13b-5a=0$.

15 Let x denote the height of the tower in yards, then the distance from A to the foot of the tower is $x\cot 15^\circ$ The observer moves so that the tower always subtends the same angle, hence he must describe the arc of a circle having its centre at the foot of the tower, and as the bearing of the tower changes from north to north-east he must describe one eighth part of the circumference, therefore

$$\frac{2\pi x\cot 15^\circ}{8}=100,\quad \text{therefore } x=\frac{400\tan 15^\circ}{\pi}$$

16 Let A denote the object which is further from the road, B that which is nearer to the road, C the point where AB subtends the greatest angle, D the second point of observation

It is known that the point C is such that a circle described round A, B, and C will touch CD at C, see *Notes on Euclid*, page 308 Therefore the angle BCD is equal to the angle BAC, denote it by θ Then the angle $ABC=\theta+\beta$, and also $=\pi-\theta-\alpha$, therefore $2\theta=\pi-\alpha-\beta$

Now $\frac{BC}{CD}=\frac{\sin\beta}{\sin(\theta+\beta)}$, therefore $BC=\frac{c\sin\beta}{\sin(\theta+\beta)}$, and $\frac{AB}{BC}=\frac{\sin\alpha}{\sin\theta}$,

therefore $AB=\frac{c\sin\alpha\sin\beta}{\sin\theta\sin(\theta+\beta)}=\frac{2c\sin\alpha\sin\beta}{\cos\beta-\cos(2\theta+\beta)}$

$$=\frac{2c\sin\alpha\sin\beta}{\cos\beta-\cos(\pi-\alpha)}=\frac{2c\sin\alpha\sin\beta}{\cos\beta+\cos\alpha}.$$

17. Let A denote the fortress, B the first position of the ship, C the second; produce BC through C to any point E Then the angle $ABC=22\frac{1}{2}^\circ$, and the angle $ACE=67\frac{1}{2}^\circ$, therefore the angle $BAC=45^\circ$.

$$\frac{AB}{BC}=\frac{\sin ACB}{\sin BAC}=\frac{\sin(180^\circ-67\frac{1}{2}^\circ)}{\sin 45^\circ}=\frac{\sqrt{(2+\sqrt{2})}}{2}\div\frac{1}{\sqrt{2}}=\frac{\sqrt{(2+\sqrt{2})}}{\sqrt{2}}=\sqrt{\frac{2+\sqrt{2}}{2}},$$

therefore $\qquad AB=4\sqrt{\frac{2+\sqrt{2}}{2}}=\sqrt{(16+8\sqrt{2})}$

And $\dfrac{AC}{BC}=\dfrac{\sin ABC}{\sin BAC}=\dfrac{\sin 22\frac{1}{2}^0}{\sin 45^0}=\dfrac{\sqrt{(2-\sqrt{2})}}{2}-\dfrac{1}{\sqrt{2}}=\sqrt{\dfrac{2-\sqrt{2}}{2}}$,

therefore $AC=4\sqrt{\dfrac{2-\sqrt{2}}{2}}=\sqrt{(16-8\sqrt{2})}$ See Example VII 18

18 Let P be the first position of the ship, A the nearer lighthouse, and B the further lighthouse, let Q be the second position of the ship Then the angle $BQP=45^0$, and the angle $AQP=22\frac{1}{2}^0$, therefore the angle $QAP=67\frac{1}{2}^0$

$$\frac{BQ}{BA}=\frac{\sin BAQ}{\sin BQA}=\frac{\sin(180^0-67\frac{1}{2}^0)}{\sin 22\frac{1}{2}^0}=\frac{\sin 67\frac{1}{2}^0}{\sin 22\frac{1}{2}^0}=\frac{\cos 22\frac{1}{2}^0}{\sin 22\frac{1}{2}^0}=\cot 22\frac{1}{2}^0$$

$=\sqrt{2}+1$, by Example VII 18, therefore $BQ=8(\sqrt{2}+1)$

And $$PQ=BQ\sin 45^0=\frac{8(\sqrt{2}+1)}{\sqrt{2}}=8+4\sqrt{2}$$

19 Let A denote the top of the lighthouse, P the top of the mast at the first observation, C the centre of the earth Draw a straight line from P to A and let it touch the earth at B

Let r denote the radius of the earth in feet, then

$PB=\sqrt{PC^2-BC^2}=\sqrt{(r+64)^2-r^2}=\sqrt{2r\times 64+(64)^2}=\sqrt{2r\times 64}$ very nearly, for r is very large compared with $(64)^2$

In precisely the same manner if Q denote the deck of the ship at the second observation, $QB=\sqrt{2r\times 16}$

Now, since PCB is a very small angle, we may, by the principle that $\tan\theta$ is nearly equal to θ when θ is very small, consider the straight line PB to be equal to the arc which measures the distance of the ship from B at the first observation, and similarly we may consider QB to be equal to the arc which measures the distance of the ship from B at the second observation Thus between the two observations the ship has sailed over $\sqrt{2r\times 64}-\sqrt{2r\times 16}$, that is, $4\sqrt{2r}$, that is, in half-an-hour it has sailed over $4\sqrt{8000\times 5280}$ feet, so that the rate is $8\sqrt{8000\times 5280}$ feet per hour, that is, $\dfrac{8\sqrt{8000\times 5280}}{5280}$ miles per hour, that is, $8\sqrt{\dfrac{800}{528}}$ miles per hour, that is, $8\sqrt{\dfrac{50}{33}}$ miles per hour, this is very nearly $8\sqrt{\dfrac{3}{2}}$ miles per hour

20 Let A denote the summit of the mountain, B the base, BC the first part of the path, CA the second part From A draw AE perpendicular to the horizontal plane which contains B, then $AE=n$

The following are the angles

$$BAE=\frac{\pi}{2}-\gamma,\qquad CBE=\alpha,\qquad CAE=\frac{\pi}{2}-\beta,$$

therefore $BAC=\beta-\gamma,\quad ABC=\gamma-\alpha,\quad ACB=\pi+\alpha-\beta.$

$$AB = \frac{AE}{\sin\gamma} = \frac{n}{\sin\gamma},$$

$$\frac{BC}{AB} = \frac{\sin BAC}{\sin ACB} = \frac{\sin(\beta-\gamma)}{\sin(\beta-\alpha)},$$

$$\frac{AC}{AB} = \frac{\sin ABC}{\sin ACB} = \frac{\sin(\gamma-\alpha)}{\sin(\beta-\alpha)},$$

therefore $$\frac{BC+AC}{AB} = \frac{\sin(\beta-\gamma)+\sin(\gamma-\alpha)}{\sin(\beta-\alpha)}$$

$$= \frac{2\sin\frac{\beta-\alpha}{2}\cos\left(\frac{\alpha+\beta}{2}-\gamma\right)}{\sin(\beta-\alpha)} = \frac{\cos\left(\frac{\alpha+\beta}{2}-\gamma\right)}{\cos\frac{\beta-\alpha}{2}},$$

therefore $$BC+AC = \frac{n}{\sin\gamma}\cdot\frac{\cos\left(\frac{\alpha+\beta}{2}-\gamma\right)}{\cos\frac{\beta-\alpha}{2}}.$$

21. Let O denote the foot of the object, and let A, B, and C denote the three points of observation Let x denote the height of the object, then $OA = x\cot\alpha$, $OB = x\cot\beta$, and $OC = x\cot\gamma$

From the triangle AOC we have

$$x^2\cot^2\alpha = x^2\cot^2\gamma + a^2 - 2ax\cot\gamma\cos ACO,$$

and from the triangle BOC we have

$$x^2\cot^2\beta = x^2\cot^2\gamma + b^2 - 2bx\cot\gamma\cos BCO$$

Multiply the first equation by b and the second by a, and add, thus

$$x^2(b\cot^2\alpha + a\cot^2\beta) = ab(a+b) + x^2(a+b)\cot^2\gamma,$$

therefore

$$x^2 = \frac{ab(a+b)\sin^2\alpha\sin^2\beta\sin^2\gamma}{a(\cos^2\beta\sin^2\gamma - \cos^2\gamma\sin^2\beta)\sin^2\alpha + b(\cos^2\alpha\sin^2\gamma - \cos^2\gamma\sin^2\alpha)\sin^2\beta}$$

$$= \frac{ab(a+b)\sin^2\alpha\sin^2\beta\sin^2\gamma}{a(\sin^2\gamma - \sin^2\beta)\sin^2\alpha + b(\sin^2\gamma - \sin^2\alpha)\sin^2\beta}$$

22 Let P be the summit of the lower hill, Q the summit of the higher hill, let A be the first point of observation, B the second, C the third. From P and Q draw PM and QN, respectively perpendicular to the horizontal plane which contains A, B, and C

Let $PM = h$, and $QN = h'$

Then $AM = h\cot\alpha$, and $AM = AB + BC + CM = c + 1 + h\cot\beta$,

therefore $h\cot\alpha = c+1+h\cot\beta$, therefore $h(\cot\alpha - \cot\beta) = c+1$,

therefore $$h = \frac{(c+1)\sin\alpha\sin\beta}{\sin(\beta-\alpha)}$$

And by similar triangles

$$\frac{h'}{h} = \frac{QN}{PM} = \frac{BN}{BM} = \frac{AN-AB}{AM-AB} = \frac{h'\cot\alpha' - c}{h\cot\alpha - c},$$

thus since h is known we can find h'

23 Let h be the height of the tower in feet, α the altitude of the sun at noon The distance between the foot of the tower and the edge of the moat is $h\cot 60^\circ$, hence the distance between the foot of the tower and the extremity of the shadow is $h\cot 60^\circ + 45$ at noon, and $h\cot 60^\circ + 120$ when the sun is due west The directions of the shadows include a right angle,

therefore $$(h\cot 60^\circ + 45)^2 + (h\cot 60^\circ + 120)^2 = (375)^2$$

Therefore $$\frac{2h^2}{3} + \frac{2h}{\sqrt{3}} \cdot 165 + (45)^2 + (120)^2 = (375)^2,$$

therefore $$\frac{2h^2}{3} + \frac{2h}{\sqrt{3}}\, 165 = 124200.$$

By solving this quadratic in the usual way we obtain $h = 180\sqrt{3}$ or $-345\sqrt{3}$, only the positive value is applicable. Then $h\cot\alpha - h\cot 60^\circ = 45$,

therefore $$\cot\alpha = \cot 60^\circ + \frac{45}{h} = \frac{1}{\sqrt{3}} + \frac{45}{180\sqrt{3}} = \frac{1}{\sqrt{3}} + \frac{1}{4\sqrt{3}} = \frac{5}{4\sqrt{3}},$$

therefore $$\tan\alpha = \frac{4\sqrt{3}}{5}$$

24. Let P denote the top of the tower Then ϕ is the angle between PA and CA produced through A Thus the angle $CPA = \phi - \alpha$, and the angle $DPC = \alpha - \beta$

Then $$\frac{DC}{CP} = \frac{\sin DPC}{\sin CDP} = \frac{\sin(\alpha-\beta)}{\sin\beta},$$

$$\frac{CA}{CP} = \frac{\sin CPA}{\sin CAP} = \frac{\sin(\phi-\alpha)}{\sin(\pi-\phi)} = \frac{\sin(\phi-\alpha)}{\sin\phi};$$

therefore $$\frac{\sin(\alpha-\beta)}{\sin\beta} = \frac{\sin(\phi-\alpha)}{\sin\phi},$$

therefore $$\sin\alpha\cot\beta - \cos\alpha = \cos\alpha - \sin\alpha\cot\phi,$$

therefore $$\cot\phi = 2\cot\alpha - \cot\beta$$

Now let α', β', and ϕ' correspond to observations made in another straight line $AC'D'$, then $\cot\phi' = 2\cot\alpha' - \cot\beta'$, but by supposition $2\tan\beta' = \tan\alpha'$, therefore $\cot\phi' = 0$, therefore $\phi' = \frac{\pi}{2}$ Thus $AC'D'$ makes a right angle with AP, and therefore $AC'D'$ is a horizontal straight line

From D draw DM perpendicular to AD', and from M draw MN perpendicular to the horizontal plane which contains D, and produce PA through A to meet the same plane at Q

Then $$\sin\theta=\frac{MN}{MD},\ \sin\gamma=\frac{DM}{DA},\ \cos\phi=\frac{AQ}{AD}=\frac{MN}{AD};$$

therefore $$\cos\phi=\sin\theta\sin\gamma.$$

25 $$\sin B=\frac{b}{a}\sin A=\frac{3\sqrt{3}}{3}\sin A=\sqrt{3}\sin A,$$

thus if $A=\frac{\pi}{6}$ we have $\sin B=\sqrt{3}\cdot\frac{1}{2}$; therefore $B=\frac{\pi}{3}$ or $\frac{2\pi}{3}$.

Suppose however that $A=\frac{\pi}{6}\pm h$, where h is the circular measure of 2″, then $\sin B=\sqrt{3}\sin\left(\frac{\pi}{6}\pm h\right)=\sqrt{3}\left\{\sin\frac{\pi}{6}\pm h\cos\frac{\pi}{6}\right\}$ very nearly Suppose that $B=\frac{\pi}{3}\pm l$; then approximately $\sin\frac{\pi}{3}\pm l\cos\frac{\pi}{3}=\sqrt{3}\left\{\sin\frac{\pi}{6}\pm h\cos\frac{\pi}{6}\right\}$,

therefore $\pm l\cos\frac{\pi}{3}=\pm h\sqrt{3}\cos\frac{\pi}{6}$, therefore $l=h\sqrt{3}\,.\cot\frac{\pi}{6}=3h$

In the same way if $B=\frac{2\pi}{3}\pm l$ we find that $l=-3h$ Thus the approximate error in B is 6 seconds

26 Let A and B be the two objects on the opposite bank of the river, and suppose P and Q two points on this bank, such that $PQ=AB$, and let P correspond to A and Q to B, so that AP is equal and parallel to BQ Let AQ and BP intersect at C

Then $\alpha=$ the angle APB, and $\beta=$ the angle $AQB=$ the angle PAQ

Therefore $$\frac{PC}{PA}=\frac{\sin\beta}{\sin(\alpha+\beta)},\quad \frac{AC}{PA}=\frac{\sin\alpha}{\sin(\alpha+\beta)},$$

but $PQ^2=PC^2+QC^2-2PC\ QC\,.\cos PCQ$, and $QC=AC$,

therefore $$c^2=PA^2\frac{\sin^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta\cos(\alpha+\beta)}{\sin^2(\alpha+\beta)}$$

Let x denote the breadth of the river, then the area of the triangle $APB=\frac{1}{2}xc$, and this area is also equal to

$$\frac{1}{2}PA\,.PB\sin APB=PA\ PC\sin\alpha=\frac{PA^2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$$
$$=\frac{c^2\sin\alpha\sin\beta\sin(\alpha+\beta)}{\sin^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta\cos(\alpha+\beta)}.$$

27 Let AB denote a side of the fort, O the position due south of A, let D be the second position, so that $CD=a$, and the angle $ACD=90^\circ$, also A, B, D, and C will lie on the circumference of a circle Let E be the third position, so that E is on CD produced through D, and $DE=b$, and the angle BED is a right angle

Let ϕ be the angle between AB produced through B and CE produced through E Then $a+b=AB\cos\phi$, therefore $AB=(a+b)\sec\phi$

And $\quad BE=EC\tan BCE$, and $=ED\tan BDE$,

therefore $\quad (a+b)\tan(90^\circ-a)=b\tan BAC=b\tan(90^\circ-\phi)$. (Euclid III 22)

28 From A draw AM perpendicular to the horizontal plane which contains the road, and draw AN perpendicular to the straight road

Then
$$\sin\alpha=\frac{AM}{AB},\text{ and }\sin\beta=\frac{AN}{AB}$$

Similarly from A' draw $A'M'$ perpendicular to the horizontal plane, and $A'N'$ perpendicular to the straight road

Then
$$\sin\alpha'=\frac{A'M'}{A'B'}\text{ and }\sin\beta'=\frac{A'N'}{A'B'}$$

Thus we have to shew that $\dfrac{AM}{AB}\cdot\dfrac{A'N'}{A'B'}=\dfrac{A'M'}{A'B'}\quad\dfrac{AN}{AB}$,

or that $\quad AM\quad A'N'=A'M'\quad AN$, or that $\dfrac{AM}{AN}=\dfrac{A'M'}{A'N'}$.

Now if A is just hidden by A' at some point of the road, the straight line $A'A$ if produced through A will intersect the road, and then AA' and the road will lie in one plane, the sine of the inclination of this plane to the horizontal plane is expressed by $\dfrac{AM}{AN}$ and also by $\dfrac{A'M'}{A'N'}$; so that these are equal

29 There are two cases Suppose the angles APQ and BPR to be on the same sides of AP and BP respectively, then the angle $QPR=$the angle $APB=a$. Suppose the angles APQ and BPR not to fall on the same sides of AP and BP respectively, then the angle $RPQ=\pi-a$ In both cases $AB=RQ$, for the diameter of the circle which goes round the five points A, B, P, Q, and $R=\dfrac{AB}{\sin APB}$ and also $=\dfrac{RQ}{\sin RPQ}$

In the former case $AB=\sqrt{(a^2+b^2-2ab\cos a)}$, and in the latter case
$$AB=\sqrt{(a^2+b^2+2ab\cos a)}$$

30 Suppose both straight lines OC and $O'C$ to fall within the angle ACB. Let $AC=a$, $ACO=\phi$, then from the triangles ACO and BCO we get
$$OC=\frac{a\sin(\phi+a)}{\sin a}\text{ and }OC=\frac{a\cos(\phi-\beta)}{\sin\beta},$$

therefore $OC \sin\alpha = a(\sin\phi\cos\alpha + \cos\phi\sin\alpha)$,

$OC\sin\beta = a(\cos\phi\cos\beta + \sin\phi\sin\beta)$

Hence $$a\sin\phi = \frac{OC\sin\alpha(\cos\beta - \sin\beta)}{\cos(\alpha+\beta)},$$

$$a\cos\phi = \frac{OC\sin\beta(\cos\alpha - \sin\alpha)}{\cos(\alpha+\beta)}.$$

Square and add, thus

$$a^2\cos^2(\alpha+\beta) = OC^2\{\sin^2\alpha(\cos\beta-\sin\beta)^2 + \sin^2\beta(\cos\alpha-\sin\alpha)^2\}$$
$$= OC^2\{\sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta\sin(\alpha+\beta)\}$$

Thus $$OC^2 = \frac{a^2\cos^2(\alpha+\beta)}{\sin^2\alpha+\sin^2\beta-2\sin\alpha\sin\beta\sin(\alpha+\beta)}$$

A similar expression will be found for $O'C^2$ in terms of α' and β' Then $O'C^2 = OC^2 + d^2$ This finds a, and then $AB = a\sqrt{2}$.

Similarly the problem may be solved for any other positions of the lines OC, $O'C$

31. Let α denote the Sun's altitude, then $\tan\alpha = \frac{150}{75} = 2$,

therefore $L\tan\alpha = 10 + \log 2 = 10\ 3010300$

$$\begin{array}{r} 10\ 3013153 \\ 10\ 3009994 \\ \hline 0003159 \end{array} \qquad \begin{array}{r} 10\ 3010300 \\ 10\ 3009994 \\ \hline 0000306 \end{array} \qquad 0003159 \quad \cdot 0000306 \quad 60'' \quad x'';$$

this gives $x=6$, therefore $\alpha = 63°\ 26'\ 6''$.

32 Take the diagram of Art 240 Here $PBC=55°$, $PAC=48°$, $AB=30$ feet

$$\frac{PB}{BA} = \frac{\sin PAB}{\sin APB} = \frac{\sin 48°}{\sin 7°}; \text{ therefore } PB = \frac{30\sin 48°}{\sin 7°},$$

$$BC = BP\cos PBC = BP\cos 55° = BP\sin 35° = \frac{30\sin 48°\sin 35°}{\sin 7°},$$

$$\log BC = \log 30 + L\sin 48° - 10 + L\sin 35° - 10 - (L\sin 7° - 10)$$
$$= 1\ 47712 + 9\ 87107 + 9\ 75859 - 9\ 08589 - 10 = 2\ 02089;$$

therefore $BC = 104\ 93$

33 Let α denote the inclination; then $\sin\alpha = \frac{100}{196} = \frac{100}{4\times 49}$,

therefore $L\sin\alpha = 10 + \log 100 - \log(4\times 49) = 12 - 2\log 2 - 2\log 7 = 9\ 70774$

$$\begin{array}{r} 9\ 70782 \\ 9\ 70761 \\ \hline 00021 \end{array} \qquad \begin{array}{r} 9\ 70774 \\ 9\ 70761 \\ \hline 00013 \end{array} \qquad 00021 \quad 00013 \quad 60'' \cdot x'',$$

this gives $x=37$, therefore $\alpha = 30°\ 40'\ 37''$.

34 Let A be the point of intersection of the hills, B the point of observation on the hill, P the top of the object, C the bottom Produce PB through B to meet at D the horizontal straight line which contains A, produce DA through A to any point E Then $AB=64$ feet, and the following are the given angles

$$CAE=60^\circ,\ BAD=40^\circ,\ BDA=70^\circ,\ BPC=90^\circ-70^\circ=20^\circ$$

Therefore $$BAC=80^\circ,\ BCA=20^\circ,\ PBC=30^\circ.$$

$$\frac{BC}{BA}=\frac{\sin BAC}{\sin BCA}=\frac{\sin 80^\circ}{\sin 20^\circ},$$

$$\frac{PC}{BC}=\frac{\sin PBC}{\sin BPC}=\frac{\sin 30^\circ}{\sin 20^\circ},\ \text{therefore}\ \frac{PC}{BA}=\frac{\sin 80^\circ \sin 30^\circ}{\sin^2 20^\circ},$$

therefore $$PC=\frac{64 \sin 80^\circ \sin 30^\circ}{\sin^2 20^\circ}=\frac{64 \sin 40^\circ \cos 40^\circ}{\sin^2 20^\circ}=\frac{128 \cos 40^\circ}{\tan 20^\circ},$$

therefore $$\log PC=7\log 2+L\cos 40^\circ-L\tan 20^\circ=2\ 4303981,$$

therefore $$PC=269\ 40031$$

35 Let A, B, C be the three successive positions of the ship from which the observations are made, let P, Q, R be the corresponding positions of the other ship

Then the straight line ABC is parallel to the straight line PQR, also $AB=BC$, and $PQ=QR$

Let θ be the angle between the North direction and the direction of sailing

From B draw a straight line parallel to AP, meeting PQ at M, then

$$\frac{QM}{BM}=\frac{\sin QBM}{\sin BQM}=\frac{\sin QBM}{\sin BQP}=\frac{\sin(\beta-\alpha)}{\sin(\theta-\beta)}$$

Again, from C draw a straight line parallel to AP, meeting QR at N, then

$$\frac{RN}{CN}=\frac{\sin RCN}{\sin CRN}=\frac{\sin RCN}{\sin CRP}=\frac{\sin(\gamma-\alpha)}{\sin(\theta-\gamma)}$$

But $BM=CN$, and $RN=2QM$, for RN is the difference of the paths of the ships in two hours, and QM is the difference in one hour.

Therefore $$\frac{2\sin(\beta-\alpha)}{\sin(\theta-\beta)}=\frac{\sin(\gamma-\alpha)}{\sin(\theta-\gamma)},$$

therefore $$2\sin(\theta-\gamma)\sin(\beta-\alpha)=\sin(\gamma-\alpha)\sin(\theta-\beta),$$

therefore

$$2(\sin\theta\cos\gamma-\cos\theta\sin\gamma)\sin(\beta-\alpha)=(\sin\theta\cos\beta-\cos\theta\sin\beta)\sin(\gamma-\alpha)$$

Divide by $\cos\theta$, thus we obtain the value of $\tan\theta$

36 If $\alpha+\beta+C=\pi$, then $x+y=\pi$; therefore $\sin x=\sin y$, therefore $\tan\phi=1$.

We might as in Art 242 say that

$$\frac{\sin x - \sin y}{\sin x + \sin y} = \tan\left(\phi - \frac{\pi}{4}\right),$$

that is
$$\frac{2\sin\frac{x-y}{2}\cos\frac{x+y}{2}}{2\sin\frac{x+y}{2}\cos\frac{x-y}{2}} = \tan\left(\phi - \frac{\pi}{4}\right)$$

But as $\cos\frac{x+y}{2}$ is now zero we cannot divide both numerator and denominator of the last fraction by it, and thus we cannot proceed further. In fact in this case a circle would go round P, A, C, and B, and P may be at any point of the arc between A and B

XVI P 145

1 Here $s=36$, $s-a=12$, $s-b=6$, $s-c=18$

The area of the triangle $=\sqrt{36\times12\times6\times18}=\sqrt{36\times36\times36}=6^3=216$

2 The third angle of the triangle $=180^\circ-60^\circ=120^\circ$.

One of the containing sides $=\frac{10\times\sin 15^\circ}{\sin 120^\circ}$, and the other $=\frac{10\times\sin 45^\circ}{\sin 120^\circ}$.

Hence the area

$$=\frac{1}{2}\frac{(10)^2\sin 15^\circ\sin 45^\circ}{\sin^2 120^\circ}\sin 120^\circ=\frac{50\sin 15^\circ\sin 45^\circ}{\sin 120^\circ}=\frac{50(\sqrt{3}-1)}{2\sqrt{2}}\times\frac{1}{\sqrt{2}}\times\frac{2}{\sqrt{3}}$$
$$=\frac{25(\sqrt{3}-1)}{\sqrt{3}}.$$

3 The area of the triangle $=\frac{1}{2}\times3\times12\times\sin 30^\circ=\frac{36}{4}=9$

Let x denote the hypotenuse of the right angled triangle, then each of the equal sides is $\frac{x}{\sqrt{2}}$, and the area is $\frac{1}{2}\times\left(\frac{x}{\sqrt{2}}\right)^2$, that is $\frac{x^2}{4}$. Hence $\frac{x^2}{4}=9$, therefore $x^2=36$, therefore $x=6$.

4
$$r=\frac{S}{s},\quad R=\frac{abc}{4S};\quad \text{therefore } \frac{R}{r}=\frac{sabc}{4S^2}.$$

Now $s=7$, $s-a=4$, $s-b=2$, $s-c=1$, therefore $S=\sqrt{7\times4\times2}$, thus

$$\frac{R}{r}=\frac{7\times3\times5\times6}{4\times7\times4\times2}=\frac{45}{16}$$

5 From the angle C of a triangle draw a perpendicular CD to the side AB, or AB produced

First suppose A and B acute, so that D is between A and B. Then $CD=b\sin A$, $AD=b\cos A$, thus the area of $ACD=\frac{1}{2}b^2\sin A\cos A=\frac{1}{4}b^2\sin 2A$. Similarly the area of $BCD=\frac{1}{2}a^2\sin B\cos B=\frac{1}{4}a^2\sin 2B$.

Therefore the area of the whole triangle $=\frac{1}{4}(a^2\sin 2B+b^2\sin 2A)$.

Next suppose the angle B obtuse, so that D falls on AB produced through D. Then as before the area of $ACD=\frac{1}{4}b^2\sin 2A$. And the area of $CBD=\frac{1}{2}a^2\sin(180^\circ-B)\cos(180^\circ-B)=\frac{a^2}{4}\sin(360^\circ-2B)$.

Therefore the area of ABC

$$=\frac{1}{4}\{b^2\sin 2A-a^2\sin(360^\circ-2B)\}=\frac{1}{4}(b^2\sin 2A+a^2\sin 2B).$$

This mode of solution shews the geometrical meaning of the two parts of the expression. We may proceed more briefly thus

$$\frac{1}{4}(a^2\sin 2B+b^2\sin 2A)$$

$=\frac{1}{2}(a\sin B\,a\cos B+b\sin A\,b\cos A)=\frac{1}{2}a\sin B(a\cos B+b\cos A)$, by Art. 214,

$=\frac{1}{2}ac\sin B$, by Art. 216, $=$ the area of the triangle by Art. 247.

6. $\dfrac{a^2-b^2}{2}\dfrac{\sin A\sin B}{\sin(A-B)}=\dfrac{\sin A\sin B}{2\sin(A-B)}\left\{\dfrac{c^2\sin^2 A}{\sin^2 C}-\dfrac{c^2\sin^2 B}{\sin^2 C}\right\}$

$=\dfrac{c^2\sin A\sin B(\sin^2 A-\sin^2 B)}{2\sin(A-B)\sin^2 C}=\dfrac{c^2\sin A\sin B\sin(A+B)\sin(A-B)}{2\sin(A-B)\sin^2 C}$

$=\dfrac{c^2\sin A\sin B}{2\sin C}=$ area of the triangle, by Art. 247.

7. $\dfrac{2abc}{a+b+c}\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}=\dfrac{2abc}{2s}\sqrt{\dfrac{s(s-a)}{bc}}\times\sqrt{\dfrac{s(s-b)}{ac}}\times\sqrt{\dfrac{s(s-c)}{ab}}$

$=\sqrt{s(s-a)(s-b)(s-c)}=S=$ the area of the triangle.

8. $$PA=\frac{EA}{\cos DAC}=\frac{AB\cos A}{\sin C}=2R\cos A.$$

Similarly $PB=2R\cos B$, $PC=2R\cos C$.

9. $$PD=PC\sin PCD=2R\cos C\cos B. \quad \text{(Ex. 8.)}$$

Similarly $PE=2R\cos A\cos C$, $PF=2R\cos A\cos B$.

10 Since $OA' = OB\cos BOA' = R\cos A$,

$$OA' = \frac{1}{2}PA \text{ (Ex 8)} = AH \text{ or } HP,$$

and OA' is parallel to AHP,

$AOA'H$ and $HOA'P$ are both parallelograms

11 F, B, D, P are concyclic, and $PDCE$ are concyclic,

$$\therefore \angle FDP = \angle FBP = 90^\circ - A,$$

and $$\angle EDP = \angle ECP = 90^\circ - A,$$

$$\therefore \angle FDE = 180^\circ - 2A$$

Similarly $$\angle DEF = 180^\circ - 2B,$$

and $$\angle DFE = 180^\circ - 2C$$

In the same way, by considering $I_1I_2I_3$ as the original triangle (Art 253),

$$A = \angle BAC = 180^\circ - 2\angle I_2I_1I_3,$$

$$\angle I_2I_1I_3 = 90^\circ - \frac{1}{2}A,$$

similarly $$\angle I_1I_2I_3 = 90^\circ - \frac{1}{2}B, \quad \angle I_1I_3I_2 = 90^\circ - \frac{1}{2}C.$$

Or, since IBI_1C are concyclic,

$$\angle I_2I_1I_3 = 180^\circ - BIC = 180^\circ - \left(A + \frac{1}{2}B + \frac{1}{2}C\right) = 90^\circ - \frac{1}{2}A$$

12 EF is a chord of the nine-points circle $\left(\text{radius } \frac{1}{2}R\right)$, subtending an angle $180^\circ - 2A$ (i e. $\angle FDE$, Ex 11) at the circumference. Therefore

$$EF = R\sin(180^\circ - 2A) = 2R\sin A\cos A = a\cos A$$

Similarly $$DF = b\cos B, \quad DE = c\cos C$$

13 The radius of the circumcircle of $I_1I_2I_3 = 2R$ (Art. 253),

the angle $I_2I_1I_3 = 90^\circ - \frac{1}{2}A$ (Ex. 11),

hence $$I_2I_3 = 2(2R)\sin I_2I_1I_3 = 4R\cos\frac{1}{2}A$$

14 I is the orthocentre of the triangle $I_1I_2I_3$; the radius of the circumcircle of $I_1I_2I_3$ is $2R$, therefore from Ex 8

$$II_1 = 2(2R)\cos I_2I_1I_3 = 4R\cos\left(90^\circ - \frac{1}{2}A\right), \qquad \text{(Ex 11)}$$

$$\therefore II_1 \operatorname{cosec}\frac{1}{2}A = 4R$$

Similarly $$II_2 \operatorname{cosec}\frac{1}{2}B = 4R = II_3 \operatorname{cosec}\frac{1}{2}C$$

15 $I_1A \;\; I_2I_3=2$ area $I_1I_2I_3$, hence from Ex 13,

$$I_1A\cos\frac{1}{2}A=\frac{2\text{ area }I_1I_2I_3}{4R},$$

$$I_1A\cos\frac{1}{2}A=I_2B\cos\frac{1}{2}B=I_3C\cos\frac{1}{2}C$$

Or, directly, from the figure, draw I_1X perpendicular to AB, then $AX=s$ (Art 250)

But $$AX=AI_1\cos XAI_1=AI_1\cos\frac{1}{2}A;$$

$$AI_1\cos\frac{1}{2}A=BI_2\cos\frac{1}{2}B=CI_3\cos\frac{1}{2}C$$

16 By Art 252 we have, since $\frac{1}{2}R$ is the radius of the circumcircle of DEF,

$$2R \;\; \text{area } DEF=DE \;\; DF \;\; EF=abc\cos A\cos B\cos C, \qquad \text{(Ex. 12)}$$

$$\text{area } DEF=2\,.\,\frac{abc}{4R}\,.\cos A\cos B\cos C$$

$$=2S\cos A\cos B\cos C$$

17 Area $I_1I_2I_3=\frac{1}{2}I_1A \;\; I_2I_3$

$$=\frac{1}{2}\cdot\frac{s}{\cos\frac{A}{2}}\cdot 4R\cos\frac{A}{2} \qquad \text{(Ex. 15 and 13)}$$

$$=2Rs=2\;\frac{abc}{4S}\cdot\frac{S}{r}=\frac{abc}{2r}.$$

Again, area $I_1I_2I_3=\frac{1}{2}I_1I_2 \;\; I_1I_3\sin I_1$

$$=\frac{1}{2}\;\; 4R\cos\frac{1}{2}C \;\; 4R\cos\frac{1}{2}B\,.\sin\left(90^0-\frac{1}{2}A\right) \qquad \text{(Ex 13 \& 11)}$$

$$=8R^2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$$

18 $OA'G$ and AGP are similar triangles, therefore

$$OG\cdot GP=OA' \;\; AP=1 \;\; 2, \qquad \text{(Ex 10)}$$

$$A'G \;\; AG=OA'\cdot AP=1 \;\; 2,$$

$$GP=2OG,$$

and $$2A'G=AG,$$

or $$3A'G=AA'.$$

19 $A'D$ is a chord of the nine-points circle, the circle is therefore on the line bisecting $A'D$ at right angles This line, which is parallel to OA' and PD, passes through the middle point of OP Again, the line bisecting

$C'F$ at right angles passes through the centre of the nine points circle and also through the middle point of OP. Hence the centre is on two straight lines which intersect in the middle point of OP, therefore the middle point of OP is the centre.

20
$$r_1+r_2+r_3-r=S\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}\right)$$
$$=S\left\{\frac{2s-a-b}{(s-a)(s-b)}+\frac{c}{s(s-c)}\right\}$$
$$=Sc\cdot\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}$$
$$=Sc\cdot\frac{2s^2-(a+b+c)s+ab}{S^2}$$
$$=\frac{abc}{S}=4R$$

21
$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{s-a}{S}+\frac{s-b}{S}+\frac{s-c}{S}$$
$$=\frac{3s-(a+b+c)}{S}$$
$$=\frac{s}{S}=\frac{1}{r}$$

By Art 249
$$\frac{r\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}=a=2R\sin A=4R\sin\frac{A}{2}\cos\frac{A}{2};$$
$$r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

By Art 251,
$$\frac{r_1\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}=a=4R\sin\frac{A}{2}\cos\frac{A}{2},$$
$$r_1=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Or
$$\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\sqrt{\frac{(s-b)(s-c)\cdot s(s-b)\cdot s(s-c)}{bc\cdot ac\cdot ab}}$$
$$=\frac{s(s-b)(s-c)}{abc}$$
$$=\frac{S^2}{s-a}\cdot\frac{1}{4RS}=\frac{r_1}{4R}$$

22 $$rr_1r_2r_3=\frac{S}{s}\cdot\frac{S}{s-a}\cdot\frac{S}{s-b}\cdot\frac{S}{s-c}=\frac{S^4}{S^2}=S^2,$$

$$\cot^2\frac{A}{2}\cot^2\frac{B}{2}\cot^2\frac{C}{2}=\frac{s(s-a)}{(s-b)(s-c)}\times\frac{s(s-b)}{(s-a)(s-c)}\times\frac{s(s-c)}{(s-a)(s-b)}$$

$$=\frac{s^3}{(s-a)(s-b)(s-c)}=\frac{s^4}{S^2}=\frac{S^2}{r^4},$$

$$\therefore S^2=r^4\cot^2\frac{A}{2}\cot^2\frac{B}{2}\cot^2\frac{C}{2}$$

23 $$r_2r_3+r_3r_1+r_1r_2$$

$$=S^2\left[\frac{1}{(s-b)(s-c)}+\frac{1}{(s-c)(s-a)}+\frac{1}{(s-a)(s-b)}\right]$$

$$=S^2\,\frac{s}{(s-a)(s-b)(s-c)}$$

$$=s^2$$

24 $$a\cot A+b\cot B+c\cot C=\frac{a}{\sin A}\cos A+\frac{b}{\sin B}\cdot\cos B+\frac{c}{\sin C}\cos C$$

$$=2R(\cos A+\cos B+\cos C)=2R\left(1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\qquad\text{(Art 114)}$$

$$=2R\left(1+\frac{r}{R}\right)\qquad\text{(Ex 21)}$$

$$=2R+2r$$

25. $$a\cos A+b\cos B+c\cos C=2R\sin A\cos A+2R\sin B\cos B+2R\sin C\cos C$$

$$=R(\sin 2A+\sin 2B+\sin 2C)=4R\sin A\sin B\sin C\qquad\text{(Art 114)}$$

26 $$\frac{rr_1}{r_2r_3}=\frac{S^2}{s(s-a)}-\frac{S^2}{(s-b)(s-c)}=\frac{(s-b)(s-c)}{s(s-a)}=\tan^2\frac{A}{2}$$

27 $$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$=1+\frac{r}{R}\qquad\text{(Ex 21)}$$

28 $$\sin A+\sin B+\sin C=\frac{a}{2R}+\frac{b}{2R}+\frac{c}{2R}=\frac{s}{R}$$

29 $$r_1+r_2=S\left(\frac{1}{s-a}+\frac{1}{s-b}\right)=\sqrt{s(s-a)(s-b)(s-c)}\left\{\frac{2s-a-b}{(s-b)(s-a)}\right\}$$

$$=c\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}=c\cot\frac{C}{2}$$

30 From Ex 27, 28 we have

$$1+\frac{r}{R}=\cos A+\cos B+\cos C$$

$$\frac{s}{R}=\sin A+\sin B+\sin C$$

Square and subtract, therefore

$$\left(1+\frac{r}{R}\right)^2-\frac{s^2}{R^2}=\cos^2 A-\sin^2 A+\quad +2(\cos A\cos B-\sin A\sin B)+$$

$$=\cos 2A+\cos 2B+\cos 2C+2\cos(A+B)+2\cos(A+C)+2\cos(B+C)$$

$$=-1-4\cos A\cos B\cos C-2(\cos A+\cos B+\cos C)\qquad\text{(Ch VIII, Ex 18)}$$

$$=-1-4\cos A\cos B\cos C-2\left(1+\frac{r}{R}\right),$$

$$4\cos A\cos B\cos C=\frac{s^2}{R^2}-\left(1+\frac{r}{R}\right)^2-2\left(1+\frac{r}{R}\right)-1$$

$$=\frac{s^2}{R^2}-\left(2+\frac{r}{R}\right)^2$$

31 $$r=\frac{S}{s},\ 4R=\frac{abc}{S},$$

$$r^2+4Rr=\frac{S^2}{s^2}+\frac{abc}{s}$$

$$=\frac{(s-a)(s-b)(s-c)+abc}{s}$$

$$=s^2-s(a+b+c)+ab+bc+ca$$

$$=\frac{1}{4}\{-(a+b+c)^2+4ab+4bc+4ca\},$$

$$\therefore\ 4r^2+16Rr=2ab+2bc+2ca-a^2-b^2-c^2$$

32 r_1, r_2, r_3 are roots of the equation

$$x^3-(r_1+r_2+r_3)x^2+(r_1r_2+r_1r_3+r_2r_3)x-r_1r_2r_3=0$$

Now $$r_1+r_2+r_3=r+4R,\qquad\text{(Ex 20)}$$

$$r_1r_2+r_1r_3+r_2r_3=s^2,\qquad\text{(Ex 23)}$$

$$r_1r_2r_3=\frac{S^3}{(s-a)(s-b)(s-c)}$$

$$=Ss=rs^2$$

Hence the above equation becomes

$$x^3-(r+4R)x^2+s^2x-s^2r=0$$

33 If the cubic equation be

$$x^3-fx^2+gx-h=0,$$

then

$$f=s-a+s-b+s-c=s,$$

$$g=(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)$$

$$=S^2\left\{\frac{1}{r_1r_2}+\frac{1}{r_2r_3}+\frac{1}{r_3r_1}\right\}$$

$$=\frac{S^2}{r_1r_2r_3}(r_1+r_2+r_3)$$

$$=r(r+4R), \qquad \text{(Ex 20, 22)}$$

$$h=(s-a)(s-b)(s-c)=\frac{S^2}{s}=r^2s$$

The required cubic equation is therefore

$$x^3-sx^2+r(r+4R)x-r^2s=0$$

34 Put $s-y$ for x in the equation of the preceding example Then when $x=s-a$, $y=a$ Hence a, b, c are roots of the equation

$$(s-y)^3-s(s-y)^2+(r^2+4Rr)(s-y)-r^2s=0,$$

or

$$-y(s^2-2sy+y^2)+(r^2+4Rr)s-(r^2+4Rr)y-r^2s=0,$$

or

$$-y^3+2sy^2-(r^2+s^2+4Rr)y+4Rrs=0,$$

or

$$y^3-2sy^2+(r^2+s^2+4Rr)y-4Rrs=0$$

35

$$s-a=AI\cos\frac{1}{2}A,$$

$$AI^2=\frac{(s-a)^2}{\cos^2\frac{1}{2}A}=\frac{bc(s-a)^2}{s(s-a)}=\frac{bc(s-a)}{s},$$

$$a\ AI^2+b\ BI^2+c\ CI^2=\frac{abc}{s}(s-a+s-b+s-c)$$

$$=abc$$

36

$$AI_1^2=\frac{r_1^2}{\sin^2\frac{A}{2}}=\frac{r_1^2bc}{(s-b)(s-c)},$$

$$BI_1^2=\frac{r_1^2}{\cos^2\frac{B}{2}}=\frac{r_1^2ca}{s(s-b)},$$

$$CI_1^2=\frac{r_1^2}{\cos^2\frac{C}{2}}=\frac{r_1^2ab}{s(s-c)}.$$

$$a\,.\,AI_1^2-b\,.\,BI_1^2-c\,.\,CI_1^2=r_1^2\,.\,abc\,.\left\{\frac{1}{(s-b)(s-c)}-\frac{1}{s(s-b)}-\frac{1}{s(s-c)}\right\}$$

$$=r_1^2\,.\,abc\;\frac{s-s+c-s+b}{s(s-b)(s-c)}$$

$$=\frac{S^2}{(s-a)^2}\,.\,abc\,.\,\frac{s-a}{s(s-b)(s-c)}$$

$$=abc$$

37 The radius of the circle inscribed in the triangle ABC

$$=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$=2R'\cos I_1\cos I_2\cos I_3, \qquad \text{(Ex 11)}$$

where R' is the radius of the circumcircle of $I_1I_2I_3$ Hence by Art 253 the radius of the circle inscribed in DEF is $2R\cos A\cos B\cos C$

This result may be found at once from the figure (Art 253) P is the centre of the inscribed circle of the triangle DEF, let ρ be the radius Then

$$\rho=PD\sin PDE=PD\sin PCE=PD\cos A$$

$$=2R\cos B\cos C\,\cos A \qquad \text{(Ex 9)}$$

38 $$\frac{PA}{BC}+\frac{2R\,BC}{CA\,.\,AB}=\frac{2R\cos A}{a}+\frac{2Ra}{bc}$$

$$=\frac{2R}{abc}(a^2+bc\cos A)$$

$$=\frac{2R}{abc}\left(a^2+\frac{-a^2+b^2+c^2}{2}\right)$$

$$=\frac{R(a^2+b^2+c^2)}{abc}$$

Similarly

$$\frac{PB}{CA}+\frac{2R\,.\,CA}{AB\,BC}=\frac{R(a^2+b^2+c^2)}{abc}=\frac{PC}{AB}+\frac{2R\,AB}{BC\,CA}$$

39 Let ρ be the radius of the inscribed circle Since R is the radius of the circumcircle we have from Ex. 21

$$\rho=4R\sin\frac{1}{2}X\sin\frac{1}{2}Y\sin\frac{1}{2}Z$$

Now $$X=\angle AXY+\angle AXZ=\angle ABY+\angle ACZ=\frac{1}{2}B+\frac{1}{2}C,$$

$$\rho=4R\sin\frac{B+C}{4}\sin\frac{C+A}{4}\sin\frac{A+B}{4}.$$

40 A circle passes round $QLCM$, LM is a chord of this circle subtending an angle C at the circumference, and QC is the diameter, therefore

$$LM = QC \sin C$$

Similarly $$MN = QA \sin A,\quad LN = QB \sin B,$$

$$MN + LN + LM = QA \sin A + QB \sin B + QC \sin C$$

41. Let AP produced meet the circumcircle again in X

$$\angle CBX = \angle CAD = 90^\circ - C = \angle CBP.$$

Thus $$DX = PD$$

Now $$PA = 2R \cos A,\quad PX = 2PD = 4R \cos B \cos C$$

And $$PA \cdot PX = R^2 - OP^2, \qquad \text{(Euc III 25)}$$

$$OP^2 = R^2 - 8R^2 \cos A \cos B \cos C$$

42 Let AXX' be the bisector of the angle A, cutting BC in X and the circumcircle at X'

$$\frac{1}{2} bc \sin A = \text{area } ABC = \text{area } ABX + \text{area } ACX$$

$$= \frac{1}{2} pc \sin \frac{1}{2} A + \frac{1}{2} pb \sin \frac{1}{2} A = \frac{1}{2} p \sin \frac{1}{2} A\,(b+c),$$

$$\frac{\cos \frac{1}{2} A}{p} = \frac{1}{2}\left(\frac{1}{b} + \frac{1}{c}\right)$$

Similarly $$\frac{\cos \frac{1}{2} B}{q} = \frac{1}{2}\left(\frac{1}{c} + \frac{1}{a}\right), \quad \frac{\cos \frac{1}{2} C}{t} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right),$$

$$\frac{\cos \frac{1}{2} A}{p} + \frac{\cos \frac{1}{2} B}{q} + \frac{\cos \frac{1}{2} C}{t} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

Again the angle subtended by AX' at the circumference

$$= \angle ABX' = \angle ABC + \angle CBX' = B + \frac{1}{2} A,$$

$$p' = 2R \sin\left(B + \frac{1}{2} A\right),$$

$$p' \cos \frac{1}{2} A = 2R \sin\left(B + \frac{1}{2} A\right) \cos \frac{1}{2} A$$

$$= R \{\sin (A+B) + \sin B\}$$

$$= R \sin C + R \sin B = \frac{1}{2}(c+b)$$

Similarly $$q' \cos \frac{1}{2} B = \frac{1}{2}(a+c), \quad t' \cos \frac{1}{2} C = \frac{1}{2}(a+b),$$

$$p' \cos \frac{1}{2} A + q' \cos \frac{1}{2} B + t' \cos \frac{1}{2} C = a + b + c$$

43 $$\angle ABX = \angle ABC + \angle CBX = \angle ABC + \angle CAD$$
$$= B + 90^0 - C$$
$$AX = 2R \sin (B + 90^0 - C) = 2R \cos (B - C)$$
$$AD = AB \sin B = 2R \sin C \sin B,$$
$$\frac{AX}{AD} = \frac{\cos (B-C)}{\sin B \sin C} = 1 + \cot B \cot C,$$
$$\frac{AX}{AD} + \frac{BY}{BE} + \frac{CZ}{CF} = 3 + \cot B \cot C + \cot C \cot A + \cot A \cot B$$
$$= 4$$

44 HK is parallel to AB (fig Art 253), hence
$$\angle KHD = \angle BAD = 90^0 - B$$
Thus KD, a chord of the nine-points circle, subtends at the circumference an angle $90^0 - B$
$$KD = R \sin (90^0 - B) = R \cos B$$
Again $$2R_1 = \frac{KD}{\sin KBD} \text{ (Art 252)} = \frac{R \cos B}{\cos C}$$
Similarly $$2R_2 = \frac{R \cos C}{\cos B},$$
$$4R_1R_2 = R^2$$

45 $$\angle BOC = 2A,$$
$$R_1 = \frac{a}{2 \sin 2A}, \qquad \text{(Art 252)}$$
$$\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} = 2 (\sin 2A + \sin 2B + \sin 2C)$$
$$= 8 \sin A \sin B \sin C \qquad \text{(Ch viii, Ex 33)}$$
$$= 8 \cdot \frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} = \frac{abc}{R^3}$$

46 $$\angle BIC = 180^0 - \frac{1}{2}(B + C),$$
$$\rho_1 = \frac{a}{2 \sin \left(180^0 - \frac{B+C}{2}\right)} = \frac{a}{2 \sin \frac{B+C}{2}} = \frac{a}{2 \cos \frac{A}{2}},$$
$$\rho_1\rho_2\rho_3 = \frac{abc}{8} \cdot \frac{1}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}$$
$$= \frac{abc}{8} \sqrt{\left\{\frac{bc}{s(s-a)} \cdot \frac{ac}{s(s-b)} \cdot \frac{ab}{s(s-c)}\right\}}$$
$$= \frac{abc \times abc}{8S \times s} = 2\left(\frac{abc}{4S}\right)^2 \times \frac{S}{s}$$
$$= 2R^2 r$$

47. Since area $DEF=2S\cos A\cos B\cos C$, we have by Art 253,

$$\text{area } ABC=2\text{ area } I_1I_2I_3\ \cos I_1\cos I_2\cos I_3$$

$$=2\text{ area } I_1I_2I_3\ \sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C, \qquad \text{(Ex 11)}$$

$$\text{area } I_1I_2I_3=\frac{1}{2}S\operatorname{cosec}\frac{1}{2}A\operatorname{cosec}\frac{1}{2}B\operatorname{cosec}\frac{1}{2}C$$

48 A, B, C are the feet of the perpendiculars drawn from I, I_3, I_2 to the sides of the triangle II_2I_3 The circumcircle of ABC is therefore the nine-points circle of the triangle II_2I_3; hence the radius (R) of the circumcircle of ABC is half the radius of the circumcircle of II_2I_3

49 Let ABC denote the right-angled isosceles triangle where C is the right angle Let F be the middle point of AB, let D be on BC, and E on AC, such that DE is parallel to AB, and the triangle DEF is equilateral

Then the angle $DEC=45^0$, and the angle $DEF=60^0$, therefore the angle $AEF=75^0$. Now $\dfrac{FE}{FA}=\dfrac{\sin FAE}{\sin FEA}=\dfrac{\sin 45^0}{\sin 75^0}$,

therefore $FE=\dfrac{FA\sin 45^0}{\sin 75^0}=\dfrac{a}{\sqrt{2}}\ \dfrac{\sin 45^0}{\cos 15^0}=\dfrac{a}{2}\ \dfrac{1}{\cos 15^0}=\dfrac{a\sin 15^0}{2\cos 15^0\sin 15^0}$

$=\dfrac{a\sin 15^0}{\sin 30^0}=2a\sin 15^0$ Therefore the area of the equilateral triangle

$$=\frac{1}{2}(2a\sin 15^0)^2\sin 60^0=2a^2\sin^2 15^0\sin 60^0$$

50 Let a denote one side of the right-angled triangle, and $a+h$ the other side, then the hypotenuse $=\sqrt{a^2+(a+h)^2}=\sqrt{h^2+2a(a+h)}$

But $S=$ half the product of the sides $=\frac{1}{2}a(a+h)$, therefore $4S=2a(a+h)$ Thus the hypotenuse $=\sqrt{(h^2+4S)}$, and the hypotenuse is a diameter of the circumscribing circle

51 The angle $ABO=$ the angle $BAO=\dfrac{\pi}{2}-C$, and therefore the angle $BOD=\pi-2C$, the angle $OBD=\dfrac{\pi}{2}-A$,

therefore the angle $BDO=2C+A-\dfrac{\pi}{2}=A+C+B+C-B-\dfrac{\pi}{2}=\dfrac{\pi}{2}+C-B$

$$\text{Thus}\qquad \frac{DO}{BO}=\frac{\sin DBO}{\sin BDO}=\frac{\sin\left(\frac{\pi}{2}-A\right)}{\sin\left(\frac{\pi}{2}+C-B\right)}=\frac{\cos A}{\cos(C-B)},$$

and $BO=AO$, therefore $DO\cos(B-C)=AO\cos A$.

52 From A draw AD perpendicular to BC, and produce AD to meet the circumference of the circle at L

Then the angle ALB = the angle $ACB = C$,

$$\alpha = DL = BD \cot ALB = BD \cot C$$
$$= \frac{c \cos B \cos C}{\sin C} = \frac{a \cos B \cos C}{\sin A},$$

therefore $$\frac{a}{\alpha} = \frac{\sin A}{\cos B \cos C} = \frac{\sin (B+C)}{\cos B \cos C} = \tan B + \tan C$$

Similarly $$\frac{b}{\beta} = \tan A + \tan C, \text{ and } \frac{c}{\gamma} = \tan C + \tan A$$

Therefore $$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 2 (\tan A + \tan B + \tan C)$$

53 The area of the inscribed circle is to the area of the triangle as πr^2 is to S, that is, as π is to $\frac{S}{r^2}$. Thus we have to shew that

$$\frac{S}{r^2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$$

Now

$$\cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} \times \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} \times \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$
$$= \frac{s\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}} = \frac{s^2}{S} = S \times \frac{s^2}{S^2} = \frac{S}{r^2}$$

54 Let the triangle constructed on BC have its vertex at L, let that constructed on CA have its vertex at M, and that constructed on AB have its vertex at N

Take the diagram of Art 252 The triangle CLB will be equal to the triangle COB in all respects, therefore the angle BCL = the angle $OCB = \frac{\pi}{2} - A$

In the same manner the angle $ACM = \frac{\pi}{2} - B$,

therefore the angle $LCM = \frac{\pi}{2} - A + \frac{\pi}{2} - B + C = 2C$

Then $(LM)^2 = R^2 + R^2 - 2R^2 \cos 2C = 2R^2 (1 - \cos 2C) = 4R^2 \sin^2 C$,

therefore $$LM = 2R \sin C = c$$

In a similar manner we find that $MN = a$, and $NL = b$ Thus the triangle LMN is in all respects equal to the triangle ABC

55. $a \cos A + b \cos B + c \cos C = 2R \sin A \cos A + 2R \sin B \cos B + 2R \sin C \cos C$
$= R (\sin 2A + \sin 2B + \sin 2C) = 4R \sin A \sin B \sin C$, by Art 114

56
$$OA'^2 = R^2\cos^2 A = \frac{a^2}{4\sin^2 A}\cos^2 A = \frac{a^2}{4}\cot^2 A,$$
$$OB'^2 = R^2\cos^2 B = \frac{b^2}{4\sin^2 B}\cos^2 B = \frac{b^2}{4}\cot^2 B,$$
$$OC'^2 = R^2\cos^2 C = \frac{c^2}{4\sin^2 C}\cos^2 C = \frac{c^2}{4}\cot^2 C,$$

therefore $4(OA'^2 + OB'^2 + OC'^2) = a^2\cot^2 A + b^2\cot^2 B + c^2\cot^2 C$

57 Take the diagram of Art 248. The circle which is to be drawn will have its centre, and its point of contact with the circle already drawn, on the straight line OA Thus the length of $OA = r + r_a + r_a \operatorname{cosec}\frac{A}{2}$, and this distance also $= r\operatorname{cosec}\frac{A}{2}$, therefore

$$r_a\left(1 + \operatorname{cosec}\frac{A}{2}\right) = r\left(\operatorname{cosec}\frac{A}{2} - 1\right),$$

therefore
$$r_a = \frac{r\left(1 - \sin\frac{A}{2}\right)}{1 + \sin\frac{A}{2}} = \frac{r\left(\cos\frac{A}{4} - \sin\frac{A}{4}\right)^2}{\left(\cos\frac{A}{4} + \sin\frac{A}{4}\right)^2}.$$

58 By Example 57 we have

$$r_a r_b = \frac{r^2\left(\cos\frac{A}{4} - \sin\frac{A}{4}\right)^2\left(\cos\frac{B}{4} - \sin\frac{B}{4}\right)^2}{\left(\cos\frac{A}{4} + \sin\frac{A}{4}\right)^2\left(\cos\frac{B}{4} + \sin\frac{B}{4}\right)^2},$$

therefore
$$\sqrt{(r_a r_b)} = \frac{r\left(\cos\frac{A}{4} - \sin\frac{A}{4}\right)\left(\cos\frac{B}{4} - \sin\frac{B}{4}\right)}{\left(\cos\frac{A}{4} + \sin\frac{A}{4}\right)\left(\cos\frac{B}{4} + \sin\frac{B}{4}\right)}$$
$$= \frac{r\left(\cos\frac{A}{4} - \sin\frac{A}{4}\right)\left(\cos\frac{B}{4} - \sin\frac{B}{4}\right)\left(\cos\frac{C}{4} + \sin\frac{C}{4}\right)}{\left(\cos\frac{A}{4} + \sin\frac{A}{4}\right)\left(\cos\frac{B}{4} + \sin\frac{B}{4}\right)\left(\cos\frac{C}{4} + \sin\frac{C}{4}\right)}$$
$$= \frac{r\cos\frac{A+\pi}{4}\cos\frac{B+\pi}{4}\cos\frac{C-\pi}{4}}{\cos\frac{A-\pi}{4}\cos\frac{B-\pi}{4}\cos\frac{C-\pi}{4}}$$
$$= \frac{r\left(\cos\frac{A}{2} + \cos\frac{B}{2} - \cos\frac{C}{2}\right)}{\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}},$$ by Examples VIII 20 and 21

Similar expressions can be found for $\sqrt{(r_b r_c)}$ and $\sqrt{(r_c r_a)}$, and the sum of the three expressions $= r$

59 $$\frac{1}{\sqrt{A}} = \frac{1}{\sqrt{(\pi r^2)}} = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{r} = \frac{1}{\sqrt{\pi}} \cdot \frac{s}{S}$$

Similarly $\dfrac{1}{\sqrt{A_1}} = \dfrac{1}{\sqrt{\pi}} \cdot \dfrac{s-a}{S}$, $\quad \dfrac{1}{\sqrt{A_2}} = \dfrac{1}{\sqrt{\pi}} \cdot \dfrac{s-b}{S}$, $\quad \dfrac{1}{\sqrt{A_3}} = \dfrac{1}{\sqrt{\pi}} \cdot \dfrac{s-c}{S}$,

therefore

$$\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}} = \frac{1}{\sqrt{\pi}}\left(\frac{s-a}{S} + \frac{s-b}{S} + \frac{s-c}{S}\right) = \frac{1}{\sqrt{\pi}} \cdot \frac{3s-a-b-c}{S} = \frac{1}{\sqrt{\pi}} \cdot \frac{s}{S}.$$

60. Suppose a, b, c to be in Arithmetical Progression, so that $2b = a + c$

The perpendicular on the mean side from the opposite angle

$$= a \sin C = \frac{ab \sin C}{b} = \frac{2S}{b}$$

The radius of the circle which touches the mean side and the other two sides produced $= \dfrac{S}{s-b} = \dfrac{2S}{a+c-b} = \dfrac{2S}{b}$

The radius of the inscribed circle $= \dfrac{S}{s} = \dfrac{2S}{a+b+c} = \dfrac{2S}{3b}$

The first and the second of these are each three times the third

61 See Ex. 37

62 Let a_1, b_1, c_1 be the sides of one triangle, S_1 its area; let a_2, b_2, c_2 be the sides of the other triangle, S_2 its area

Then, by hypothesis, $\dfrac{S_1}{b_1+c_1-a_1} = \dfrac{S_2}{a_2+c_2-b_2}$, therefore

$$\frac{S_1}{S_2} = \frac{b_1+c_1-a_1}{a_2+c_2-b_2} = \frac{a_1\dfrac{\sin B}{\sin A} + \dfrac{a_1 \sin C}{\sin A} - a_1}{a_2 + \dfrac{a_2 \sin C}{\sin A} - \dfrac{a_2 \sin B}{\sin A}}$$

$$= \frac{a_1}{a_2} \cdot \frac{\sin B + \sin C - \sin A}{\sin A + \sin C - \sin B}.$$

But the areas of similar triangles are as the squares of their homologous sides, thus $\dfrac{S_1}{S_2} = \dfrac{a_1^2}{a_2^2}$, therefore, finally,

$$\frac{a_1}{a_2} = \frac{\sin B + \sin C - \sin A}{\sin A + \sin C - \sin B}.$$

63 We have

$$r'=\frac{\text{area of } I_1I_2I_3}{\text{semiperimeter of } I_1I_2I_3}=\frac{8R^2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}{2R\left(\cos\frac{1}{2}A+\cos\frac{1}{2}B+\cos\frac{1}{2}C\right)},$$

by Examples 17 and 13, therefore by Ex 21

$$r'=\frac{r\cot\frac{1}{2}A\cot\frac{1}{2}B\cot\frac{1}{2}C}{\cos\frac{1}{2}A+\cos\frac{1}{2}B+\cos\frac{1}{2}C}$$

64. By Art 248

$$rs=\text{area } ABC,$$

$$r's'=\text{area } I_1I_2I_3,$$

$$\frac{rs}{r's'}=\frac{S}{abc}\frac{2r}{}, \text{ by Example 17}$$

$$=\frac{2S^2}{abcs}.$$

And $2\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$

$$=2\sqrt{\frac{(s-b)(s-c)}{bc}\times\frac{(s-c)(s-a)}{ca}\times\frac{(s-a)(s-b)}{ab}}$$

$$=\frac{2S^2}{abcs}=\frac{rs}{r's'}$$

65 We have

$$\alpha=r\operatorname{cosec}\frac{A}{2},\qquad \alpha_1=r_1\operatorname{cosec}\frac{A}{2},$$

$$\beta=r\operatorname{cosec}\frac{B}{2},\qquad \beta_1=r_2\operatorname{cosec}\frac{B}{2},$$

$$\gamma=r\operatorname{cosec}\frac{C}{2},\qquad \gamma_1=r_3\operatorname{cosec}\frac{C}{2},$$

therefore $\alpha\beta\gamma\alpha_1\beta_1\gamma_1=r^3r_1r_2r_3\operatorname{cosec}^2\frac{A}{2}\operatorname{cosec}^2\frac{B}{2}\operatorname{cosec}^2\frac{C}{2}$

$$=\frac{S^3}{s^3}\times\frac{S^3}{(s-a)(s-b)(s-c)}\times\frac{bc}{(s-c)(s-b)}\times\frac{ca}{(s-a)(s-c)}\times\frac{ab}{(s-a)(s-b)}$$

$$=\frac{S^6a^2b^2c^2}{S^6}=a^2b^2c^2$$

66 $$\frac{bc}{\alpha_1^2}+\frac{ca}{\beta_1^2}+\frac{ab}{\gamma_1^2}=\frac{bc\sin^2\frac{A}{2}}{r_1^2}+\frac{ca\sin^2\frac{B}{2}}{r_2^2}+\frac{ab\sin^2\frac{C}{2}}{r_3^2}$$

$$=\frac{1}{s^2}\left\{bc\cos^2\frac{A}{2}+ca\cos^2\frac{B}{2}+ab\cos^2\frac{C}{2}\right\}, \text{ by Art 251,}$$

$$=\frac{1}{s^2}\{s(s-a)+s(s-b)+s(s-c)\}=\frac{1}{s}(s-a+s-b+s-c)$$

$$=\frac{1}{s}(3s-a-b-c)=1$$

67 $$\alpha^2\left(\frac{1}{c}-\frac{1}{b}\right)+\beta^2\left(\frac{1}{a}-\frac{1}{c}\right)+\gamma^2\left(\frac{1}{b}-\frac{1}{a}\right)$$

$$=\frac{r^2(b-c)}{bc\sin^2\frac{A}{2}}+\frac{r^2(c-a)}{ca\sin^2\frac{B}{2}}+\frac{r^2(a-b)}{ab\sin^2\frac{C}{2}}$$

$$=\frac{r^2}{abc}\left\{\frac{a(b-c)}{\sin^2\frac{A}{2}}+\frac{b(c-a)}{\sin^2\frac{B}{2}}+\frac{c(a-b)}{\sin^2\frac{C}{2}}\right\}$$

$$=\frac{4Rr^2}{abc}\left\{(b-c)\cot\frac{A}{2}+(c-a)\cot\frac{B}{2}+(a-b)\cot\frac{C}{2}\right\}$$

$=0$, by Example XIII 29

68 $$\frac{b-c}{a\alpha_1^2}+\frac{c-a}{b\beta_1^2}+\frac{a-b}{c\gamma_1^2}$$

$$=\frac{b-c}{ar_1^2}\sin^2\frac{A}{2}+\frac{c-a}{br_2^2}\sin^2\frac{B}{2}+\frac{a-b}{cr_3^2}\sin^2\frac{C}{2}$$

$$=\frac{1}{s^2}\left\{\frac{b-c}{a}\cos^2\frac{A}{2}+\frac{c-a}{b}\cos^2\frac{B}{2}+\frac{a-b}{c}\cos^2\frac{C}{2}\right\}, \text{ by Art 251,}$$

$$=\frac{1}{4Rs^2}\left\{(b-c)\cot\frac{A}{2}+(c-a)\cot\frac{B}{2}+(a-b)\cot\frac{C}{2}\right\}$$

$=0$, by Example XIII. 29

69 Let $C'A'$ intersect AB at E and CB at F

The angle $A'FC$ is equal to the sum of the angles $FC'C$ and FCC', that is to the sum of the angles $A'AC$ and FCC', that is to $\frac{1}{2}A+\frac{1}{2}C$, the angle $BCA'=$the angle $BAA'=\frac{1}{2}A$.

Thus $$\frac{FA'}{CA'}=\frac{\sin\frac{1}{2}A}{\sin\frac{1}{2}(A+C)}=\frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}B}.$$

Let R be the radius of the circle, then

$$A'C=2R\sin\frac{1}{2}A, \text{ therefore } FA'=\frac{2R\sin^2\frac{A}{2}}{\cos\frac{B}{2}}$$

In the same manner $$EC'=\frac{2R\sin^2\frac{C}{2}}{\cos\frac{B}{2}}$$

And $$A'C'=2R\sin\frac{1}{2}(A+C)=2R\cos\frac{B}{2}$$

Therefore $$EF=2R\cos\frac{B}{2}-\frac{2R\left(\sin^2\frac{A}{2}+\sin^2\frac{C}{2}\right)}{\cos\frac{B}{2}}$$

$$=\frac{2R}{\cos\frac{B}{2}}\left\{\cos^2\frac{B}{2}-\sin^2\frac{A}{2}-\sin^2\frac{C}{2}\right\}=\frac{R}{\cos\frac{B}{2}}\{1+\cos B-(1-\cos A)-(1-\cos C)\}$$

$$=\frac{R}{\cos\frac{B}{2}}(\cos A+\cos B+\cos C-1)=\frac{2R}{\cos\frac{B}{2}}\times 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}, \text{ by Art 114.}$$

70 Take the diagram of Art 248, draw FD, DE, and EF

The angle $FDB=\frac{1}{2}(\pi-B)$, the angle $EDC=\frac{1}{2}(\pi-C)$, therefore the angle $FDE=\frac{1}{2}(B+C)$ Similarly the angle $DEF=\frac{1}{2}(C+A)$, and the angle $EFD=\frac{1}{2}(A+B)$

Suppose A, B, C in ascending order of magnitude, then

$$\frac{1}{2}(A+B),\qquad \frac{1}{2}(A+C),\qquad \frac{1}{2}(B+C),$$

are in ascending order of magnitude, and

$$\frac{1}{2}(B+C)-\frac{1}{2}(A+B)=\frac{1}{2}(C-A)$$

Thus the difference between the greatest and least angles of the first derived triangle is *half* the difference between the greatest and least angles

of the original triangle In like manner the difference between the greatest and least angles of the second derived triangle is *half* the difference between the greatest and least angles of the first derived triangle, and therefore a *fourth* of the difference between the greatest and least angles of the original triangle Proceeding in this way we see that the triangles thus formed ultimately become equilateral

71 Let A, B, C, D, E be five consecutive angles of the hexagon, draw AC, BD, CE, let AC and BD intersect at P, and let BD and CE intersect at Q Then PQ is the side of the second regular hexagon

The angle DBC is half of the angle which DC would subtend at the centre of the circle circumscribing the regular hexagon, and is therefore $\frac{\pi}{6}$ Similarly the angle ACB is $\frac{\pi}{6}$.

$$\text{Then } \frac{PC}{BC}=\frac{\sin\frac{\pi}{6}}{\sin\left(\pi-\frac{2\pi}{6}\right)}=\frac{\sin\frac{\pi}{6}}{\sin\frac{2\pi}{6}}=\frac{1}{2\cos\frac{\pi}{6}}, \text{ therefore } PC=\frac{BC}{2\cos\frac{\pi}{6}}$$

$$\text{And} \qquad PQ=2PC\sin\frac{1}{2}PCQ=2PC\sin\frac{\pi}{6}=BC\tan\frac{\pi}{6}$$

Thus $PQ=\frac{BC}{\sqrt{3}}$ And the areas of similar polygons are as the squares of their homologous sides, so that if S denote the area of the first hexagon the area of the second is $\frac{S}{3}$ In like manner the area of the next hexagon is $\frac{1}{3}$ of $\frac{S}{3}$, that is $\frac{S}{9}$, and so on Hence the sum of the areas of all the derived figures is $\frac{S}{3}+\frac{S}{9}+\frac{S}{27}+\dots$, that is $\frac{1}{3}\frac{S}{1-\frac{1}{3}}$, that is $\frac{S}{2}$

72 Suppose that the original figure instead of being a hexagon is a regular polygon of n sides Proceed as before and we have

$$\frac{PC}{BC}=\frac{\sin\frac{\pi}{n}}{\sin\left(\pi-\frac{2\pi}{n}\right)}=\frac{\sin\frac{\pi}{n}}{\sin\frac{2\pi}{n}}=\frac{1}{2\cos\frac{\pi}{n}}$$

$$\text{Then} \qquad PQ=2PC\sin\frac{1}{2}PCQ,$$

and the angle $PCQ=(n-4)\frac{\pi}{n}$, therefore $PQ=2PC\sin(n-4)\frac{\pi}{2n}$

$$=2PC\sin\left(\frac{\pi}{2}-\frac{2\pi}{n}\right)=2PC\cos\frac{2\pi}{n}=BC\frac{\cos\frac{2\pi}{n}}{\cos\frac{\pi}{n}}$$

Thus the area of the second polygon is $\dfrac{S\cos^2\dfrac{2\pi}{n}}{\cos^2\dfrac{\pi}{n}}$,

and $\Sigma = S\{m+m^2+m^3+ \quad \}$ where m stands for $\dfrac{\cos^2\dfrac{2\pi}{n}}{\cos^2\dfrac{\pi}{n}}$,

$$\text{thus}\quad \Sigma=\frac{Sm}{1-m}=\frac{S\cos^2\dfrac{2\pi}{n}}{\cos^2\dfrac{\pi}{n}-\cos^2\dfrac{2\pi}{n}}=\frac{S\cos^2\dfrac{2\pi}{n}}{\sin^2\dfrac{2\pi}{n}-\sin^2\dfrac{\pi}{n}}=\frac{S\cos^2\dfrac{2\pi}{n}}{\sin\dfrac{3\pi}{n}\sin\dfrac{\pi}{n}}.$$

If $n=3$ this becomes infinite, for $\sin\pi=0$, in this case the original figure is a triangle, and the second figure is the *same* triangle, and so on thus the sum of the areas is infinite

If $n=4$ the expression vanishes, for $\cos\dfrac{2\pi}{4}=0$, in this case the original figure is a square, and the second figure is only a point, and so on thus the sum of the areas is zero.

73. In order that it may be possible to inscribe a circle within a quadrilateral the sum of one pair of opposite sides must be equal to the sum of the other pair Now if we take the point O of the diagram of Art 248, we see that the condition is satisfied for $OFAE$, $OECD$, and $ODBF$, since $OE+AF=OF+AE$, and so on We have then to shew that no other point but O can be taken

Take any other point P, from it draw PM perpendicular to AC and PN perpendicular to AB The centre of a circle inscribed within $PMAN$ must be on the straight line which bisects the angle A, and also on the straight line which bisects the angle NPM, but unless P is on AO, the latter straight line will be *parallel* to AO, the former straight line, and therefore cannot meet it. Thus P must be on AO, similarly it must be on BO and on CO.

Then take the circle inscribed in $OFAE$, and draw perpendiculars from the centre on the sides of the quadrilateral Thus we have

$$\rho_1(AF+FO+OE+EA)=\text{twice the area of } OFAE,$$

therefore
$$\rho_1\left\{\rho+\rho\cot\frac{A}{2}\right\}=\rho^2\cot\frac{A}{2},$$

therefore
$$\rho_1=\frac{\rho\cot\dfrac{A}{2}}{1+\cot\dfrac{A}{2}},\quad\text{therefore }\frac{1}{\rho_1}=\frac{1+\cot\dfrac{A}{2}}{\rho\cot\dfrac{A}{2}}$$

Similarly
$$\frac{1}{\rho_2}=\frac{1+\cot\dfrac{B}{2}}{\rho\cot\dfrac{B}{2}}$$

Thus $\left(\frac{1}{\rho_1}-\frac{1}{\rho}\right)\left(\frac{1}{\rho_2}-\frac{1}{\rho}\right)=\frac{1}{\rho^2\cot\frac{A}{2}\cot\frac{B}{2}}=\frac{1}{\rho^2}\tan\frac{A}{2}\tan\frac{B}{2}$

In this manner we find that the proposed expression

$$=\frac{1}{\rho^2}\left\{\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}\right\}$$

$$=\frac{1}{\rho^2}, \text{ by Example VIII 25}$$

74 As in Example 57 we shall find that the radii of the circles successively inscribed in the angle A are lr, l^2r, l^3r, where

$$l=\frac{1-\sin\frac{A}{2}}{1+\sin\frac{A}{2}}.$$

Hence the sum of the areas of all these circles is $\pi(l^2r^2+l^4r^2+l^6r^2+\quad)$,

$$\text{that is } \frac{\pi l^2r^2}{1-l^2}, \qquad \text{that is } \frac{\pi\left(1-\sin\frac{A}{2}\right)^2r^2}{4\sin\frac{A}{2}},$$

$$\text{that is } \frac{\pi\left(1-\cos\frac{B+C}{2}\right)^2r^2}{4\sin\frac{A}{2}}, \qquad \text{that is } \pi r^2\sin^4\frac{B+C}{4}\operatorname{cosec}\frac{A}{2}$$

Similarly we find the areas of the circles inscribed within the angles B and C Thus the sum of all the areas is

$$\pi r^2\left\{\sin^4\frac{B+C}{4}\operatorname{cosec}\frac{A}{2}+\sin^4\frac{C+A}{4}\operatorname{cosec}\frac{B}{2}+\sin^4\frac{A+B}{4}\operatorname{cosec}\frac{C}{2}\right\}$$

75 Since the angles at B' and C' are right angles it will follow that A will be on the circumference of the circle which is described round $PB'C'$, and that PA is a diameter of the circle Let O_1 denote the centre of the circle, then $PO_1=\frac{1}{2}PA$

In a similar manner if O_2 is the centre of the circle round $PC'A'$, and O_3 the centre of the circle round $PA'B'$, we have

$$PO_2=\frac{1}{2}PB, \text{ and } PO_3=\frac{1}{2}PC.$$

Then in the triangle PO_2O_3 we have

$$O_2O_3{}^2=PO_2{}^2+PO_3{}^2-2PO_2PO_3\cos O_2PO_3;$$

and in the triangle PBC we have

$$BC^2=PB^2+PC^2-2PB\,.\,PC\cos BPC$$

Hence $O_2O_3=\frac{1}{2}BC$ Or this might be obtained by Euclid VI. 2, and VI 4

Similarly $O_3O_1=\frac{1}{2}CA$, and $O_1O_2=\frac{1}{2}AB$ Thus the area of $O_1O_2O_3$ is one-fourth of the area of ABC

76 Let r_1, r_2, r_3 denote the radii of the circles, then the sides of the triangle are respectively r_2+r_3, r_3+r_1, and r_1+r_2 Thus

$$s=r_1+r_2+r_3,\quad s-a=r_1,\quad s-b=r_2,\quad s-c=r_3$$

Therefore $$S^2=(r_1+r_2+r_3)\,r_1r_2r_3$$

77 Suppose a, b, c in Geometrical Progression, so that $b^2=ac$; let p_1, p_2, p_3 denote the perpendiculars from the opposite angles on a, b, c respectively

Then $\frac{1}{2}p_1a=S$, so that $p_1=\frac{2S}{a}$, similarly $p_2=\frac{2S}{b}$, and $p_3=\frac{2S}{c}$

Let A_1, B_1, C_1 be the angles opposite p_1, p_2, p_3 respectively in the new triangle

Then $$\cos A_1=\frac{p_2^2+p_3^2-p_1^2}{2p_2p_3}=\frac{\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a^2}}{\frac{2}{bc}}=\frac{\frac{b^2+c^2}{b^2c^2}-\frac{1}{a^2}}{\frac{2}{bc}}$$

$$=\frac{a^2(b^2+c^2)-b^2c^2}{2a^2bc}=\frac{b^2(a^2-c^2)+a^2c^2}{2a^2bc}=\frac{a^2-c^2+ac}{2ab}$$

$$=\frac{a^2+b^2-c^2}{2ab}=\cos C$$

Thus $A_1=C$ Similarly $C_1=A$. Therefore $B_1=B$.

78 Here $\alpha=\frac{a}{c\sin B}=\frac{\sin A}{\sin B\sin C}$.

Similarly $\beta=\frac{\sin B}{\sin C\sin A}$, and $\gamma=\frac{\sin C}{\sin A\sin B}$.

Therefore $2(\beta\gamma+\gamma\alpha+\alpha\beta)-\alpha^2-\beta^2-\gamma^2=$ the product of $\frac{1}{\sin^2 A\sin^2 B\sin^2 C}$ into $\{2\sin^2 B\sin^2 C+2\sin^2 C\sin^2 A+2\sin^2 A\sin^2 B-\sin^4 A-\sin^4 B-\sin^4 C\}$

The expression within brackets is equal to

$(\sin A+\sin B+\sin C)(\sin A+\sin B-\sin C)(\sin A-\sin B+\sin C)$ $(\sin B+\sin C-\sin A)$,

as we know from a similar process in Art 218

Then, by Examples VIII 16 and 17, we obtain

$$4^4 \sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}, \text{ that is, } 4\sin^2 A\sin^2 B\sin^2 C.$$

Hence $$2(\beta\gamma+\gamma\alpha+\alpha\beta)-\alpha^2-\beta^2-\gamma^2=4,$$

and therefore $$\alpha^2+\beta^2+\gamma^2-2(\beta\gamma+\gamma\alpha+\alpha\beta)+4=0.$$

79 Let P, Q, R be the centres of the equilateral triangles described on BC, CA, AB respectively.

Then $$PQ^2=PC^2+QC^2-2PC.QC\cos PCQ;$$

also $$PC=\frac{a}{\sqrt{3}}, \text{ and } QC=\frac{b}{\sqrt{3}}.$$

Thus
$$\begin{aligned}3PQ^2&=a^2+b^2-2ab\cos(C+60^\circ)\\&=a^2+b^2-2ab(\cos C\cos 60^\circ-\sin C\sin 60^\circ)\\&=a^2+b^2-ab\cos C+ab\sin C\sqrt{3}\\&=a^2+b^2-\frac{a^2+b^2-c^2}{2}+ab\sin C\sqrt{3}\\&=\frac{a^2+b^2+c^2}{2}+2S\sqrt{3}\end{aligned}$$

We shall obtain the same symmetrical expression for $3QR^2$ and $3RP^2$. Thus $PQ=QR=RP$.

80 We have $$\tan\frac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\frac{A}{2};$$

therefore $$\cot\frac{A}{2}=\frac{65+25}{65-25}\tan 30^\circ=\frac{9}{4}\cdot\frac{1}{\sqrt{3}}=\frac{3\sqrt{3}}{4};$$

therefore $$L\cot\frac{A}{2}=10+\frac{3}{2}\log 3-2\log 2=10\ 1136219$$

$$\begin{array}{r}10\ 1137122\\ 10\ 1134508\\ \hline 0002614\end{array}\qquad\begin{array}{r}10\ 1136219\\ 10\ 1134508\\ \hline 0001711\end{array}\qquad 0002614 . 0001711 \cdot 60'' \cdot x'',$$

this gives $x=39$; therefore $\frac{A}{2}=37^\circ\,36'-39''=37^\circ\,35'\,21''$. Therefore $A=75^\circ\,10'\,42''$ Thus $B+C=180^\circ-75^\circ\,10'\,42''$; and $B-C=60^\circ$. Therefore $B=82^\circ\,24'\,39''$ and $C=22^\circ\,24'\,39'$

81 In the solution of Example 12 it is shewn that the sides of the new triangle are $a\cos A$, $b\cos B$, and $c\cos C$ respectively.

In the solution of Example 11 it is shewn that the angles of the new triangle are $\pi-2A$, $\pi-2B$, and $\pi-2C$ respectively. Then, by Art 215,

$$\cos(\pi-2A)=\frac{b^2\cos^2 B+c^2\cos^2 C-a^2\cos^2 A}{2bc\cos B\cos C};$$

but $\cos(\pi-2A)=-\cos 2A$. Therefore

$$\cos 2A=\frac{a^2\cos^2 A-b^2\cos^2 B-c^2\cos^2 C}{2bc\cos B\cos C}$$

82 Let ρ_1 denote the radius of the circle which touches BD, BF and the arc DF in the diagram of Art 250 Let ρ_2 denote the radius of the circle which touches CD, CE, and the arc DE.

The angle $DBF=\pi-B$ Hence, by the method of Example 57, we have

$$\rho_1=r_1\frac{1-\sin\frac{\pi-B}{2}}{1+\sin\frac{\pi-B}{2}}=r_1\frac{1-\cos\frac{B}{2}}{1+\cos\frac{B}{2}}=r_1\tan^2\frac{B}{4}$$

Similarly $$\rho_2=r_1\tan^2\frac{C}{4}$$

In this way we see that the product of three of the radii

$$=r_1\tan^2\frac{B}{4}\times r_2\tan^2\frac{C}{4}\times r_3\tan^2\frac{A}{4},$$

and the product of the other three

$$=r_1\tan^2\frac{C}{4}\times r_2\tan^2\frac{A}{4}\times r_3\tan^2\frac{B}{4}$$

The two products are equal

83. $\frac{AB'}{AP}=\frac{\sin APB'}{\sin AB'P}$; therefore $AB'=\frac{AP\sin APB'}{\sin AB'P}$

Similarly $$BC'=\frac{BP\sin BPC'}{\sin BC'P},\text{ and } CA'=\frac{CP\sin CPA'}{\sin CA'P}.$$

Thus $$AB'\,BC'\,CA'=\frac{AP\;BP\,.\,CP\sin APB'\sin BPC'\sin CPA'}{\sin AB'P\,.\sin BC'P\;\sin CA'P}$$

In like manner

$$AC'\;BA'\,.\,CB'=\frac{AP\;BP\;CP\sin APC'\;\sin BPA'\;\sin CPB'}{\sin AC'P\;\sin BA'P\;\sin CB'P}$$

The two expressions are obviously equal, for $\sin APB'=\sin BPA'$, $\sin BPC'=\sin B'PC$, and $\sin CPA'=\sin C'PA$. Also, $\sin AB'P=\sin CB'P$, and so on.

84 Let P denote the intersection of AA' and BB', then, if CC' does not pass through P, let a straight line be drawn from C through P, and let it meet AB at C_1

Then, by the Example, we have

$$AB'.BC_1.CA'=AC_1\;BA'.CB'$$

But by hypothesis,

$$AB'.BC'\;CA'=AC'.BA'.CB'$$

Therefore $$\frac{BC_1}{BC'}=\frac{AC_1}{AC'};$$

therefore $$\frac{BC'-C_1C'}{BC'}=\frac{AC'+C_1C'}{AC'},$$

therefore $$-\frac{C_1C'}{BC'}=\frac{C_1C'}{AC'};$$

therefore $$C_1C'=0;$$

therefore C_1 must coincide with C'.

85 Let the feet of the perpendiculars from A, B, C be denoted by A', B' C' respectively. If all the angles are acute, we have

$$AB'=c\cos A,\quad BC'=a\cos B,\quad CA'=b\cos C,$$

$$AC'=b\cos A,\quad BA'=c\cos B,\quad CB'=a\cos C,$$

thus $$AB'.BC'\ CA'=AC'\ BA'.CB'.$$

Therefore, by Example 84, the straight lines AA', BB', and CC' meet at a point

Next suppose one angle obtuse, say C Then

$$CA'=b\cos(180^0-C),\text{ and } CB'=a\cos(180^0-C),$$

the other expressions remain as before, and the result holds as before.

86 Let the straight lines which bisect the angles A, B, C respectively meet the opposite sides at A', B', C' respectively Then

$$\frac{AB'}{BB'}=\frac{\sin\frac{1}{2}B}{\sin A},\quad \frac{BC'}{CC'}=\frac{\sin\frac{1}{2}C}{\sin B},\quad \frac{CA'}{AA'}=\frac{\sin\frac{1}{2}A}{\sin C};$$

therefore $$AB'.BC'.CA'=AA'.BB'.CC'\frac{\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}{\sin A\sin B\sin C},$$

the same value may be obtained for $AC'.BA'.CB'$.

Therefore, by Example 84, the straight lines AA', BB', and CC' meet at a point

87 Let A', B', C' denote the middle points of BC, CA, AB respectively Then

$$AB'.BC'\ CA'=\frac{1}{2}b\times\frac{1}{2}c\times\frac{1}{2}a=\frac{1}{8}abc.$$

Similarly $$AC'.BA'.CB'=\frac{1}{8}abc.$$

Therefore, by Example 84, the straight lines AA', BB', and CC' meet at a point

88 Let the points of contact opposite to A, B, C respectively be denoted by A', B', C' respectively

Then $$AB' = r\cot\frac{A}{2},\quad BC' = r\cot\frac{B}{2},\quad CA' = r\cot\frac{C}{2}.$$

Thus $$AB'.BC'\ CA' = r^3\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

Similarly $$AC'\ BA'.CB' = r^3\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

Therefore, by Example 84, the straight lines AA', BB', and CC' meet at a point

89 Let the points of contact opposite to A, B, C respectively be denoted by A', B', C' respectively.

Then $$AB' = r_2\cot\frac{1}{2}(\pi - A) = r_2\tan\frac{1}{2}A,$$

$$BC' = r_3\tan\frac{1}{2}B,$$

$$CA' = r_1\tan\frac{1}{2}C,$$

therefore $$AB'\ BC'.CA' = r_1r_2r_3\tan\frac{1}{2}A\tan\frac{1}{2}B\tan\frac{1}{2}C.$$

Similarly $$AC'.BA'.CB' = r_1r_2r_3\tan\frac{1}{2}A\tan\frac{1}{2}B\tan\frac{1}{2}C$$

Therefore, by Example 84, the straight lines AA', BB', and CC' meet at a point

90 Here $AE = AF$, $CE = CD$, $BD = BF$, therefore
$$AE\ BF.CD = AF.BD.CE.$$
Therefore, by Example 84, the straight lines AD, BE, and CF meet at a point

91 Let $ABCD$ be the quadrilateral figure Then, denoting by A, B, C, and D the internal angles of the figure, we have

$$r_a\left(\cot\frac{\pi - B}{2} + \cot\frac{\pi - C}{2}\right) = BC,$$

therefore $$r_a\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) = BC$$

Again, in like manner we have

$$r\left(\cot\frac{A}{2} + \cot\frac{D}{2}\right) = DA,$$

that is $$r\left(\tan\frac{C}{2}+\tan\frac{B}{2}\right)=DA,$$

for $A+C=\pi$, and $B+D=\pi$, by Euclid III 22

Hence $$\frac{r_a}{r}=\frac{BC}{DA}.$$

In the same manner we can show that

$$\frac{r_b}{r}=\frac{CD}{AB},\ \frac{r_c}{r}=\frac{AD}{BC},\ \text{and}\ \frac{r_d}{r}=\frac{AB}{DC}.$$

Therefore $$\frac{r_a r_c}{r^2}=1,\quad\text{and}\ \frac{r_b r_d}{r^2}=1;$$

therefore $$r_a r_b r_c r_d=r^4$$

92 $$\frac{QA}{QB}=\frac{\sin QBA}{\sin QAB}=\frac{\sin\beta'}{\sin\alpha},$$

$$\frac{QB}{QC}=\frac{\sin QCB}{\sin QBC}=\frac{\sin\gamma'}{\sin\beta},$$

$$\frac{QC}{QA}=\frac{\sin QAC}{\sin QCA}=\frac{\sin\alpha'}{\sin\gamma}$$

Therefore, by multiplication,

$$1=\frac{\sin\alpha'\sin\beta'\sin\gamma'}{\sin\alpha\sin\beta\sin\gamma},$$

or, $$\sin\alpha\sin\beta\sin\gamma=\sin\alpha'\sin\beta'\sin\gamma'$$

93 $\angle QAC=A-\omega$, $\angle QCA=\omega$, $\therefore \angle AQC=\pi-A$.

Similarly $\angle BQA=\pi-B$, $\angle BQC=\pi-C$.

Now $$\frac{\sin(\pi-A)}{\sin\omega}=\frac{\sin AQC}{\sin QCA}=\frac{AC}{AQ}=\frac{AC}{AB}\cdot\frac{AB}{AQ}$$

$$=\frac{\sin B}{\sin C}\cdot\frac{\sin(\pi-B)}{\sin(B-\omega)}.$$

$$\frac{\sin(B-\omega)}{\sin B\sin\omega}=\frac{\sin B}{\sin A\sin C}=\frac{\sin(A+C)}{\sin A\sin C}.$$

$$\therefore\ \cot\omega-\cot B=\cot A+\cot C;$$

$$\therefore\ \cot\omega=\cot A+\cot B+\cot C.$$

This equation may also be deduced from the equation of Example 92, which becomes in this case

$$\sin^3\omega=\sin(A-\omega)\sin(B-\omega)\sin(C-\omega)$$

If $x=\cot\omega$ this equation reduces to

$$x(x^2+1)-(x^2+1)(\cot A+\cot B+\cot C)=0.$$

94 Let AXY, BYZ, CZX cut the sides of the triangle ABC in D, E, F Then $\angle XYZ = \angle BAY + \angle ABY = \angle YBC + \angle ABY = B$

Hence the triangles XYZ and ABC are similar, and therefore their areas are in the ratio of YZ^2 to BC^2

Let Q be the point determined in Ex. 93 Then

$$\angle QAY = \omega - \phi = \angle QBY,$$

Q, A, B, Y are concyclic

$$\angle BQY = \angle BAY = \phi,$$

$$\therefore \angle QYZ = \angle YBQ + \angle YQB = \omega - \phi + \phi = \omega$$

Similarly, $\angle QZY = \omega = \angle QXY$

Hence the triangles QYZ and QBC are similar;

$$\frac{YZ}{BC} = \frac{YQ}{BQ} = \frac{\sin YBQ}{\sin BYQ} = \frac{\sin(\omega - \phi)}{\sin \omega},$$

$$\frac{\text{area } XYZ}{\text{area } ABC} = \frac{YZ^2}{BC^2} = \frac{\sin^2(\omega - \phi)}{\sin^2 \omega}.$$

95 Let AO cut BC in F, AF being the crease When ABF is doubled over, let AB cut FC in G We have to find the area of AFG.

$$\angle FAG = \angle BAF = 90^\circ - C.$$

$$\frac{AF}{AB} = \frac{\sin ABF}{\sin AFB} = \frac{\sin B}{\sin(ABF + BAF)} = \frac{\sin B}{\sin(B + 90^\circ - C)}$$

$$AF = \frac{c \sin B}{\cos(C - B)}.$$

$$\frac{AG}{AB} = \frac{\sin ABF}{\sin AGB} = \frac{\sin B}{\sin(ABG + BAG)} = \frac{\sin B}{\sin(B + 180^\circ - 2C)}$$

$$\therefore AG = \frac{c \sin B}{\sin(2C - B)}.$$

$$\text{area } AFG = \frac{1}{2} AF \cdot AG \sin FAG = \frac{1}{2} \frac{c^2 \sin^2 B \sin(90^\circ - C)}{\cos(C - B) \sin(2C - B)}$$

$$= \frac{1}{2} b^2 \sin^2 C \cos C \sec(C - B) \operatorname{cosec}(2C - B)$$

96. $$\frac{XQ}{BQ} = \frac{\sin XBQ}{\sin BXQ} = \frac{\sin XBQ}{\sin BAY} = \frac{\sin XBA}{\cos ABY}.$$

$$\frac{YQ}{BQ} = \frac{\sin YBQ}{\sin BYQ} = \frac{\sin YBQ}{\sin BAX} = \frac{\sin YBA}{\cos XBA}.$$

Therefore, by multiplication,

$$\tan XBA \tan YBA = \frac{QX \cdot QY}{BQ^2} = \frac{BQ \cdot AQ}{BQ^2} = \frac{QA}{QB},$$

or $$\tan \frac{1}{2} XCA \tan \frac{1}{2} YCA = \frac{QA}{QB}$$

97 Let O, O' be the centres, r and r' the radii; PTQ a common tangent cutting OO' in T; and let $\angle POT=\theta$ Then

$$\frac{r}{\cos\theta}+\frac{r'}{\cos\theta}=OT+O'T=2a$$

But $$r+r'=a,$$

$$\therefore \cos\theta=\frac{1}{2}.$$

$$\theta=\frac{\pi}{3}.$$

The length of the string

$$=(2\pi-2\theta)r+(2\pi-2\theta)r'+2PQ$$

$$=\left(2\pi-\frac{2\pi}{3}\right)a+2PT+2QT$$

$$=\frac{4\pi a}{3}+2r\tan\theta+2r'\tan\theta$$

$$=\frac{4\pi a}{3}+2a.\sqrt{3}$$

98 For convenience of reference let Q be within the angle AOC, let $\angle AOQ=\theta$, so that $\angle BOQ=\angle BOA+\theta=2C+\theta$, $\angle COQ=2B-\theta$

Then $l^2=AO^2+OQ^2-2AO.OQ\cos\theta \quad =R^2+d^2-2Rd\cos\theta,$

$$m^2=BO^2+OQ^2-2BO.OQ\cos BOQ=R^2+d^2-2Rd\cos(2C+\theta),$$

$$n^2=CO^2+OQ^2-2CO.OQ\cos COQ=R^2+d^2-2Rd\cos(2B-\theta)$$

Multiply these equations by $\sin 2A$, $\sin 2B$, $\sin 2C$ and add, the coefficient of $-2Rd$ is

$$\cos\theta\sin 2A+\cos(2C+\theta)\sin 2B+\cos(2B-\theta)\sin 2C$$

$$=\cos\theta(\sin 2A+\sin 2B\cos 2C+\cos 2B\sin 2C)$$

$$+\sin\theta(\sin 2C\sin 2B-\sin 2B\sin 2C)$$

$$=\cos\theta\{\sin 2A+\sin(360^\circ-2A)\}=0$$

$$l^2\sin 2A+m^2\sin 2B+n^2\sin 2C=(R^2+d^2)(\sin 2A+\sin 2B+\sin 2C)$$

$$=4(R^2+d^2)\sin A\sin B\sin C.$$

99 Let $\angle APQ=\theta$, Q being within the angle APC, where P is the orthocentre. We have

$$\angle BPQ=\theta+\angle APB=180^\circ-(C-\theta),$$

$$\angle CPQ=180^\circ-(\angle APQ+\angle CPD)=180^\circ-(B+\theta)$$

$$l^2=PA^2+d^2-2PA.d\cos\theta \quad =4R^2\cos^2 A+d^2-4Rd\cos A\cos\theta,$$

$$m^2=PB^2+d^2+2PB.d\cos(C-\theta)=4R^2\cos^2 B+d^2+4Rd\cos B\cos(C-\theta),$$

$$n^2=PC^2+d^2+2PC.d\cos(B+\theta)=4R^2\cos^2 C+d^2+4Rd\cos C\cos(B+\theta)$$

Multiply these by $\tan A$, $\tan B$, $\tan C$, the coefficient of $4Rd$ is

$$-\sin A\cos\theta+\sin B\cos(C-\theta)+\sin C\cos(B+\theta)$$

$$=\cos\theta(-\sin A+\sin B\cos C+\sin C\cos B)+\sin\theta(\sin B\sin C-\sin B\sin C)=0.$$

Hence, $$l^2\tan A+m^2\tan B+n^2\tan C$$

$$=4R^2(\sin A\cos A+\sin B\cos B+\sin C\cos C)+d^2(\tan A+\tan B+\tan C)$$

$$=2R^2(\sin 2A+\sin 2B+\sin 2C)+d^2\tan A\tan B\tan C$$

$$=8R^2\sin A\sin B\sin C+d^2\tan A\tan B\tan C$$

100 Let $QA=l$, $QB=m$, $QC=n$, $QO=d$, $QP=d'$, O being the centre of the circumcircle and P the orthocentre From Examples 98, 99,

$$l^2\sin 2A+m^2\sin 2B+n^2\sin 2C=4(R^2+d^2)\sin A\sin B\sin C,$$

$$l^2\tan A+m^2\tan B+n^2\tan C=8R^2\sin A\sin B\sin C+d'^2\tan A\tan B\tan C$$

Now $$\frac{\sin 2A}{\sin A\sin B\sin C}=\frac{2\cos A}{\sin B\sin C}=\frac{-2\cos(B+C)}{\sin B\sin C}=2-2\cot B\cot C.$$

The first equation therefore becomes

$$(1-\cot B\cot C)\,l^2+(1-\cot C\cot A)\,m^2+(1-\cot A\cot B)\,n^2=2R^2+2d^2$$

Dividing the second equation by $\frac{1}{2}\tan A\tan B\tan C$ we obtain

$$2\cot B\cot C\,.\,l^2+2\cot C\cot A\;\;m^2+2\cot A\cot B\;\;n^2$$
$$=16R^2\cos A\cos B\cos C+2d'^2.$$

Therefore by addition

$$(1+\cot B\cot C)\,l^2+(1+\cot C\cot A)\,m^2+(1+\cot A\cot B)\,n^2$$
$$=16R^2\cos A\cos B\cos C+2R^2+2(d^2+d'^2).$$

But if N be the centre of the nine points circle (the middle point of OP) we have $$d^2+d'^2=QO^2+QP^2=2QN^2+2ON^2,$$

$$2(d^2+d'^2)=4QN^2+PO^2$$

The least value of this is when $QN=0$, that is when Q is the centre of the nine points circle

$$(1+\cot B\cot C)\,l^2+(1+\cot C\cot A)\,m^2+(1+\cot A\cot B)\,n^2$$

is a minimum when Q is the centre of the nine points circle

101 Draw B_1N perpendicular to CC_1 Then $B_1C_1{}^2=B_1N^2+C_1N^2$

Therefore $$a_1{}^2=a^2+(y-z)^2$$

Similarly $$b_1{}^2=b^2+(z-x)^2,\quad c_1{}^2=c^2+(x-y)^2$$

Let $$\xi=y-z,\quad \eta=z-x,\quad \zeta=x-y;$$

then $$\xi+\eta+\zeta=0,$$

$$a_1{}^2=a^2+\xi^2,\quad b_1{}^2=b^2+\eta^2,\quad c_1{}^2=c^2+\zeta^2.$$

Now $$16\Delta_1^2=2b_1^2c_1^2+2c_1^2a_1^2+2a_1^2b_1^2-a_1^4-b_1^4-c_1^4$$
$$=2(b^2+\eta^2)(c^2+\zeta^2)+2(c^2+\zeta^2)(a^2+\xi^2)+2(a^2+\xi^2)(b^2+\eta^2)$$
$$-(a^2+\xi^2)^2-(b^2+\eta^2)^2-(c^2+\zeta^2)^2$$
$$=2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$$
$$+2(c^2\eta^2+b^2\zeta^2+a^2\zeta^2+c^2\xi^2+b^2\xi^2+\eta^2a^2-a^2\xi^2-b^2\eta^2-c^2\zeta^2)$$
$$+2\eta^2\zeta^2+2\zeta^2\xi^2+2\xi^2\eta^2-\xi^4-\eta^4-\zeta^4$$

The terms in the last line are divisible by $\xi+\eta+\zeta$ and therefore vanish. Hence $16\Delta_1^2=16\Delta^2+2a^2(\eta^2+\zeta^2-\xi^2)+2b^2(\xi^2-\eta^2+\zeta^2)+2c^2(\xi^2+\eta^2-\zeta^2)$.

Now $\eta^2+\zeta^2-\xi^2=-2\eta\zeta=2(x-y)(x-z)$ &c ; therefore

$$\Delta_1^2-\Delta^2=\frac{1}{4}[a^2(x-y)(x-z)+b^2(y-z)(y-x)+c^2(z-y)(z-x)].$$

Again since $a^2=a_1^2-\xi^2,\quad b^2=b_1^2-\eta^2,\quad c^2=c_1^2-\zeta^2,$

and $16\Delta^2=2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4,$

we shall obtain by proceeding as above

$$16\Delta^2=16\Delta_1^2-2a_1^2(\eta^2+\zeta^2-\xi^2)-2b_1^2(\xi^2-\eta^2+\zeta^2)-2c_1^2(\xi^2+\eta^2-\zeta^2),$$
$$\Delta_1^2-\Delta^2=\frac{1}{4}[a_1^2(x-y)(x-z)+b_1^2(y-z)(y-x)+c_1^2(z-y)(z-x)]$$

102. Since the triangles EAD and EBC are similar

$$\frac{EB}{b}=\frac{EA}{d}, \text{ and } \frac{EC}{b}=\frac{ED}{d}.$$

Since the triangles EAB and ECD are similar

$$\frac{EA}{a}=\frac{ED}{c}, \text{ and } \frac{EB}{a}=\frac{EC}{c}.$$

These relations are equivalent to

$$\frac{EA}{ad}=\frac{EB}{ab}=\frac{EC}{bc}=\frac{ED}{cd}$$

Let each of these fractions $=\kappa$, then

$$EA=\kappa ad,\quad EB=\kappa ab,\quad EC=\kappa bc,\quad ED=\kappa cd,$$
$$\therefore AC=\kappa(ad+bc),\quad BD=\kappa(ab+cd).$$

Now $$AC.BD=AB.CD+BC.AD,$$
$$\kappa^2(ad+bc)(ab+cd)=ac+bd,$$
$$\kappa=\sqrt{\frac{ac+bd}{(ad+bc)(ab+cd)}}.$$
$$AC=\sqrt{(ac+bd)}.\sqrt{\frac{ad+bc}{ab+cd}},$$
$$BD=\sqrt{(ac+bd)}\sqrt{\frac{ab+cd}{ad+bc}}$$

103 Let θ be the angle between the diagonals

$$p_1a = 2 \text{ area } EAB = EA \cdot EB \sin\theta$$

$$= \kappa^2 a^2 bd \sin\theta \qquad \text{(Ex. 102.)}$$

Now

$$S = \frac{1}{2} BD \cdot AC \sin\theta \qquad \text{(Ex. 113)}$$

$$= \frac{1}{2}\kappa^2 (ad+bc)(ab+cd) \sin\theta \qquad \text{(Ex. 102.)}$$

$$\frac{p_1}{S} = \frac{2abd}{(ad+bc)(ab+cd)},$$

$$p_1c = \frac{2abcd}{(ad+bc)(ab+cd)} \cdot S$$

Similarly p_2d, p_3a, p_4b are each equal to the same expression

104 The product of the segments of any chord through E

$$= EA \cdot EC$$

$$= \kappa ad \cdot \kappa bc \qquad \text{(Ex. 102)}$$

$$= abcd \cdot \frac{ac+bd}{(ad+bc)(ab+cd)}. \qquad \text{(Ex. 102)}$$

105 Let $BF = x$, $CF = y$

Since the triangles BCF, ADF are similar, we have

$$\frac{x}{b} = \frac{y+c}{d}, \text{ and } \frac{y}{b} = \frac{x+a}{d}.$$

$$dx - by = bc,$$

$$dy - bx = ab.$$

Solving these equations we get

$$x = \frac{ab+cd}{d^2-b^2} \cdot b, \quad y = \frac{bc+ad}{d^2-b^2} \cdot b$$

106 Let 2σ = the perimeter of the triangle FBC.
By Example 105 we have

$$\frac{2\sigma}{b} = 1 + \frac{bc+ad}{d^2-b^2} + \frac{ab+cd}{d^2-b^2}$$

$$= \frac{d^2-b^2+(b+d)(a+c)}{d^2-b^2} = \frac{d-b+a+c}{d-b}$$

$$= \frac{2(s-b)}{d-b}$$

$$\frac{\sigma}{b} = \frac{s-b}{d-b}.$$

$$\frac{\sigma-b}{b}=\frac{s-d}{d-b}$$

$$\cos^2\frac{1}{2}F=\frac{\sigma(\sigma-b)}{FC\,.\,FB}=\frac{(s-b)(s-d)b^2}{(d-b)^2}\times\frac{(d^2-b^2)^2}{b^2(ab+cd)(bc+ad)}\ \text{(Ex. 105)}$$
$$=\frac{(s-b)(s-d)(b+d)^2}{(ab+cd)(bc+ad)}.$$

107 Area EFG = area AFG − area AEF − area AEG

$$=\text{area } AFG-\frac{1}{2}p_1\,.\,AF-\frac{1}{2}p_4\ AG,$$

where p_1 and p_4 are perpendiculars from E on AB, AD; the values of p_1 and p_4 are found in Ex 103.

From Example 105

$$BF=\frac{ab+cd}{d^2-b^2}\,.\,b,$$

$$AF=a+\frac{ab+cd}{d^2-b^2}\,.\,b=\frac{ad+bc}{d^2-b^2}\,.\,d$$

Similarly, $AG=\dfrac{ad+bc}{a^2-c^2}\,.\,a,$

$$\therefore\ \text{area } AFG=\frac{1}{2}\ AG\,.\,AF\sin A$$
$$=\frac{ad(ad+bc)}{(a^2-c^2)(d^2-b^2)}\left\{\frac{1}{2}ad\sin A+\frac{1}{2}bc\sin A\right\}$$
$$=\frac{ad(ad+bc)}{(a^2-c^2)(d^2-b^2)}(\text{area } ABD+\text{area } BDC)$$
$$=\frac{ad(ad+bc)}{(a^2-c^2)(d^2-b^2)}\,.\,S,$$

$$\frac{1}{2}p_1\ AF=\frac{abd}{(ab+cd)(ad+bc)}\ S\times\frac{ad+bc}{d^2-b^2}\,.\,d \qquad \text{(Ex 103)}$$
$$=\frac{abd^2}{(ab+cd)(d^2-b^2)}\,.\,S,$$

$$\frac{1}{2}p_4\ AG=\frac{acd}{(ab+cd)(ad+bc)}\,.\,S\times\frac{ad+bc}{a^2-c^2}\,.\,a$$
$$=\frac{a^2cd}{(ab+cd)(a^2-c^2)}\ S$$

$$\text{area } EFG=\frac{adS}{(ab+cd)(a^2-c^2)(d^2-b^2)}\{(ab+cd)(ad+bc)-bd(a^2-c^2)-ac(d^2-b^2)\}$$
$$=\frac{adS}{(ab+cd)(a^2-c^2)(d^2-b^2)}\,.\,2bc(ab+cd)$$
$$=\frac{2abcd}{(a^2-c^2)(d^2-b^2)}\ S$$

108 As in Example 105,

$$CF=\frac{bc+ad}{d^2-b^2}\cdot b, \quad CG=\frac{bc+ad}{a^2-c^2}\ c$$

$$\cos FCG=\cos C=\frac{b^2+c^2-a^2-d^2}{2(bc+ad)}, \text{ as in Art } 255$$

$$FG^2=CF^2+CG^2-2CF\ CG\cdot\cos FCG$$

$$=(bc+ad)^2\left[\frac{b^2}{(d^2-b^2)^2}+\frac{c^2}{(a^2-c^2)^2}-\frac{bc(b^2+c^2-a^2-d^2)}{(d^2-b^2)(a^2-c^2)(bc+ad)}\right]$$

$$=(bc+ad)^2\left[\frac{b^2}{(d^2-b^2)^2}+\frac{c^2}{(a^2-c^2)^2}+\frac{bc}{(a^2-c^2)(bc+ad)}+\frac{bc}{(d^2-b^2)(bc+ad)}\right].$$

$$\frac{b^2}{(d^2-b^2)^2}+\frac{bc}{(d^2-b^2)(bc+ad)}=\frac{b^2(bc+ad)+bc(d^2-b^2)}{(d^2-b^2)^2(bc+ad)}$$

$$=\frac{bd(ab+cd)}{(d^2-b^2)^2(bc+ad)}$$

$$\frac{c^2}{(a^2-c^2)^2}+\frac{bc}{(a^2-c^2)(bc+ad)}=\frac{c^2(bc+ad)+bc(a^2-c^2)}{(a^2-c^2)^2(bc+ad)}$$

$$=\frac{ac(ab+cd)}{(a^2-c^2)^2(bc+ad)}.$$

$$FG^2=(bc+ad)(ab+cd)\left[\frac{bd}{(d^2-b^2)^2}+\frac{ac}{(a^2-c^2)^2}\right].$$

109 CC' is parallel to BD, therefore $BC'=CD$ and $C'D=BC$

Now $\quad BD\ CC'+CD\ BC'=BC\ C'D,$ (Euc vi Prop D)

$$CC'=\frac{BC^2-CD^2}{BD}=(b^2-c^2)\sqrt{\frac{ad+bc}{(ab+cd)(ac+bd)}}.$$

Let $\theta=\angle ACC'=$angle between the diagonals.

If R be the radius of the circumcircle,

$$AC'=2R\sin\theta, \text{ and } S=\frac{1}{2}AC\ BD\sin\theta, \qquad \text{(Ex 113)}$$

$$R=\frac{AC'}{2\sin\theta}=\frac{AC'\ AC\ BD}{4S}.$$

110 Let $\angle ADB=\theta$, $\angle BDC=\phi$ Then

$$\text{area } ACD=\frac{1}{2}AD\ DC\ \sin D$$

$$=\frac{1}{2}BD\cos\theta\ BD\cos\phi\ \sin D,$$

$$\text{area } ABC = \frac{1}{2} AB \,.\, BC \,.\, \sin ABC$$

$$= \frac{1}{2} BD \sin\theta \,.\, BD \sin\phi \quad \sin D,$$

$$\therefore \text{ area } ACD - \text{area } ABC = \frac{1}{2} BD^2 \sin D (\cos\theta\cos\phi - \sin\theta\sin\phi)$$

$$= \frac{1}{2} BD^2 \sin D \,.\, \cos(\theta+\phi) = \frac{1}{2} BD^2 \sin D \,.\, \cos D = \frac{1}{4} BD^2 \sin 2D$$

Again, we have $\qquad a^2 + d^2 = BD^2 = b^2 + c^2,$

$$4(s-a)(s-d) = (-a+b+c+d)(a+b+c-d)$$

$$= b^2 + 2bc + c^2 - a^2 + 2ad - d^2 = 2(bc+ad)$$

$$(s-a)(s-d) = \frac{1}{2} bc + \frac{1}{2} ad = \text{area } BCD + \text{area } ABD$$

$$= \text{area } ABCD.$$

Similarly $(s-b)(s-c) = \text{area } ABCD$

111 Let a, b, c, d be the four given straight lines any three of which are supposed to be greater than the fourth.

The sides adjacent to a must be (i) b, d, or (ii) b, c; or (iii) c, d With any one of these arrangements a quadrilateral inscribable in a circle can be formed by making the sum of two opposite angles equal to two right angles

In each case

$$\text{the area } S = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where $\qquad 2s = a+b+c+d,$

and $\qquad R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}.$

The lengths of the diagonals are

in (i) $\sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$ and $\sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}$,

in (ii) $\sqrt{\frac{(ad+bc)(ac+bd)}{ab+cd}}$ and $\sqrt{\frac{(ad+bc)(ab+cd)}{ac+bd}}$,

in (iii) $\sqrt{\frac{(ab+cd)(ad+bc)}{ac+bd}}$ and $\sqrt{\frac{(ab+cd)(ac+bd)}{ad+bc}}$

There are here only three different expressions, if these be x, y, z we have

$$xyz = \sqrt{(ac+bd)(ad+bc)(ab+cd)},$$

$$R = \frac{xyz}{4S}$$

112 In this case $a+c=b+d=s$, and $A+C=180^0$, therefore from Art 256,

$$S=\sqrt{abcd}$$

If r=radius of the inscribed circle,

$$\frac{1}{2}(ra+rb+rc+rd)=\text{area of quadrilateral},$$

$$r=\frac{\sqrt{abcd}}{\frac{1}{2}(a+b+c+d)},$$

$$r=\frac{\sqrt{abcd}}{a+c},$$

or

$$r=\frac{\sqrt{abcd}}{b+d}.$$

$$r^2=\frac{abcd}{(a+c)(b+d)}$$

113 Let the diagonals intersect in E, then

$$S=\text{sum of areas } AEB, BEC, CED, DEA$$

$$=\frac{1}{2}(AE\ EB+BE . EC+EC\ ED+ED\ EA)\sin\theta$$

$$=\frac{1}{2}(AE+EC)(BE+BD)\sin\theta$$

$$=\frac{1}{2}AC\ BD\sin\theta$$

114 Let those sides of the square which pass through A, B meet in H, $ABCD$ being the quadrilateral

Let $\angle CAH=\alpha$, $\angle DBH=\beta$, θ the angle between the diagonals which is opposite H

Then $$x\sin\alpha=p,\quad y\sin\beta=p \qquad (1),$$

and $$\alpha+\beta+\theta=270^0$$

$$\cos(\alpha+\beta)=\cos(270^0-\theta)=-\sin\theta,$$

$$xy\cos\alpha\cos\beta-xy\sin\alpha\sin\beta=-xy\sin\theta$$

$$x^2y^2\cos^2\alpha\cos^2\beta=(p^2-xy\sin\theta)^2, \qquad \text{from (1)},$$

from (1), $$x^2y^2\left(1-\frac{p^2}{x^2}\right)\left(1-\frac{p^2}{y^2}\right)=p^4-2p^2xy\sin\theta+x^2y^2\sin^2\theta$$

$$p^2(x^2-2xy\sin\theta+y^2)=x^2y^2\cos^2\theta$$

115 If a, b, c, d be the sides and S the area, we have

$$S^2=(s-a)(s-b)(s-c)(s-d)-abcd\cos^2\frac{1}{2}(A+C)$$

This is greatest when $\cos\frac{1}{2}(A+C)=0$,

that is, when $A+C=180^0$

The area is therefore greatest when the quadrilateral can be inscribed in a circle

116 Let θ be the angle between the diagonals,

$$\text{area } ABC \,.\, \text{area } ABD=\frac{1}{2}AC\,.\,AB\sin CAB\,.\,\frac{1}{2}DB\,.\,BA\sin DBA$$
$$=\frac{1}{2}AC\,.\,BD\sin\theta\times\frac{1}{2}AB^2\frac{\sin EAB\sin EBA}{\sin\theta}$$
$$=\text{area } ABCD\times\frac{1}{2}AB^2.\frac{EB}{AB}\sin EBA$$
$$=\text{area } ABCD\,.\,\text{area } AEB$$

117 Let E be the intersection of diagonals, θ the angle AEB Then

$$a^2=AE^2+EB^2-2AE\,.\,EB\cos\theta,$$
$$c^2=EC^2+ED^2-2EC\ \ ED\cos\theta,$$
$$b^2=EB^2+EC^2+2EB\,.\,EC\cos\theta,$$
$$d^2=ED^2+EA^2+2ED\ \ EA\cos\theta$$

Subtract the sum of the first two from the sum of the other two,

$$b^2+d^2-a^2-c^2=2\,(AE\,.\,EB+EC\ \ ED+EB\,.\,EC+ED\ \ EA)\cos\theta$$
$$=2\,(EA+EC)(EB+ED)\cos\theta$$
$$=2fg\cos\theta;$$
$$\cos\theta=\frac{b^2+d^2-a^2-c^2}{2fg}.$$

118. By Example 113,

$$S=\frac{1}{2}fg\sin\theta,$$
$$S^2=\frac{1}{4}f^2g^2\sin^2\theta=\frac{1}{4}(f^2g^2-f^2g^2\cos^2\theta)$$
$$=\frac{1}{4}\left\{f^2g^2-\frac{1}{4}(b^2+d^2-a^2-c^2)^2\right\}. \qquad \text{(Ex. 117)}$$
$$S=\frac{1}{4}\sqrt{\{4f^2g^2-(b^2+d^2-a^2-c^2)^2\}}$$

119 Let AB, DC be produced to meet at F, let θ be the angle at F,

$$FA=\alpha,\ FB=\beta,\ FC=\gamma,\ FD=\delta$$

Then

$$b^2=\beta^2+\gamma^2-2\beta\gamma\cos\theta,\quad d^2=\alpha^2+\delta^2-2\alpha\delta\cos\theta,$$
$$f^2=\alpha^2+\gamma^2-2\alpha\gamma\cos\theta,\quad g^2=\beta^2+\delta^2-2\beta\delta\cos\theta,$$
$$f^2+g^2-b^2-d^2=2\cos\theta\,(\beta\gamma+\alpha\delta-\alpha\gamma-\beta\delta)$$
$$=2\cos\theta\,(\alpha-\beta)(\delta-\gamma)=2ac\cos\theta$$

$$\frac{1}{4}\sqrt{\{4a^2c^2-(f^2+g^2-b^2-d^2)^2\}}=\frac{1}{2}ac\sin\theta$$
$$=\frac{1}{2}(\alpha-\beta)(\delta-\gamma)\sin\theta=\frac{1}{2}(\alpha\delta+\beta\gamma-\beta\delta-\alpha\gamma)\sin\theta$$
$$=AFD+FBC-FBD-FCA$$
$$=ACD-BCD=AED-BEC$$

120 Let BA, CD produced meet in F, and let $FA=x$, $FD=y$

Then $$2S=2\,(\text{area } FBC-\text{area } FAD)$$
$$=\{(x+a)(y+c)-xy\}\sin F$$
$$=(ay+cx+ac)\sin(\theta+\phi)$$

Now $$\frac{x+a}{\sin\phi}=\frac{y+c}{\sin\theta}=\frac{b}{\sin(\theta+\phi)},$$
$$x\sin(\theta+\phi)=b\sin\phi-a\sin(\theta+\phi),$$
$$y\sin(\theta+\phi)=b\sin\theta-c\sin(\theta+\phi);$$
$$(cx+ay)\sin(\theta+\phi)=bc\sin\phi+ab\sin\theta-2ac\sin(\theta+\phi)$$

Hence $$S=\frac{1}{2}\{bc\sin\phi+ab\sin\theta-ac\sin(\theta+\phi)\}$$

If f, g be the diagonals we have from Ex 119,
$$f^2+g^2-b^2-d^2=2ac\cos F=-2ac\cos(\theta+\phi)$$
Now $$f^2=b^2+c^2-2bc\cos\phi,$$
$$g^2=a^2+b^2-2ab\cos\theta$$
Hence $$a^2+b^2+c^2-d^2-2bc\cos\phi-2ab\cos\theta=-2ac\cos(\theta+\phi),$$
or $$d^2=a^2+b^2+c^2-2bc\cos\phi-2ab\cos\theta+2ac\cos(\theta+\phi)$$

121 Let $AC=f$, then by Art 217,
$$4ab\sin^2\frac{1}{2}B=(-a+b+f)(a-b+f)=f^2-(a-b)^2,$$
$$4cd\cos^2\frac{1}{2}D=(c+d+f)(c+d-f)=(c+d)^2-f^2,$$
$$4ab\sin^2\frac{1}{2}B+4cd\cos^2\frac{1}{2}D=(c+d)^2-(a-b)^2$$
$$=(c+d-a+b)(c+d+a-b)=4(s-a)(s-b)$$
Again, $$4ab\cos^2\frac{1}{2}B=(a+b+f)(a+b-f)=(a+b)^2-f^2,$$
$$4cd\sin^2\frac{1}{2}D=(-c+d+f)(c-d+f)=f^2-(c-d)^2,$$
$$4ab\cos^2\frac{1}{2}B+4cd\sin^2\frac{1}{2}D=(a+b)^2-(c-d)^2$$
$$=4(s-c)(s-d).$$

122 Let AD, BC produced meet in G; then

$$AB^2=GA^2+GB^2-2GA\ GB\cos\theta,\quad CD^2=GC^2+GD^2-2GC\,.\,GD\cos\theta,$$

$$AC^2=GA^2+GC^2-2GA\ GC\cos\theta,\quad BD^2=GB^2+GD^2-2GB\ GD\cos\theta,$$

$$\begin{aligned}AC^2+BD^2-(AB^2+CD^2)&=2\,(GA\ GB+GC\ GD-GA\ GC-GB\ GD)\cos\theta\\&=2\,(GA-GD)(GB-GC)\cos\theta\\&=2AD\ BC\cos\theta.\end{aligned}$$

$$AB^2+CD^2=AC^2+BD^2-2AD\,.\,BC\cos\theta$$

123 From Example 118 we have

$$4f^2g^2=(a^2+c^2-b^2-d^2)^2+16S^2$$

Hence from Art 256,

$$\begin{aligned}4f^2g^2&=(a^2+c^2-b^2-d^2)^2-(a^2+d^2-b^2-c^2)^2\\&\qquad+4a^2d^2+4b^2c^2-8abcd\cos(A+C)\\&=4\,(a^2-b^2)(c^2-d^2)+4a^2d^2+4b^2c^2-8abcd\cos(A+C),\end{aligned}$$

$$f^2g^2=a^2c^2+b^2d^2-2abcd\cos(A+C)$$

124. Let x, y be the sides of any of the rectangles Let f, g be the diagonals of the quadrilateral, θ the angle between them, also, let f make with one side of the rectangle an angle a, then g makes with the same side an angle $\theta-a$ We have therefore

$$f\sin a=y,\quad g\cos(\theta-a)=x$$

Now $\dfrac{x}{y}$ is constant, since the rectangles are all similar

$$\frac{\cos(\theta-a)}{\sin a}\text{ is constant for all values of } a,$$

i e $\qquad \cos\theta\cot a+\sin\theta$ is constant for all values of a,

$$\cos\theta=0.$$

Hence the diagonals of the quadrilateral are at right angles If E be their point of intersection

$$a^2+c^2=EA^2+EB^2+EC^2+ED^2=b^2+d^2$$

125 Let P, Q, R, S be the middle points of the sides Then $PQRS$ is a parallelogram whose area is $\frac{1}{2}\delta$ Let $PQ=a$, $PS=\beta$, then a and β are equal to half the diagonals of the quadrilateral, therefore

$$a+\beta=p,\quad a-\beta=q$$

Let $\qquad QS=x_1,\quad PR=x_2$

Expressing the area of the triangle PQS in terms of the sides, we have

$$\frac{1}{4}\delta=\frac{1}{4}\sqrt{(x_1+a+\beta)(x_1-a+\beta)(x_1+a-\beta)(-x_1+a+\beta)},$$

$$\delta^2=(x_1+p)(x_1-q)(x_1+q)(-x_1+p)$$

$$(x_1^2-p^2)(x_1^2-q^2)+\delta^2=0$$

Similarly, $\qquad (x_2^2-p^2)(x_2^2-q^2)+\delta^2=0$

Hence x_1 and x_2 are roots of the equation

$$(x^2-p^2)(x^2-q^2)+\delta^2=0$$

XVII

1. Let $\tan^{-1}\frac{1}{3}=\theta$, then $\tan\theta=\frac{1}{3}$; therefore

$$\tan 2\theta=\frac{\frac{2}{3}}{1-\frac{1}{9}}=\frac{6}{8}=\frac{3}{4},$$

therefore $\quad 2\theta=\tan^{-1}\frac{3}{4}$. Therefore $\tan^{-1}\frac{3}{4}=2\tan^{-1}\frac{1}{3}$

2 Let $\quad \sin^{-1}\frac{1}{2}=\theta$, and $\cos^{-1}\frac{1}{2}=\phi$,

therefore $\quad \sin\theta=\frac{1}{2}$, and $\cos\theta=\frac{\sqrt{3}}{2}$,

and $\quad \cos\phi=\frac{1}{2}$, and $\sin\phi=\frac{\sqrt{3}}{2}$

Therefore $\quad \sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi=\frac{1}{4}+\frac{3}{4}=1$

3 Let $\sin^{-1}\frac{3}{5}=\alpha$, and $\sin^{-1}\frac{8}{17}=\beta$,

then $\quad \sin\alpha=\frac{3}{5}, \qquad \cos\alpha=\sqrt{\left(1-\frac{9}{25}\right)}=\frac{4}{5};$

and $\quad \sin\beta=\frac{8}{17}, \qquad \cos\beta=\sqrt{\left(1-\frac{64}{289}\right)}=\frac{15}{17};$

therefore $\quad \sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$$=\frac{3\times 15}{5\times 17}+\frac{4\times 8}{5\times 17}=\frac{45+32}{85}=\frac{77}{85},$$

therefore $\quad \alpha+\beta=\sin^{-1}\frac{77}{85}$.

4. Let $\alpha=\tan^{-1}x$, and $\beta=\cot^{-1}x$;

then $\quad \tan\alpha=x$, and $\cot\beta=x$; therefore $\tan\beta=\frac{1}{x}$,

and $\quad \tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\dfrac{x+\frac{1}{x}}{1-1}=\dfrac{x+\frac{1}{x}}{0}$

Thus $\tan(\alpha+\beta)$ is infinite

5 Let $\tan^{-1}\frac{1}{3}=\alpha$, $\tan^{-1}\frac{1}{5}=\beta$, $\tan^{-1}\frac{1}{7}=\gamma$, $\tan^{-1}\frac{1}{8}=\delta$

Thus $\tan\alpha=\frac{1}{3}$, and $\tan\beta=\frac{1}{5}$;

therefore $$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{3}\times\frac{1}{5}}=\frac{8}{14}=\frac{4}{7}$$

And $$\tan(\gamma+\delta)=\frac{\tan\gamma+\tan\delta}{1-\tan\gamma\tan\delta}=\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}}=\frac{15}{55}=\frac{3}{11}.$$

Then $$\tan(\alpha+\beta+\gamma+\delta)=\frac{\tan(\alpha+\beta)+\tan(\gamma+\delta)}{1-\tan(\alpha+\beta)\tan(\gamma+\delta)}=\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{4}{7}\times\frac{3}{11}}=\frac{65}{65}=1,$$

therefore $$\alpha+\beta+\gamma+\delta=\frac{\pi}{4}.$$

6 Let $\tan^{-1}a=\theta$, and $\tan^{-1}b=\phi$,

then $a=\tan\theta$, and $b=\tan\phi$,

and $$\tan(\theta-\phi)=\frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}=\frac{a-b}{1+ab}.$$

Thus $$\tan^{-1}\frac{a-b}{1+ab}=\tan^{-1}a-\tan^{-1}b$$

Similarly $$\tan^{-1}\frac{b-c}{1+bc}=\tan^{-1}b-\tan^{-1}c.$$

Therefore $$\tan^{-1}\frac{a-b}{1+ab}+\tan^{-1}\frac{b-c}{1+bc}=\tan^{-1}a-\tan^{-1}c,$$

and $$\tan^{-1}\frac{a-b}{1+ab}+\tan^{-1}\frac{b-c}{1+bc}+\tan^{-1}c=\tan^{-1}a$$

7. Let $\alpha=\tan^{-1}\frac{1}{7}$, $\beta=\tan^{-1}\frac{1}{3}$, $\gamma=\tan^{-1}\frac{1}{26}$.

$$\tan 3\alpha=\frac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}=\frac{\frac{3}{7}-\frac{1}{7^3}}{1-\frac{3}{7^2}}=\frac{146}{322}=\frac{73}{161}.$$

$$\tan(\beta+\gamma)=\frac{\tan\beta+\tan\gamma}{1-\tan\beta\tan\gamma}=\frac{\frac{1}{8}+\frac{1}{26}}{1-\frac{1}{8\times 26}}=\frac{29}{77}.$$

$$\tan(3\alpha+\beta+\gamma)=\frac{\tan 3\alpha+\tan(\beta+\gamma)}{1-\tan 3\alpha\tan(\beta+\gamma)}=\frac{\frac{73}{161}+\frac{29}{77}}{1-\frac{73}{161}\times\frac{29}{77}}$$

$$=\frac{10290}{10280}=\frac{1029}{1028}$$

$$\tan\left(3\alpha+\beta+\gamma-\frac{\pi}{4}\right)=\frac{\tan(3\alpha+\beta+\gamma)-1}{1+\tan(3\alpha+\beta+\gamma)}=\frac{\frac{1029}{1028}-1}{1+\frac{1029}{1028}}=\frac{1}{2057}$$

8 We see as in the solution of Example 6 that

$$\tan^{-1}\{(\sqrt{2}+1)\tan\alpha\}-\tan^{-1}\{(\sqrt{2}-1)\tan\alpha\}$$

$$=\tan^{-1}\frac{(\sqrt{2}+1)\tan\alpha-(\sqrt{2}-1)\tan\alpha}{1+(\sqrt{2}+1)(\sqrt{2}-1)\tan^2\alpha}$$

$$=\tan^{-1}\frac{2\tan\alpha}{1+\tan^2\alpha}=\tan^{-1}(\sin 2\alpha).$$

9. $\tan(\theta-\alpha)\tan(\theta-\beta)=\tan^2\theta$,

therefore $$\frac{\sin(\theta-\alpha)\sin(\theta-\beta)}{\cos(\theta-\alpha)\cos(\theta-\beta)}=\frac{1-\cos 2\theta}{1+\cos 2\theta};$$

therefore $$\frac{\cos(\alpha-\beta)-\cos(2\theta-\alpha-\beta)}{\cos(\alpha-\beta)+\cos(2\theta-\alpha-\beta)}=\frac{1-\cos 2\theta}{1+\cos 2\theta},$$

therefore $$\cos(\alpha-\beta)\cos 2\theta=\cos(2\theta-\alpha-\beta)$$

$$=\cos 2\theta\cos(\alpha+\beta)+\sin 2\theta\sin(\alpha+\beta),$$

therefore $$\tan 2\theta\sin(\alpha+\beta)=\cos(\alpha-\beta)-\cos(\alpha+\beta)=2\sin\alpha\sin\beta,$$

therefore $$\tan 2\theta=\frac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)},$$

therefore $$2\theta=\tan^{-1}\frac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$$

10 Let $\alpha=\cos^{-1}\frac{9}{\sqrt{(82)}}$, and $\beta=\operatorname{cosec}^{-1}\frac{\sqrt{(41)}}{4}$,

then
$$\cos\alpha=\frac{9}{\sqrt{(82)}},\quad\text{and }\sin\alpha=\frac{1}{\sqrt{(82)}};$$
$$\sin\beta=\frac{4}{\sqrt{(41)}},\quad\cos\beta=\frac{5}{\sqrt{(41)}}$$

Therefore
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$=\frac{45-4}{\sqrt{(82)}\times\sqrt{(41)}}=\frac{41}{41\sqrt{2}}=\frac{1}{\sqrt{2}}.$$

Therefore
$$\alpha+\beta=\frac{\pi}{4}$$

11 Let $\alpha=\sin^{-1}\frac{4}{5}$, $\beta=\sin^{-1}\frac{5}{13}$, $\gamma=\sin^{-1}\frac{16}{65}$,

then
$$\sin\alpha=\frac{4}{5},\quad\sin\beta=\frac{5}{13},\quad\sin\gamma=\frac{16}{65},$$
and
$$\cos\alpha=\frac{3}{5},\quad\cos\beta=\frac{12}{13},\quad\cos\gamma=\frac{63}{65}.$$

Then
$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta=\frac{48+15}{65}=\frac{63}{65},$$

thus
$$\sin(\alpha+\beta)=\cos\gamma,\quad\text{so that }\alpha+\beta+\gamma=\frac{\pi}{2}.$$

12 Let $\alpha=\tan^{-1}\frac{1}{4}$, and $\beta=\tan^{-1}\frac{1}{20}$.

$$\tan 3\alpha=\frac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}=\frac{\frac{3}{4}-\frac{1}{4^3}}{1-\frac{3}{4^2}}=\frac{47}{52}.$$

$$\tan(3\alpha+\beta)=\frac{\tan 3\alpha+\tan\beta}{1-\tan 3\alpha\tan\beta}=\frac{\frac{47}{52}+\frac{1}{20}}{1-\frac{47}{52\times 20}}=\frac{992}{993}.$$

Again, let $\gamma=\tan^{-1}\frac{1}{1985}$, then

$$\tan\left(\frac{\pi}{4}-\gamma\right)=\frac{1-\frac{1}{1985}}{1+\frac{1}{1985}}=\frac{1984}{1986}=\frac{992}{993}.$$

Therefore
$$3\alpha+\beta=\frac{\pi}{4}-\gamma.$$

13 Let $\theta=\tan^{-1}\frac{2a-b}{b\sqrt{3}}$, $\phi=\tan^{-1}\frac{2b-a}{a\sqrt{3}}$,

then $$\tan(\theta+\phi)=\frac{\frac{2a-b}{b\sqrt{3}}+\frac{2b-a}{a\sqrt{3}}}{1-\frac{(2a-b)(2b-a)}{3ab}}=\frac{a(2a-b)+b(2b-a)}{3ab-(2a-b)(2b-a)}\sqrt{3}$$

$$=\frac{2(a^2+b^2)-2ab}{2(a^2+b^2)-2ab}\sqrt{3}=\sqrt{3}$$

Thus $$\theta+\phi=\frac{\pi}{3}.$$

14 Let $\tan^{-1}a=\theta$, and $\tan^{-1}a^3=\phi$

Then $$\tan\theta=a, \text{ and } \tan 2\theta=\frac{2a}{1-a^2}.$$

Also $$\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}=\frac{a+a^3}{1-a^4}=\frac{a}{1-a^2}.$$

Therefore $\tan(2\tan^{-1}a)=2\tan(\tan^{-1}a+\tan^{-1}a^3)$

15. Let $\tan^{-1}\left(\frac{1}{2}\tan 2A\right)=\alpha$, then $\tan\alpha=\frac{1}{2}\tan 2A$,

let $$\tan^{-1}(\cot A)=\beta, \text{ then } \tan\beta=\cot A,$$

let $$\tan^{-1}(\cot^3 A)=\gamma, \text{ then } \tan\gamma=\cot^3 A$$

Thus $$\tan(\beta+\gamma)=\frac{\tan\beta+\tan\gamma}{1-\tan\beta\tan\gamma}=\frac{\cot A+\cot^3 A}{1-\cot^4 A}=\frac{\cot A}{1-\cot^2 A}$$

$$=\frac{\frac{1}{\tan A}}{1-\frac{1}{\tan^2 A}}=\frac{\tan A}{\tan^2 A-1}=-\frac{1}{2}\tan 2A=-\tan\alpha.$$

Therefore $$\tan(\alpha+\beta+\gamma)=\frac{\tan\alpha+\tan(\beta+\gamma)}{1-\tan\alpha\tan(\beta+\gamma)}=0.$$

Thus $$\alpha+\beta+\gamma=0$$

16 Let $\cos^{-1}\frac{a}{b}=\theta$, then $\cos\theta=\frac{a}{b}$.

And $$\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}+\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$$

$$=\frac{\left(1-\tan\frac{\theta}{2}\right)^2+\left(1+\tan\frac{\theta}{2}\right)^2}{1-\tan^2\frac{\theta}{2}}=2\frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}=2\frac{1}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}$$

$$=\frac{2}{\cos\theta}=\frac{2b}{a}.$$

17 Let $\tan^{-1}\frac{a}{b}=\theta$, then $\tan\theta=\frac{a}{b}$,

$$\operatorname{cosec}^2\frac{\theta}{2}=\frac{1}{\sin^2\frac{\theta}{2}}=\frac{2}{1-\cos\theta}=\frac{2}{1-\frac{b}{\sqrt{(a^2+b^2)}}}=\frac{2\sqrt{(a^2+b^2)}}{\sqrt{(a^2+b^2)}-b}$$

$$=\frac{2\sqrt{(a^2+b^2)}}{\sqrt{(a^2+b^2)}-b}\times\frac{\sqrt{(a^2+b^2)}+b}{\sqrt{(a^2+b^2)}+b}=\frac{2(a^2+b^2)+2b\sqrt{(a^2+b^2)}}{a^2},$$

therefore $$\frac{a^3}{2}\operatorname{cosec}^2\frac{\theta}{2}=a(a^2+b^2)+ab\sqrt{(a^2+b^2)}$$

Let $\tan^{-1}\frac{b}{a}=\phi$, then $\tan\phi=\frac{b}{a}$

$$\sec^2\frac{\phi}{2}=\frac{1}{\cos^2\frac{\phi}{2}}=\frac{2}{1+\cos\phi}=\frac{2}{1+\frac{a}{\sqrt{(a^2+b^2)}}}=\frac{2\sqrt{(a^2+b^2)}}{\sqrt{(a^2+b^2)}+a}$$

$$=\frac{2\sqrt{(a^2+b^2)}}{\sqrt{(a^2+b^2)}+a}\times\frac{\sqrt{(a^2+b^2)}-a}{\sqrt{(a^2+b^2)}-a}=\frac{2(a^2+b^2)-2a\sqrt{(a^2+b^2)}}{b^2},$$

therefore $$\frac{b^3}{2}\sec^2\frac{\phi}{2}=b(a^2+b^2)-ab\sqrt{(a^2+b^2)}$$

Therefore $$\frac{a^3}{2}\operatorname{cosec}^2\frac{\theta}{2}+\frac{b^3}{2}\sec^2\frac{\phi}{2}=(a+b)(a^2+b^2)$$

18 $\sin^{-1}x+\sin^{-1}\frac{x}{2}=\frac{\pi}{4}$,

therefore $$\sin^{-1}\frac{x}{2}=\frac{\pi}{4}-\sin^{-1}x$$

Take the sines of both sides; thus

$$\frac{x}{2}=\sin\left(\frac{\pi}{4}-\sin^{-1}x\right)=\sin\frac{\pi}{4}\cdot\sqrt{(1-x^2)}-\cos\frac{\pi}{4}\ x$$

$$=\frac{\sqrt{(1-x^2)}-x}{\sqrt{2}},$$

therefore $$x\left(\frac{1}{\sqrt{2}}+1\right)=\sqrt{(1-x^2)},$$

therefore $$x^2\left(\frac{1}{\sqrt{2}}+1\right)^2=1-x^2,$$

therefore $$x^2\left(\frac{5}{2}+\frac{2}{\sqrt{2}}\right)=1,$$

therefore $$x^2(5+2\sqrt{2})=2,$$

therefore $$x^2=\frac{2}{5+2\sqrt{2}}=\frac{2}{5+2\sqrt{2}}\cdot\frac{5-2\sqrt{2}}{5-2\sqrt{2}}=\frac{2}{17}(5-2\sqrt{2})$$

19. We shall first shew that $\sin^{-1}\dfrac{2a}{1+a^2}=2\tan^{-1}a$

Let $\tan^{-1}a=\theta$, then $\tan\theta=a$, and $\sin 2\theta=\dfrac{2\tan\theta}{1+\tan^2\theta}=\dfrac{2a}{1+a^2}$,

therefore $$\sin^{-1}\frac{2a}{1+a^2}=2\theta=2\tan^{-1}a$$

Similarly $$\sin^{-1}\frac{2b}{1+b^2}=2\tan^{-1}b$$

Hence the equation may be written

$$2\tan^{-1}a+2\tan^{-1}b=2\tan^{-1}x;$$

therefore $$\tan^{-1}x=\tan^{-1}a+\tan^{-1}b$$

Take the tangents of both sides, thus

$$x=\tan(\tan^{-1}a+\tan^{-1}b)=\frac{a+b}{1-ab}.$$

20 Let $\tan^{-1}(x-1)=\alpha$, $\tan^{-1}x=\beta$, $\tan^{-1}(x+1)=\gamma$.

Thus $$\tan^{-1}3x=\alpha+\beta+\gamma$$

Take the tangents of both sides, thus

$$3x=\tan(\alpha+\beta+\gamma)=\frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\beta\tan\gamma-\tan\gamma\tan\alpha-\tan\alpha\tan\beta}$$

$$=\frac{3x-x(x^2-1)}{1-x(x+1)-(x+1)(x-1)-x(x-1)}=\frac{4x-x^3}{2-3x^2}.$$

Therefore either $x=0$, or $3(2-3x^2)=4-x^2$, the latter gives $8x^2=2$, therefore $x^2=\frac{1}{4}$; therefore $x=\pm\frac{1}{2}$.

21 $\sin^{-1}2x-\sin^{-1}x\sqrt{3}=\sin^{-1}x$.

Take the sines of both sides, thus

$$2x\sqrt{(1-3x^2)}-x\sqrt{3}\times\sqrt{(1-4x^2)}=x.$$

Thus either $x=0$, or $2\sqrt{(1-3x^2)}-\sqrt{3}\times\sqrt{(1-4x^2)}=1$

Transpose, thus $2\sqrt{(1-3x^2)}=1+\sqrt{3}\times\sqrt{(1-4x^2)}$

Square, $4(1-3x^2)=1+2\sqrt{3}\times\sqrt{(1-4x^2)}+3(1-4x^2)$,

therefore $2\sqrt{3}\times\sqrt{(1-4x^2)}=0$;

therefore $1-4x^2=0$, therefore $x=\pm\frac{1}{2}$

22 $\tan^{-1}\frac{1}{4}+2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{6}+\tan^{-1}\frac{1}{x}=\frac{\pi}{4}$

Let $\tan^{-1}\frac{1}{4}=\alpha,\quad \tan^{-1}\frac{1}{5}=\beta,\quad \tan^{-1}\frac{1}{6}=\gamma$

Thus the equation may be written

$$\tan^{-1}\frac{1}{x}=\frac{\pi}{4}-(\alpha+2\beta+\gamma),$$

therefore $$\frac{1}{x}=\frac{1-\tan(\alpha+2\beta+\gamma)}{1+\tan(\alpha+2\beta+\gamma)}$$

Now $$\tan(\alpha+\beta)=\frac{\frac{1}{4}+\frac{1}{5}}{1-\frac{1}{4}\times\frac{1}{5}}=\frac{9}{19},$$

$$\tan(\beta+\gamma)=\frac{\frac{1}{5}+\frac{1}{6}}{1-\frac{1}{5}\times\frac{1}{6}}=\frac{11}{29},$$

therefore $$\tan(\alpha+\beta+\beta+\gamma)=\frac{\frac{9}{19}+\frac{11}{29}}{1-\frac{9}{19}\times\frac{11}{29}}=\frac{470}{452}.$$

Hence $$\frac{1}{x}=\frac{1-\frac{470}{452}}{1+\frac{470}{452}}=-\frac{18}{922}=-\frac{9}{461}$$

23. Let $\tan^{-1}x=\theta$, then $\tan\theta=x$, $\cot 2\theta=\frac{1}{\tan 2\theta}=\frac{1-x^2}{2x}$

Thus the equation may be written

$$\sin 2\cos^{-1}\frac{1-x^2}{2x}=0.$$

Now since $2\cos^{-1}\frac{1-x^2}{2x}$ has zero for its sine, the angle must be of the form $n\pi$, where n is zero or some integer.

Thus $\quad 2\cos^{-1}\frac{1-x^2}{2x}=n\pi,\quad$ therefore $\cos^{-1}\frac{1-x^2}{2x}=\frac{n\pi}{2}$,

therefore
$$\frac{1-x^2}{2x}=\cos\frac{n\pi}{2}$$

Since n is zero or an integer we have $\cos\frac{n\pi}{2}=0$, or 1, or -1

If $\frac{1-x^2}{2x}=0$, then $x=\pm1$

If $\frac{1-x^2}{2x}=1$, then $x^2+2x=1$, and from this we deduce $x=-1\pm\sqrt{2}$

If $\frac{1-x^2}{2x}=-1$, then $x^2-2x=1$, and from this we deduce $x=1\pm\sqrt{2}$.

24.
$$\tan^{-1}\frac{1}{a-1}=\tan^{-1}\frac{1}{x}+\tan^{-1}\frac{1}{a^2-x+1},$$

therefore
$$\tan^{-1}\frac{1}{a-1}-\tan^{-1}\frac{1}{x}=\tan^{-1}\frac{1}{a^2-x+1}.$$

Take the tangents of both sides; thus
$$\frac{\frac{1}{a-1}-\frac{1}{x}}{1+\frac{1}{(a-1)x}}=\frac{1}{a^2-x+1},$$

therefore
$$\frac{x-a+1}{ax-x+1}=\frac{1}{a^2-x+1},$$

therefore
$$(x-a+1)(a^2-x+1)=ax-x+1;$$

therefore
$$-x^2+x(a^2+a)-a^3+a^2-a+1=ax-x+1;$$

therefore
$$x^2-x(a^2+1)+a^3-a^2+a=0$$

By solving this quadratic in the ordinary way we obtain $x=a$ or a^2-a+1

25 $\quad \sec\theta-\operatorname{cosec}\theta=\frac{4}{3}$;

therefore
$$\frac{1}{\cos\theta}-\frac{1}{\sin\theta}=\frac{4}{3},$$

therefore
$$\sin\theta-\cos\theta=\frac{4}{3}\sin\theta\cos\theta=\frac{2}{3}\sin 2\theta$$

Square, thus $$1-\sin 2\theta=\frac{4}{9}\sin^2 2\theta$$

By solving this quadratic in the usual way we obtain $\sin 2\theta=\frac{3}{4}$, or -3, the former value is alone applicable Thus $\sin 2\theta=\frac{3}{4}$,

therefore $$2\theta=\sin^{-1}\frac{3}{4},\quad \text{therefore } \theta=\frac{1}{2}\sin^{-1}\frac{3}{4}.$$

26 $\sin(\pi\cos\theta)=\cos(\pi\sin\theta)$,

therefore $$\cos\left(\frac{\pi}{2}-\pi\cos\theta\right)=\cos(\pi\sin\theta).$$

Hence, by Art 67 the solutions are comprised in

$$\frac{\pi}{2}-\pi\cos\theta=2n\pi\pm\pi\sin\theta,$$

therefore $$\cos\theta\pm\sin\theta=\frac{1}{2}-2n$$

Square, thus $$1\pm\sin 2\theta=\left(\frac{1}{2}-2n\right)^2$$

If we give to n any integral value, positive or negative, the value of $\sin 2\theta$ is greater than unity Thus we must have n zero Then $1\pm\sin 2\theta=\frac{1}{4}$, and therefore $\sin 2\theta=\pm\frac{3}{4}$; thus $2\theta=\pm\sin^{-1}\frac{3}{4}$, and $\theta=\pm\frac{1}{2}\sin^{-1}\frac{3}{4}$

27. Let $\psi=\sin^{-1}(\sin\theta+\sin\phi)+\sin^{-1}(\sin\theta-\sin\phi)$

Take the cosines of both sides; thus

$$\cos\psi=\sqrt{1-(\sin\theta+\sin\phi)^2}\sqrt{1-(\sin\theta-\sin\phi)^2}-(\sin\theta+\sin\phi)(\sin\theta-\sin\phi)$$

$$=\sqrt{1-\frac{1}{2}-2\sin\theta\sin\phi}\sqrt{1-\frac{1}{2}+2\sin\theta\sin\phi}-(\sin^2\theta-\sin^2\phi)$$

$$=\sqrt{\frac{1}{4}-4\sin^2\theta\sin^2\phi}-(\sin^2\theta-\sin^2\phi)$$

Now $$\frac{1}{2}=\sin^2\theta+\sin^2\phi,\quad \text{therefore } \frac{1}{4}=(\sin^2\theta+\sin^2\phi)^2,$$

therefore $$\frac{1}{4}-4\sin^2\theta\sin^2\phi=(\sin^2\theta-\sin^2\phi)^2.$$

Thus $$\cos\psi=\pm(\sin^2\theta-\sin^2\phi)-(\sin^2\theta-\sin^2\phi)$$

Taking the upper sign we have $\cos\psi=0$, and therefore $\psi=(2n+1)\frac{\pi}{2}$, where n is any integer

28
$$3\tan^{-1}\frac{1}{2+\sqrt{3}}-\tan^{-1}\frac{1}{x}=\tan^{-1}\frac{1}{3}$$

Now let $\tan^{-1}\dfrac{1}{2+\sqrt{3}}=\theta$, then $\tan\theta=\dfrac{1}{2+\sqrt{3}}$,

therefore
$$\tan 3\theta=\frac{\dfrac{3}{2+\sqrt{3}}-\dfrac{1}{(2+\sqrt{3})^3}}{1-\dfrac{3}{(2+\sqrt{3})^2}}=\frac{3(2+\sqrt{3})^2-1}{(2+\sqrt{3})^3-3(2+\sqrt{3})}$$

$$=\frac{20+12\sqrt{3}}{20+12\sqrt{3}}=1, \quad \text{therefore } 3\theta=\tan^{-1}1.$$

This might also have been inferred from the fact that

$$\frac{1}{2+\sqrt{3}}=\tan 15^\circ=\tan\frac{\pi}{12}, \quad \text{so that } \theta=\frac{\pi}{12}$$

The equation may now be written

$$\tan^{-1}1-\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{1}{x}$$

Take the tangents of both sides, thus

$$\frac{1-\dfrac{1}{3}}{1+\dfrac{1}{3}}=\frac{1}{x},$$

therefore $\dfrac{1}{x}=\dfrac{1}{2}$, therefore $x=2$

29 Let $\sin^{-1}\sqrt{\left(\dfrac{a+b}{a+c}\right)}=\theta$, then $\sin\theta=\sqrt{\left(\dfrac{a+b}{a+c}\right)}$,

then
$$\cos 2\theta=1-2\sin^2\theta=1-\frac{2(a+b)}{a+c}=\frac{c-a-2b}{a+c}$$

and
$$2\theta=\cos^{-1}\frac{c-a-2b}{a+c}$$

Thus the proposed expression is

$$\sin^{-1}\frac{2b+a-c}{a+c}\pm\cos^{-1}\frac{c-a-2b}{a+c},$$

that is
$$\sin^{-1}p\pm\cos^{-1}(-p),$$

when p is put for $\dfrac{2b+a-c}{a+c}$

Now $\cos\{\sin^{-1}p \pm \cos^{-1}(-p)\} = -p\sqrt{(1-p^2)} \mp p\sqrt{(1-p^2)}$;
thus zero is one of the values of the cosine, and the corresponding angle is an odd multiple of $\frac{\pi}{2}$

30. $$\tan^{-1}x + \cot^{-1}y = \tan^{-1}3,$$

therefore $$\tan^{-1}x + \tan^{-1}\frac{1}{y} = \tan^{-1}3$$

Take the tangents of both sides, thus

$$\frac{x+\frac{1}{y}}{1-\frac{x}{y}} = 3;$$

therefore $$3(y-x) = yx + 1,$$

therefore $$x = \frac{3y-1}{y+3} = \frac{3y+9-10}{y+3} = 3 - \frac{10}{y+3}.$$

Thus if x and y are to be positive integers $y+3$ must be a divisor of 10 Try in succession the various cases, namely $y+3=1$ or 2 or 5 or 10 It will be found that the only admissible cases are $y+3=5$, and $y+3=10$ These give $y=2$ or 7, and the corresponding values of x are 1 and 2

31 $$\tan^{-1}x + \tan^{-1}y = \tan^{-1}c$$

Take the tangents of both sides, thus $\frac{x+y}{1-xy} = c$,

therefore $x + y = c(1-xy)$, therefore $x = \frac{c-y}{1+cy}$.

It is obvious that if c and y are positive integers x is either a positive or negative proper fraction, and cannot be a positive integer

Next take $$\cot^{-1}x + \cot^{-1}y = \cot^{-1}c$$

Take the cotangents of both sides; thus $\frac{xy-1}{x+y} = c$;

therefore $$xy - 1 = c(x+y),$$

therefore $$x = \frac{cy+1}{y-c} = \frac{cy-c^2+c^2+1}{y-c}$$
$$= c + \frac{c^2+1}{y-c}.$$

Thus if a denote any divisor of c^2+1 we may put $y-c=a$, so that $y=c+a$, and then $x = c + \frac{c^2+1}{a}$

Hence we see that there are as many solutions in positive integers as there are divisors of c^2+1

32 $$\tan^{-1}\frac{c_1x-y}{c_1y+x}=\tan^{-1}\frac{\frac{x}{y}-\frac{1}{c_1}}{1+\frac{x}{c_1y}}=\tan^{-1}\frac{x}{y}-\tan^{-1}\frac{1}{c_1},$$

as in the solution of Example 6

Similarly $$\tan^{-1}\frac{c_2-c_1}{c_2c_1+1}=\tan^{-1}\frac{1}{c_1}-\tan^{-1}\frac{1}{c_2},$$

$$\tan^{-1}\frac{c_3-c_2}{c_3c_2+1}=\tan^{-1}\frac{1}{c_2}-\tan^{-1}\frac{1}{c_3},$$

and so on

Thus the sum of the terms on the right-hand side of the proposed expression is $\tan^{-1}\frac{x}{y}$

33 Let $\sin^{-1}\frac{2ab}{a^2+b^2}=\theta$, and $\sin^{-1}\frac{2a'b'}{a'^2+b'^2}=\phi$,

then $$\sin\theta=\frac{2ab}{a^2+b^2},\text{ and }\sin\phi=\frac{2a'b'}{a'^2+b'^2},$$

therefore $$\cos\theta=\frac{a^2-b^2}{a^2+b^2},\text{ and }\cos\phi=\frac{a'^2-b'^2}{a'^2+b'^2},$$

therefore $$\sin(\theta+\phi)=\frac{2ab(a'^2-b'^2)+2a'b'(a^2-b^2)}{(a^2+b^2)(a'^2+b'^2)}$$

$$=\frac{2ab(a'^2-b'^2)+2a'b'(a^2-b^2)}{(ab'+a'b)^2+(aa'-bb')^2}=\frac{2(ab'+a'b)(aa'-bb')}{(ab'+a'b)^2+(aa'-bb')^2};$$

therefore $$\theta+\phi=\sin^{-1}\frac{2pq}{p^2+q^2},$$

where p and q are rational expressions

Then if there be another angle $\sin^{-1}\frac{2a''b''}{a''^2+b''^2}$, we may denote it by ψ, then $\sin\{(\theta+\phi)+\psi\}$ will take the form $\frac{2rs}{r^2+s^2}$ where r and s are rational And so on

34 We may take for the simplest value of $\sin^{-1}\frac{(-1)^m}{2}$ the angle $(-1)^m\frac{\pi}{6}$, as is evident by supposing m first even and then odd This will be the α of Art 66, and the general solution is $n\pi+(-1)^n\alpha$, that is $n\pi+(-1)^{m+n}\frac{\pi}{6}$

Or we may take the form $(m+n)\pi+(-1)^n\frac{\pi}{6}$

For the sine of this angle

$$=\sin m\pi\cos\left\{n\pi+(-1)^n\frac{\pi}{6}\right\}+\cos m\pi\sin\left\{n\pi+(-1)^n\frac{\pi}{6}\right\}$$

$$=\cos m\pi\sin\left\{n\pi+(-1)^n\frac{\pi}{6}\right\}=\cos m\pi\times\frac{1}{2}=(-1)^m\times\frac{1}{2}.$$

35. If m be even the value is $\cos^{-1}\frac{1}{2}$, that is $2n\pi\pm\frac{\pi}{3}$

If m be odd the value is $\cos^{-1}\left(-\frac{1}{2}\right)$, that is $2n\pi\pm\left(\pi+\frac{\pi}{3}\right)$.

Both forms may be comprised in $(2p+m)\pi\pm\frac{\pi}{3}$, where p is any integer

For $2n\pi\pm\frac{\pi}{3}$ consists of an *even* multiple of π augmented by $\pm\frac{\pi}{3}$, and $2n\pi\pm\left(\pi+\frac{\pi}{3}\right)$ consists of an *odd* multiple of π augmented by $\pm\frac{\pi}{3}$

36 If m be even the value is $\tan^{-1}1$, that is $n\pi+\frac{\pi}{4}$

If m be odd the value is $\tan^{-1}(-1)$, that is $n\pi-\frac{\pi}{4}$.

Both forms may be comprised in $n\pi+(-1)^m\frac{\pi}{4}$

XVIII

1 In the figure of Art. 266, let the angle $PAB=45°-A$, then

$$\angle PBA=45°+A;\quad \angle CPM=90°-\angle PCM=90°-2\angle PAB=2A,$$

$$\tan(45°+A)-\tan(45°-A)=\frac{PM}{MB}-\frac{PM}{MA}=\frac{PM^2}{PM\cdot MB}-\frac{PM^2}{PM\cdot MA}$$

$$=\frac{AM\cdot MB}{PM\cdot MB}-\frac{AM\cdot MB}{PM\cdot MA}=\frac{MA-MB}{PM}=\frac{2CM}{PM}=2\tan 2A$$

2 Let BAC be an angle of $45°$, draw BC at right angles to AC, produce CB to P, join AP and draw PN at right angles to AB produced.

Let $AC=b=BC$, $CP=a$ Then $BP=a-b$, therefore

$$PN=BN=\frac{a-b}{\sqrt{2}},\text{ also } AB=\frac{2b}{\sqrt{2}},\text{ therefore } AN=\frac{a+b}{\sqrt{2}}.$$

$$\tan PAN=\frac{PN}{AN}=\frac{a-b}{a+b},$$

and
$$\tan PAC=\frac{BC}{AC}=\frac{a}{b}$$

$$\text{since } \angle BAC=\angle PAC-\angle PAN,$$

$$\frac{\pi}{4}=\tan^{-1}\frac{a}{b}-\tan^{-1}\frac{a-b}{a+b}$$

3 From I and I_1 the centres of the inscribed and escribed circles of triangle ABC draw IX, I_1X_1, and IY, I_1Y_1 perpendicular to AB and respectively By Chapter XVI, $AX=s-a$, $AX_1=s$, $CY_1=s-b$, $CY=s-c$

Now
$$AX\tan\frac{A}{2}=IX=IY=CY\tan\frac{C}{2},$$

$$AX_1\tan\frac{A}{2}=I_1X_1=I_1Y_1=CY_1\cot\frac{C}{2}$$

multiplying,
$$s(s-a)\tan^2\frac{A}{2}=(s-b)(s-c)$$

4 Through B draw BP cutting AC produced in P so that the a $CBP=\theta$, then the angle $ABP=B+\theta$, $APB=C-\theta$ Draw AF, CG per dicular to BP, and CH perpendicular to AF

$$BG=BF+FG=BF+HC$$

Hence
$$BC\cos\theta=BA\cos(B+\theta)+AC\sin FAP,$$

or
$$a\cos\theta=c\cos(B+\theta)+b\cos(C-\theta)$$

5 Draw straight lines ON, OP_1, OP_2, OP_{n+1} so that

$$\angle P_1ON=\angle P_2OP_1=\angle P_3OP_2=\quad=\alpha$$

Let $NP_1P_2\ \ P_{n+1}$ be a straight line perpendicular to ON, cutting t lines in $N, P_1, P_2,\ \ P_{n+1}$ Then $\angle P_rON=r\alpha$, and $\sec r\alpha=\dfrac{OP_r}{ON}$. Hence sum of the given series

$$=(OP_1\ OP_2+OP_2\ OP_3+\quad)\frac{1}{ON^2}$$

$$=\frac{2}{\sin\alpha}\left(\frac{1}{2}OP_1\ OP_2\sin\alpha+\frac{1}{2}OP_2\ OP_3\sin\alpha+\quad\right)\frac{1}{ON^2}$$

$$=\frac{2}{\sin\alpha}(\Delta P_1OP_2+\Delta P_2OP_3+\quad)\frac{1}{ON^2}$$

$$= \frac{2}{\sin\alpha}\ \Delta P_1 O P_{n+1}\ \frac{1}{ON^2}$$

$$= \frac{1}{\sin\alpha} \cdot OP_1\ \ OP_{n+1}\ \sin n\alpha\ \ \frac{1}{ON^2}$$

$$= \frac{\sin n\alpha\ \ \sec(n+1)\alpha}{\sin\alpha \cdot \cos\alpha} = 2\sin n\alpha \sec(n+1)\alpha \operatorname{cosec} 2\alpha$$

6 In a circle whose centre is O place n equal straight lines AB, BC, MN, each subtending an angle β at O, and let AB make an angle α with a straight line AX The exterior angle between successive sides is β, so that BC, CD, MN make angles $\alpha+\beta$, $\alpha+2\beta$, $\alpha+\overline{n-1}\beta$ with AX, therefore the sum of the projections of AB, BC, MN on AX is

$$AB\{\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ \ +\cos(\alpha+\overline{n-1}\beta)\}$$

This must be equal to the projection of AN on AX, namely $AN\cos NAX$

Now $\qquad \angle NOA = n\beta,\ \ \angle NOB = (n-1)\beta;$

therefore $\qquad \angle NAB = \frac{1}{2}(n-1)\beta$, and $\angle NAX = \alpha + \frac{1}{2}(n-1)\beta$

From the triangle ABN $\left(\text{whose angles } NAB, ANB, NBA, \text{ are } \frac{1}{2}(n-1)\beta,\ \frac{1}{2}\beta,\ \pi-\frac{1}{2}n\beta\right)$ we have

$$AN = AB\ \frac{\sin\frac{1}{2}n\beta}{\sin\frac{\beta}{2}},$$

$$\text{the given series} = \frac{\sin\frac{1}{2}n\beta}{\sin\frac{\beta}{2}}\ \cos NAX$$

$$= \frac{\cos\left(\alpha+\frac{1}{2}\overline{n-1}\beta\right)\sin\frac{1}{2}n\beta}{\sin\frac{\beta}{2}}$$

7 If a chord of a circle of radius R subtend an angle θ at the circumference, the length of the chord is $2R\sin\theta$

Let $ABCD$ be a quadrilateral inscribed in a circle, let $\angle ABD=\alpha$, $\angle DBC=\beta$, $\angle BAC=\gamma$, then $\angle CAD=\beta$ and $\angle ADB=\pi-(\alpha+\beta+\gamma)$, BD, AC subtend angles $\beta+\gamma$, and $\alpha+\beta$ at the circumference

Now by the proposition

$$AC\ \ BD = AD\ \ BC + CD\ \ AB$$

$$2R\sin(\alpha+\beta)\ \ 2R\sin(\beta+\gamma) = 2R\sin\alpha\ \ 2R\sin\gamma + 2R\sin\beta\ \ 2R\sin(\alpha+\beta+\gamma)$$

Next suppose that BD is a diameter Then

$$AB = BD\cos\alpha,\quad CD = BD\sin\beta,\quad AD = BD\sin\alpha,\quad BC = BD\cos\beta,$$

and
$$AC = 2R\sin(\alpha+\beta) = BD\sin(\alpha+\beta)$$

Substituting these we obtain equation (ii)

8 $\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha+\beta+\gamma)$

$$= 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta) - 2\cos\frac{1}{2}(\alpha+\beta+2\gamma)\ \sin\frac{1}{2}(\alpha+\beta)$$

$$= 2\sin\frac{1}{2}(\alpha+\beta)\left\{\cos\frac{1}{2}(\alpha-\beta) - \cos\frac{1}{2}(\alpha+\beta+2\gamma)\right\}$$

$$= 4\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha+\gamma)\sin\frac{1}{2}(\beta+\gamma).$$

(i) Put $\alpha+\beta+\gamma=\pi$, then $\sin\frac{1}{2}(\beta+\gamma) = \sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) = \cos\frac{\alpha}{2}$

Hence we obtain the result of Ex. 16, Ch. VIII

(ii) In the last result put $\pi-\alpha$ for α, $\pi-\beta$ for β, $\pi-\gamma$ for γ (this may be done since the sum of the three angles is still π) This gives Ex 17

(iii) For α, β, γ write $2A - \frac{\pi}{2}$, $2B - \frac{\pi}{2}$, $2C - \frac{\pi}{2}$, then $\alpha+\beta+\gamma = \frac{\pi}{2}$,

$$\sin\frac{1}{2}(\beta+\gamma) = \sin\left(B + C - \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} - A\right) = \cos A \qquad \text{This gives Ex 18}$$

(iv) For α, β, γ write $\frac{A}{2}$, $\frac{B}{2}$, $\frac{C}{2}$, then $\alpha+\beta+\gamma = \frac{\pi}{2}$, and

$$\sin\frac{1}{2}(\beta+\gamma) = \sin\frac{1}{2}\left(\frac{\pi}{2} - \frac{A}{2}\right) \qquad \text{This gives Ex 22}$$

9 (i) $\sin 3A = 3\sin A - 4\sin^3 A$,

$$\therefore \sin^3 A + \sin^3 B + \sin^3 C = \frac{3}{4}(\sin A + \sin B + \sin C)$$
$$- \frac{1}{4}(\sin 3A + \sin 3B + \sin 3C)$$

$$= 3\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + \cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2}. \quad \text{[Ch. VIII, Ex. 16 and 32]}$$

(ii) $4\sin^4 A = (1 - \cos 2A)^2 = 1 - 2\cos 2A + \cos^2 2A$

$$= 1 - 2\cos 2A + \frac{1}{2}(1 + \cos 4A),$$

$$\sin^4 A = \frac{1}{8}(3 - 4\cos 2A + \cos 4A),$$

and similar expressions for $\sin^4 B$, $\sin^4 C$ may be written down.

From Ch. VIII, Ex. 18 and 19,

$$\cos 2A + \cos 2B - \cos 2C = -1 - 4\cos A\cos B\cos C,$$

$$\cos 4A + \cos 4B + \cos 4C = -1 + 4\cos 2A\cos 2B\cos 2C,$$

$\therefore \sin^4 A - \sin^4 B + \sin^4 C$

$$= \frac{1}{8}(9 - 4 + 16\cos A\cos B\cos C - 1 + 4\cos 2A\cos 2B\cos 2C)$$

$$= \frac{3}{2} + 2\cos A\cos B\cos C + \frac{1}{2}\cos 2A\cos 2B\cos 2C$$

10 Multiplying out, the right-hand side

$$= 2\Sigma\sin\alpha + 2\Sigma(\sin\beta\cos\gamma + \sin\gamma\cos\beta) + 2\Sigma\sin\alpha\cos\alpha$$

$$= 2\Sigma\sin\alpha + 2\Sigma\sin(\beta+\gamma) + \Sigma\sin 2\alpha$$

$$= 2\Sigma\sin\alpha - 2\Sigma\sin\alpha + \Sigma\sin 2\alpha$$

$$= \Sigma\sin 2\alpha.$$

11 See Ex. 12, Ch. VIII.

12 Since $(\beta-\gamma) - (\gamma-\alpha) + (\alpha-\beta) = 0$ we have, as in Art. 114,

$$\tan(\beta-\gamma) + \tan(\gamma-\alpha) + \tan(\alpha-\beta) = \tan(\beta-\gamma)\tan(\gamma-\alpha)\tan(\alpha-\beta).$$

Substitute this and transpose; we have to prove that

$$\tan(\beta-\gamma)(1+\tan\beta\tan\gamma) + \tan(\gamma-\alpha)(1+\tan\gamma\tan\alpha) + \tan(\alpha-\beta)(1+\tan\alpha\tan\beta) = 0,$$

or, since $\tan(\beta-\gamma) = \dfrac{\tan\beta - \tan\gamma}{1 - \tan\beta\tan\gamma}$,

that $\tan\beta - \tan\gamma + \tan\gamma - \tan\alpha - \tan\alpha - \tan\beta = 0.$

13 Put $\sin^2\alpha = a^2$, $\sin^2\beta = b^2$, $\sin^2\gamma = c^2$;

then $\cos 4\alpha = 2\cos^2 2\alpha - 1 = 2(1-2a^2)^2 - 1 = 1 - 8a^2 + 8a^4$,

$$\sin(\beta-\gamma)\sin(\beta+\gamma) = \sin^2\beta - \sin^2\gamma = b^2 - c^2$$

the left-hand side $= (1 - 8a^2 + 8a^4)(b^2 - c^2) +$ two similar terms

$$= 8\{a^4(b^2-c^2) + b^4(c^2-a^2) + c^4(a^2-b^2)\}$$

$$= -8(b^2-c^2)(c^2-a^2)(a^2-b^2)$$

$$= -8\sin(\beta-\gamma)\sin(\beta+\gamma)\ \sin(\gamma-\alpha)\sin(\gamma+\alpha)\ \sin(\alpha-\beta)\sin(\alpha+\beta)$$

14 Let $\sin(\beta+\gamma-\alpha) = a$, $\sin(\gamma+\alpha-\beta) = b$, $\sin(\alpha+\beta-\gamma) = c$

$$2\cos\alpha \,.\, \sin(\beta-\gamma) = \sin(\alpha+\beta-\gamma) - \sin(\alpha-\beta+\gamma) = c - b \qquad .(1).$$

$\cos 2(\beta+\gamma-\alpha)\sin(\beta-\gamma)\cos\alpha +$

$$= \frac{1}{2}(1-2a^2)(c-b) + \frac{1}{2}(1-2b^2)(a-c) + \frac{1}{2}(1-2c^2)(b-a)$$

$$= a^2(b-c) + b^2(c-a) + c^2(a-b)$$

$$= -(a-b)(b-c)(c-a)$$

$$= 8\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)\cos\alpha\cos\beta\cos\gamma, \text{ from (1)}$$

15 The left-hand side $=\Sigma \sin\alpha\,(\sin^2\alpha+\sin^2\beta+\sin^2\gamma-\sin^2\alpha)\sin(\beta-\gamma)$

$$=(\sin^2\alpha+\sin^2\beta+\sin^2\gamma)\,\Sigma\sin\alpha\sin(\beta-\gamma)-\Sigma\sin^3\alpha\sin(\beta-\gamma).$$

Now $\qquad \Sigma\sin\alpha\sin(\beta-\gamma)=0$ (Ch VIII, Ex 8),

and $\qquad -\Sigma\sin^3\alpha\sin(\beta-\gamma)=\frac{1}{4}\Sigma(\sin 3\alpha-3\sin\alpha)\sin(\beta-\gamma)$

$$=\frac{1}{4}\Sigma\sin 3\alpha\sin(\beta-\gamma)-\frac{3}{4}\Sigma\sin\alpha\sin(\beta-\gamma)$$

$$=\frac{1}{4}\Sigma\sin 3\alpha\sin(\beta-\gamma)=\frac{1}{8}\Sigma\{\cos(3\alpha-\beta+\gamma)-\cos(3\alpha+\beta-\gamma)\}$$

Add up the various terms in this expression as follows

$$\cos(3\alpha-\beta+\gamma)-\cos(3\beta+\gamma-\alpha)=2\sin(\alpha+\beta+\gamma)\sin(2\beta-2\alpha),$$
$$\cos(3\beta-\gamma+\alpha)-\cos(3\gamma+\alpha-\beta)=2\sin(\alpha+\beta+\gamma)\sin(2\gamma-2\beta),$$
$$\cos(3\gamma-\alpha+\beta)-\cos(3\alpha+\beta-\gamma)=2\sin(\alpha+\beta+\gamma)\sin(2\alpha-2\gamma)$$

Thus we obtain, $-\Sigma\sin^3\alpha\sin(\beta-\gamma)$

$$=\frac{1}{4}\sin(\alpha+\beta+\gamma)\{\sin(2\beta-2\alpha)+\sin(2\gamma-2\beta)+\sin(2\alpha-2\gamma)\}$$

$$=\sin(\alpha+\beta+\gamma)\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha) \qquad \text{[Ex 3, Ch VIII]}$$

16 $\cos(\alpha-\theta)\cos(\alpha-\phi)=\frac{1}{2}\{\cos(2\alpha-\theta-\phi)+\cos(\theta-\phi)\}$

$$=\frac{1}{2}\cos 2\alpha\cos(\theta+\phi)+\frac{1}{2}\sin 2\alpha\sin(\theta+\phi)+\frac{1}{2}\cos(\theta-\phi)$$

since $\Sigma(\sin 2\beta-\sin 2\gamma)=0$ and $\Sigma\sin 2\alpha(\sin 2\beta-\sin 2\gamma)=0$,

the left-hand side is

$$=\frac{1}{2}\cos(\theta+\phi)\{\cos 2\alpha(\sin 2\beta-\sin 2\gamma)+\cos 2\beta(\sin 2\gamma-\sin 2\alpha)$$
$$+\cos 2\gamma(\sin 2\alpha-\sin 2\beta)\}$$

$$=\frac{1}{2}\cos(\theta+\phi)\{\sin 2(\beta-\alpha)+\sin 2(\alpha-\gamma)+\sin 2(\gamma-\beta)\}$$

$$=2\cos(\theta+\phi)\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha) \qquad \text{(Ex. 3, Ch VIII)}$$

17 Multiply out the factors on the left-hand side and notice that wherever the product of two cosines occurs the corresponding product of two sines also occurs, combining these, we get

$$\cos 3\alpha+\cos(\alpha+2\beta)+\cos(\alpha+2\gamma)-\cos(\alpha+\beta+\gamma)-\cos(\gamma+2\alpha)-\cos(2\alpha+\beta)$$
$$+\cos(2\alpha+\beta)+\cos 3\beta+\cos(2\gamma+\beta)-\cos(2\beta+\gamma)-\cos(\alpha+\beta+\gamma)-\cos(\alpha+2\beta)$$
$$+\cos(2\alpha+\gamma)+\cos(2\beta+\gamma)+\cos 3\gamma-\cos(\beta+2\gamma)-\cos(2\gamma+\alpha)-\cos(\alpha+\beta+\gamma)$$
$$=\cos 3\alpha+\cos 3\beta+\cos 3\gamma-3\cos(\alpha+\beta+\gamma).$$

18 By Ex. 8, $\sin x+\sin y+\sin z-\sin(x+y+z)$

$$=4\sin\frac{1}{2}(y+z)\sin\frac{1}{2}(z+x)\sin\frac{1}{2}(x+y) \quad . \qquad (1)$$

Also $\quad 1+\cos(y-z)+\cos(z-x)+\cos(x-y)$

$$=2\cos^2\frac{1}{2}(y-z)+2\cos\frac{1}{2}(z-y)\cos\frac{1}{2}(z-2x+y)$$

$$=4\cos\frac{1}{2}(y-z)\cos\frac{1}{2}(z-x)\cos\frac{1}{2}(x-y) \qquad (2)$$

Since $\quad 2\sin\frac{1}{2}(y+z)\cos\frac{1}{2}(y-z)=\sin y+\sin z,$

the product of (1) and (2) is

$$2(\sin y+\sin z)(\sin z+\sin x)(\sin x+\sin y).$$

Hence, putting $\sin x=a$ &c , the right-hand side of the given identity is

$$3(a+b+c)-4\{(a+b+c)^3-3(b+c)(c+a)(a+b)\}$$
$$=3(a+b+c)-4(a^3+b^3+c^3)$$
$$=3a-4a^3+3b-4b^3+3c-4c^3$$
$$=\sin 3x+\sin 3y+\sin 3z$$

19 $(\sin A+\cos A)(\sin B+\cos B)(\sin C+\cos C)$

$=\sin A\sin B\sin C+\cos A\cos B\cos C+\Sigma\sin A\cos B\cos C+\Sigma\cos A\sin B\sin C$

$=2\sin A\sin B\sin C+2\cos A\cos B\cos C+\sin(A+B+C)-\cos(A+B+C),$

by Art. 113

If $A+B+C=(2m+1)\pi$, $\sin(A+B+C)=0$, $\quad\cos(A+B+C)=-1$

If $A+B+C=2m\pi+\frac{\pi}{2}$, $\sin(A+B+C)=1$, $\quad\cos(A+B+C)=0$

If $A+B+C=2m\pi$, $\quad\sin(A+B+C)=0$, $\quad\cos(A+B+C)=1.$

If $A+B+C=2m\pi-\frac{\pi}{2}$, $\sin(A+B+C)=-1$, $\cos(A+B+C)=0.$

20 An equation holding between the sines and cosines of the angles A, B, C of any triangle depends on the condition that the sum of A, B, C shall be equal to π, therefore the equation will hold if we substitute for A, B, C three angles whose sum is π

Now the sum of $\pi-2A$, $\pi-2B$, $\pi-2C$ is π, therefore we may substitute $\pi-2A$ for A, &c Since $\cos(\pi-2A)=-\cos 2A$, and $\sin(\pi-2A)=\sin 2A$, we may substitute $2A$ for A, &c., provided we change the sign of the cosines

Again, the sum of the angles $2\pi-5A$, $2\pi-5B$, $2\pi-5C$ is π; therefore the equation will hold if we substitute $2\pi-5A$ for A, &c , that is, since $\sin(2\pi-5A)=-\sin 5A$ and $\cos(2\pi-5A)=\cos 5A$, if we substitute $5A$ for A, &c , and change the signs of the sines.

Since $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$ (Ch VIII, Ex 33), we have by the above argument,

$$-\sin 10A-\sin 10B-\sin 10C=-4\sin 5A\sin 5B\sin 5C$$

Let $$\alpha=\frac{5\pi+A}{2^5},\quad \beta=\frac{5\pi+B}{2^5},\quad \gamma=\frac{5\pi+C}{2^5},$$

then $$\alpha+\beta+\gamma=\frac{15\pi+\pi}{32}=\frac{\pi}{2},$$

$$\cot\alpha+\cot\beta+\cot\gamma=\cot\alpha\cot\beta\cot\gamma$$

21 If in any identical relation we substitute for A, B, C

$$2n\pi-pA-qB-rC=A',$$
$$2n\pi-qA-rB-pC=B',$$
$$2n\pi-rA-pB-qC=C',$$

the relation will hold provided $A'+B'+C'=\pi$,

that is, if $$6n\pi-(p+q+r)\pi=\pi,$$

or if $$p+q+r+1 \text{ is a multiple of } 6$$

Since $$\sin(2n\pi-pA-qB-rC)=-\sin(pA+qB+rC)$$

and $$\cos(2n\pi-pA-qB-rC)=\cos(pA+qB+rC),$$

we may substitute $pA+qB+rC$ for A, &c, provided we change the signs of the sines

Again, for A, B, C substitute

$(2n+1)\pi-pA-qB-rC$, $(2n+1)\pi-qA-rB-pC$, $(2n+1)\pi-rA-pB-qC$,

the given relation will hold if the sum of these angles is π,

that is, if $$(6n+3)\pi-(p+q+r)\pi=\pi,$$

or, if $$p+q+r-2 \text{ is a multiple of } 6$$

Since $$\cos\{(2n+1)\pi-pA-qB-rC\}=-\cos(pA+qB+rC),$$

we may substitute $pA+qB+rC$ for A, &c, provided we change the signs of the cosines

In the formula $\tan A+\tan B+\tan C=\tan A\tan B\tan C$, write for A,

$$2A+\frac{3}{2}B+\frac{3}{2}C,\quad \text{for } B,\ \frac{3}{2}A+2B+\frac{3}{2}C,\quad \text{for } C,\ \frac{3}{2}A+\frac{3}{2}B+2C$$

Here $p+q+r+1=6$, so that we must change the signs of all the sines Therefore the identity given is true

22 $$4\cos(S-A)\cos(S-B)\cos(S-C)\cos(S-D)$$
$$=\{\cos(C+D)+\cos(A-B)\}\{\cos(A+B)+\cos(C-D)\},$$
$$4\sin(S-A)\sin(S-B)\sin(S-C)\sin(S-D)$$
$$=\{\cos(C+D)-\cos(A-B)\}\{\cos(A+B)-\cos(C-D)\}$$

Add these, the right-hand side

$$=2\cos(C+D)\cos(A+B)+2\cos(A-B)\cos(C-D)$$

$$=2(\cos C\cos D-\sin C\sin D)(\cos A\cos B-\sin A\sin B)$$

$$+2(\cos C\cos D+\sin C\sin D)(\cos A\cos B+\sin A\sin B)$$

$$=4\cos A\cos B\cos C\cos D+4\sin A\sin B\sin C\sin D.$$

23 In Ex 22 put $D=A+B+C$, $S=A+B+C$, $S-A=B+C$, &c Then the identity of Ex 22 reduces to that required

24 $\cos\alpha\cos\beta\sin\gamma\sin\delta-\cos\gamma\cos\delta\sin\alpha\sin\beta$

$$=\frac{1}{4}\{\cos(\alpha-\beta)+\cos(\alpha+\beta)\}\{\cos(\gamma-\delta)-\cos(\gamma+\delta)\}$$

$$-\frac{1}{4}\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}\{\cos(\gamma-\delta)+\cos(\gamma+\delta)\}$$

$$=\frac{1}{2}\{\cos(\alpha+\beta)\cos(\gamma-\delta)-\cos(\gamma+\delta)\cos(\alpha-\beta)\}$$

$$=\frac{1}{2}\cos(\alpha+\beta)\{\cos(\gamma-\delta)-\cos(n\pi-\overline{\alpha-\beta})\}$$

$$=\cos(\alpha+\beta)\sin\frac{1}{2}(n\pi-\alpha+\beta-\gamma+\delta)\sin\frac{1}{2}(n\pi-\alpha+\beta+\gamma-\delta)$$

$$=\cos(\alpha+\beta)\sin\frac{1}{2}(2n\pi-2\alpha-2\gamma)\sin\frac{1}{2}(2n\pi-2\alpha-2\delta)$$

$$=\cos(\alpha+\beta)\sin(\alpha+\gamma)\sin(\alpha+\delta)$$

25 $\cos(\alpha+\beta)=\cos(\overline{2n+1}\pi-\gamma-\delta)=-\cos(\gamma+\delta)$,

$$\cos\alpha\cos\beta+\cos\gamma\cos\delta=\sin\alpha\sin\beta+\sin\gamma\sin\delta$$

$$\cos\alpha\cos\beta+\cos\gamma\cos\delta=\frac{1}{2}(\cos\alpha\cos\beta+\cos\gamma\cos\delta+\sin\alpha\sin\beta+\sin\gamma\sin\delta)$$

$$=\frac{1}{2}\{\cos(\alpha-\beta)+\cos(\gamma-\delta)\}$$

$$=\cos\frac{1}{2}(\alpha-\beta+\gamma-\delta)\cos\frac{1}{2}(\alpha-\beta-\gamma+\delta)$$

Similarly

$$\cos\beta\cos\gamma+\cos\alpha\cos\delta=\cos\frac{1}{2}(\beta-\gamma+\alpha-\delta)\cos\frac{1}{2}(\beta-\gamma-\alpha+\delta)$$

$$\cos\gamma\cos\alpha+\cos\beta\cos\delta=\cos\frac{1}{2}(\gamma-\alpha+\beta-\delta)\cos\frac{1}{2}(\gamma-\alpha-\beta+\delta)$$

The square root of the product of these is

$$\pm\cos\frac{1}{2}(\alpha-\beta+\gamma-\delta)\cos\frac{1}{2}(\alpha-\beta-\gamma+\delta)\cos\frac{1}{2}(\alpha+\beta-\gamma-\delta)$$

$$=\pm\sin(\alpha+\gamma)\sin(\beta+\gamma)\sin(\alpha+\beta)$$

$$=\pm\frac{1}{4}\{\sin 2\alpha+\sin 2\beta+\sin 2\gamma-\sin(2\alpha+2\beta+2\gamma)\} \qquad \text{[Ex 8]}$$

$$=\pm\frac{1}{4}(\sin 2\alpha+\sin 2\beta+\sin 2\gamma+\sin 2\delta)$$

26 $\quad 2\cos\frac{1}{2}\alpha\cos\frac{1}{2}\gamma\cos\frac{1}{2}(\beta-\delta)=\left\{\cos\frac{1}{2}(\alpha+\gamma)+\cos\frac{1}{2}(\alpha-\gamma)\right\}\cos\frac{1}{2}(\beta-\delta)$

$$=\frac{1}{2}\left\{\cos\frac{1}{2}(\alpha+\beta+\gamma-\delta)+\cos\frac{1}{2}(\alpha+\gamma-\beta+\delta)\right\}+\cos\frac{1}{2}(\alpha-\gamma)\cos\frac{1}{2}(\beta-\delta)$$

$$=\frac{1}{2}\{\cos(\pi-\delta)+\cos(\pi-\beta)\}+\cos\frac{1}{2}(\alpha-\gamma)\cos\frac{1}{2}(\beta-\delta)$$

$$=1-\cos^2\frac{1}{2}\delta-\cos^2\frac{1}{2}\beta+\cos\frac{1}{2}(\alpha-\gamma)\cos\frac{1}{2}(\beta-\delta)$$

Similarly

$$2\cos\frac{1}{2}\beta\cos\frac{1}{2}\delta\cos\frac{1}{2}(\alpha-\gamma)=1-\cos^2\frac{1}{2}\alpha-\cos^2\frac{1}{2}\gamma+\cos\frac{1}{2}(\alpha-\gamma)\cos\frac{1}{2}(\beta-\delta)$$

Subtracting we obtain the required amount

27 $\quad \sin(\alpha\pm\beta\pm\gamma \quad \pm\mu+\nu)+\sin(\alpha\pm\beta\pm\mu \quad \pm\mu-\nu)$

$$=2\cos\nu\sin(\alpha\pm\beta\pm\gamma \quad \pm\mu),$$

$$\Sigma\sin(\alpha\pm\beta\pm\gamma \quad \pm\mu\pm\nu)=2\cos\nu\Sigma\sin(\alpha\pm\beta\pm\gamma \quad \pm\mu)$$

Similarly $\Sigma\sin(\alpha\pm\beta\pm\gamma \quad \pm\lambda\pm\mu)=2\cos\mu\Sigma\sin(\alpha\pm\beta\pm\gamma \quad \pm\lambda)$

and so on,

$$\Sigma\sin(\alpha\pm\beta)=2\cos\beta \quad \sin\alpha$$

Hence $\qquad \Sigma\sin(\alpha\pm\beta\pm\gamma \quad)=2^{n-1}\sin\alpha\cos\beta\cos\gamma$

Again, $\qquad \cos(\alpha\pm\beta \quad \pm\mu+\nu)+\cos(\alpha\pm\beta \quad \pm\mu-\nu)$

$$=2\cos\mu\cos(\alpha\pm\beta \quad \pm\mu),$$

therefore, proceeding as before,

$$\Sigma\cos(\alpha\pm\beta\pm\gamma \quad)=2^{n-1}\cos\alpha\cos\beta\cos\gamma \quad ..$$

28 $$\theta+\phi-\psi=k\pi+(-1)^k(\alpha+\beta),$$

$$\theta-\phi+\psi=2l\pi\pm(\gamma+\alpha),$$

$$-\theta+\phi+\psi=m\pi+(\beta+\gamma).$$

Adding these in pairs, we obtain

$$2\theta=p\pi+(-1)^p(\alpha+\beta)\pm(\gamma+\alpha)$$
$$2\phi=q\pi+(-1)^p(\alpha+\beta)+\beta+\gamma,$$
$$2\psi=r\pi+\beta+\gamma\pm(\gamma+\alpha),$$

where $p=2l+\lambda$, &c., p is odd when λ is odd.

29. $\sin(\theta+\phi)=\sin\left(\frac{\pi}{2}-\alpha+\beta\right)$, $\cos(\theta-\phi)=\cos\left(\frac{\pi}{2}-\alpha-\beta\right)$

$$\theta+\phi=2n\pi+\frac{\pi}{2}-\alpha+\beta \text{ or } (2n+1)\pi-\frac{\pi}{2}+\alpha-\beta,$$

$$\theta-\phi=2m\pi+\frac{\pi}{2}-\alpha-\beta \text{ or } 2m\pi-\frac{\pi}{2}+\alpha+\beta$$

Either set of values of $\theta+\phi$ may be taken with either set of values of $\theta-\phi$. Adding and subtracting we get these corresponding values of 2θ and 2ϕ:

$$\left.\begin{array}{l}(2m+2n+1)\pi-2\alpha\\(2n-2m)\pi+2\beta\end{array}\right\},\quad\left.\begin{array}{l}(2n+2m)\pi+2\alpha\\(2n+1-2m)\pi-2\beta\end{array}\right\},$$
$$\left.\begin{array}{l}(2m+2n+1)\pi-2\beta\\(2n-2m)\pi+2\alpha\end{array}\right\},\quad\left.\begin{array}{l}(2m+2n)\pi+2\beta\\(2n-1-2m)\pi-2\alpha\end{array}\right\}$$

These sets of values will be seen to agree with those given in the question.

30. The relation $\cos(y-z)+\cos(z-x)+\cos(x-y)=-\frac{3}{2}$ is equivalent to

$$(\cos x+\cos y+\cos z)^2+(\sin x+\sin y+\sin z)^2=0$$

$$\therefore\ \cos x+\cos y+\cos z=0,$$

and $$\sin x+\sin y+\sin z=0$$

One factor of

$$\cos^3(x+\theta)+\cos^3(y+\theta)+\cos^3(z+\theta)-3\cos(x+\theta)\cos(y+\theta)\cos(z+\theta)$$

is $$\cos(x+\theta)+\cos(y+\theta)+\cos(z+\theta),$$

or $$\cos\theta(\cos x+\cos y+\cos z)+\sin\theta(\sin x+\sin y+\sin z)$$

This vanishes for all values of θ.

31. $$\tan\frac{1}{4}(B+C-A)=\tan\frac{1}{4}(180^\circ-2A)=\tan\left(45^\circ-\frac{A}{2}\right)$$

$$=\frac{\cos A}{1+\sin A}=\frac{1-\sin A}{\cos A}.$$

Hence the given relation becomes

$$\cos A\cos B\cos C=(1+\sin A)(1+\sin B)(1+\sin C),$$

or $$\cos A\cos B\cos C=(1-\sin A)(1-\sin B)(1-\sin C)$$

Subtract, therefore

$$\sin A+\sin B+\sin C+\sin A\sin B\sin C=0$$

$$4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+\sin A\sin B\sin C=0$$

$$1+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=0$$

$$2+2\sin\frac{A}{2}\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right)=0$$

$$1+1-2\sin^2\frac{A}{2}+2\cos\frac{B+C}{2}\cos\frac{B-C}{2}=0$$

$$1+\cos A+\cos B+\cos C=0$$

32 We have from the given relation

$$\cos A\cos(B+C)-\sin A\sin(B+C)=\cos A\cos B\cos C$$

$$\sin A\sin(B+C)=\cos A\{\cos(B+C)-\cos B\cos C\}$$

$$\sin(B+C)=-\frac{\sin B\sin C}{\sin A}\cos A$$

Similarly $$\sin(C+A)=-\frac{\sin C\sin A}{\sin B}\cos B,$$

and $$\sin(A+B)=-\frac{\sin A\sin B}{\sin C}\cos C,$$

$$\sin(B+C)\sin(C+A)\sin(A+B)=-\sin A\sin B\sin C\cos A\cos B\cos C$$

$$=-\frac{1}{8}\sin 2A\sin 2B\sin 2C$$

33 From the given equation

$$\cos\alpha=\frac{-\cos\beta-\cos\gamma}{1+\cos\beta\cos\gamma}.$$

$$1-\cos\alpha=\frac{(1+\cos\beta)(1+\cos\gamma)}{1+\cos\beta\cos\gamma},$$

and $$1+\cos\alpha=\frac{(1-\cos\beta)(1-\cos\gamma)}{1+\cos\beta\cos\gamma}$$

Multiplying these, we have

$$\sin^2\alpha=\frac{\sin^2\beta\sin^2\gamma}{(1+\cos\beta\cos\gamma)^2}.$$

$$\sin^4\alpha(1+\cos\beta\cos\gamma)^2=\sin^2\alpha\sin^2\beta\sin^2\gamma$$

Similarly $$\sin^4\beta(1+\cos\gamma\cos\alpha)^2=\sin^2\alpha\sin^2\beta\sin^2\gamma,$$

and $$\sin^4\gamma(1+\cos\alpha\cos\beta)^2=\sin^2\alpha\sin^2\beta\sin^2\gamma$$

Multiply these and take the square root; therefore

$$(1+\cos\beta\cos\gamma)(1+\cos\gamma\cos\alpha)(1+\cos\alpha\cos\beta)=\pm\sin\alpha\sin\beta\sin\gamma$$

$$1+\cos\beta\cos\gamma+\cos\gamma\cos\alpha+\cos\alpha\cos\beta+\cos\alpha\cos\beta\cos\gamma(\cos\alpha+\cos\beta+\cos\gamma)$$
$$+\cos^2\alpha\cos^2\beta\cos^2\gamma=\pm\sin\alpha\sin\beta\sin\gamma$$

Hence, from the given equation

$$1+\cos\beta\cos\gamma+\cos\gamma\cos\alpha+\cos\alpha\cos\beta=\pm\sin\alpha\sin\beta\sin\gamma \qquad (1)$$

Now $\quad (\cos\alpha+\cos\beta+\cos\gamma)^2=\cos^2\alpha\cos^2\beta\cos^2\gamma$

$$=(1-\sin^2\alpha)(1-\sin^2\beta)(1-\sin^2\gamma)$$

$$\therefore \cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta+2\cos\alpha\cos\gamma+2\cos\beta\cos\gamma$$
$$=1-\sin^2\alpha-\sin^2\beta-\sin^2\gamma+\sin^2\alpha\sin^2\beta+\sin^2\beta\sin^2\gamma+\sin^2\gamma\sin^2\alpha$$
$$-\sin^2\alpha\sin^2\beta\sin^2\gamma$$

Transpose the first four terms of the last expression and use equation (1)

$$\therefore \quad \pm 2\sin\alpha\sin\beta\sin\gamma=\sin^2\alpha\sin^2\beta+\sin^2\beta\sin^2\gamma+\sin^2\gamma\sin^2\alpha$$
$$-\sin^2\alpha\sin^2\beta\sin^2\gamma$$

Dividing by $\sin^2\alpha\sin^2\beta\sin^2\gamma$ we obtain the required result

34 Let each expression $=\kappa$, therefore

$$\left.\begin{aligned}\kappa\cos(\beta+\gamma)&=\cos 2\alpha\sin(\beta-\gamma)\\ \kappa\cos(\gamma+\alpha)&=\cos 2\beta\sin(\gamma-\alpha)\\ \kappa\cos(\alpha+\beta)&=\cos 2\gamma\sin(\alpha-\beta)\end{aligned}\right\} \qquad . \qquad (1)$$

Multiply these equations by $2\cos(\beta-\gamma)$, $2\cos(\gamma-\alpha)$, $2\cos(\alpha-\beta)$ respectively and add, therefore

$$\kappa\{\cos 2\beta+\cos 2\gamma+\cos 2\gamma+\cos 2\alpha+\cos 2\alpha+\cos 2\beta\}$$
$$=\cos 2\alpha\sin(2\beta-2\gamma)+\cos 2\beta\sin(2\gamma-2\alpha)+\cos 2\gamma\sin(2\alpha-2\beta).$$

$$\therefore 2\kappa(\cos 2\alpha+\cos 2\beta+\cos 2\gamma)$$
$$=\frac{1}{2}\Big[\sin(2\alpha+2\beta-2\gamma)-\sin(2\alpha-2\beta+2\gamma)+\sin(2\beta+2\gamma-2\alpha)$$
$$-\sin(2\beta-2\gamma+2\alpha)+\sin(2\gamma+2\alpha-2\beta)-\sin(2\gamma-2\alpha+2\beta)\Big]=0$$

$$\cos 2\alpha+\cos 2\beta+\cos 2\gamma=0$$

Again, multiply equations (1) by $2\sin(\beta+\gamma)$, $2\sin(\gamma+\alpha)$, $2\sin(\alpha+\beta)$ respectively, and add

$$\kappa[\sin 2(\beta+\gamma)+\sin 2(\gamma+\alpha)+\sin 2(\alpha+\beta)]$$
$$=\cos 2\alpha(\sin^2\beta-\sin^2\gamma)+\cos 2\beta(\sin^2\gamma-\sin^2\alpha)+\cos 2\gamma(\sin^2\alpha-\sin^2\beta)$$
$$=(1-2\sin^2\alpha)(\sin^2\beta-\sin^2\gamma)+\text{two similar terms}=0,$$

$$\sin 2(\beta+\gamma)+\sin 2(\gamma+\alpha)+\sin 2(\alpha+\beta)=0$$

35 $$\frac{\sin(2\alpha-\beta-\gamma)}{\cos(2\alpha+\beta+\gamma)}=\frac{\sin(2\beta-\gamma-\alpha)}{\cos(2\beta+\gamma+\alpha)}$$

$$2\sin(2\alpha-\beta-\gamma)\cos(2\beta+\gamma+\alpha)=2\sin(2\beta-\gamma-\alpha)\cos(2\alpha+\beta+\gamma),$$

$$\sin(3\alpha+\beta)+\sin(\alpha-3\beta-2\gamma)=\sin(3\beta+\alpha)+\sin(\beta-2\gamma-3\alpha),$$

$$\sin(3\alpha+\beta)-\sin(3\beta+\alpha)=\sin(\beta-2\gamma-3\alpha)-\sin(\alpha-3\beta-2\gamma),$$

$$\cos(2\alpha+2\beta)\sin(\alpha-\beta)=\cos(-2\gamma-\alpha-\beta)\sin(-2\alpha+2\beta)$$

Now $\alpha-\beta$ is not zero, and since α and β are less than π, $\alpha-\beta$ is not a multiple of π, we may divide through by $\sin(\alpha-\beta)$

$$\cos(2\alpha+2\beta)=-2\cos(\alpha+\beta+2\gamma)\cos(\alpha-\beta)$$

$$=-\cos(2\alpha+2\gamma)-\cos(2\beta+2\gamma)$$

Thus the system of equations is equivalent to the single equation,

$$\cos(2\alpha+2\beta)+\cos(2\alpha+2\gamma)+\cos(2\beta+2\gamma)=0$$

36 $\cos 2A+\cos 2B+\cos 2C+\cos 2D$

$$=2\cos(A+B)\cos(A-B)+2\cos(C+D)\cos(C-D)$$

$$=2\cos(A+B)\{\cos(A-B)+\cos(C-D)\}$$

$$=4\cos(A+B)\cos\tfrac{1}{2}(A+C-B-D)\cos\tfrac{1}{2}(A-B-C+D)$$

$$=4\cos(A+B)\cos(A+C)\cos(A+D)$$

Similarly

$$\sin 2A+\sin 2B+\sin 2C+\sin 2D=-4\sin(A+B)\sin(A+C)\sin(A+D)$$

Expand the cosines in the given equation and use the above identity,

$$(\cos x-1)(\cos 2A+\cos 2B+\cos 2C+\cos 2D)$$
$$=\sin x(\sin 2A+\sin 2B+\sin 2C+\sin 2D)$$

Multiply both sides by $\sin x$, write $1-\cos^2 x$ for $\sin^2 x$ and divide both sides by $1-\cos x$, therefore

$$-\sin x(\cos 2A+\cos 2B+\cos 2C+\cos 2D)$$
$$=(1+\cos x)(\sin 2A+\sin 2B+\sin 2C+\sin 2D)$$

Transposing, we obtain

$$\sin(x+2A)+\sin(x+2B)+\sin(x+2C)+\sin(x+2D)$$
$$=-(\sin 2A+\sin 2B+\sin 2C+\sin 2D)=4\sin(A+B)\sin(A+C)\sin(A+D)$$

37 Put each fraction $=\lambda$, therefore

$$\cos\alpha+\cos\beta+\cos\gamma=\lambda\cos(\alpha+\beta+\gamma),$$

$$\sin\alpha+\sin\beta+\sin\gamma=\lambda\sin(\alpha+\beta+\gamma)$$

Multiply the first of these by $\cos\alpha$, the second by $\sin\alpha$ and add,

$$1+\cos(\beta-\alpha)+\cos(\gamma-\alpha)=\lambda\cos(\beta+\gamma) \qquad (1)$$

Similarly $$1+\cos(\alpha-\beta)+\cos(\gamma-\beta)=\kappa\cos(\gamma+\alpha) \qquad \text{(ii)},$$

and $$1+\cos(\beta-\gamma)+\cos(\alpha-\gamma)=\kappa\cos(\alpha+\beta) \qquad \text{(iii)}$$

$$\begin{aligned}
&\kappa[\cos(\beta+\gamma)+\cos(\gamma+\alpha)+\cos(\alpha+\beta)]\\
&=3+2\cos(\beta-\alpha)+2\cos(\gamma-\alpha)+2\cos(\beta-\gamma)\\
&=(\cos\alpha+\cos\beta+\cos\gamma)^2+(\sin\alpha+\sin\beta+\sin\gamma)^2\\
&=\kappa^2\cos^2(\alpha+\beta+\gamma)+\kappa^2\sin^2(\alpha+\beta+\gamma)=\kappa^2
\end{aligned}$$

$$\kappa=\cos(\beta+\gamma)+\cos(\gamma+\alpha)+\cos(\alpha+\beta)$$

Again, subtracting equations (ii) and (iii),

$$2\sin\left(\alpha-\frac{\beta+\gamma}{2}\right)\sin\frac{\beta-\gamma}{2}=2\kappa\sin\left(\frac{\beta+\gamma}{2}+\alpha\right)\sin\frac{\beta-\gamma}{2}$$

$$\kappa=\frac{\sin\left(\alpha-\frac{\beta+\gamma}{2}\right)}{\sin\left(\alpha+\frac{\beta+\gamma}{2}\right)}$$

$$=\frac{\sin\alpha\cos\frac{1}{2}(\beta+\gamma)-\cos\alpha\sin\frac{1}{2}(\beta+\gamma)}{\sin\alpha\cos\frac{1}{2}(\beta+\gamma)+\cos\alpha\sin\frac{1}{2}(\beta+\gamma)}$$

$$=\frac{\tan\alpha-\tan\frac{1}{2}(\beta+\gamma)}{\tan\alpha+\tan\frac{1}{2}(\beta+\gamma)}$$

38 Multiply the first equation by $\sin 6\alpha$, the second by $\sin 2\alpha$ and subtract,

$$x(\sin 6\alpha\sin\alpha-\sin 2\alpha\sin 3\alpha)=\sin 3\alpha\sin 6\alpha-\sin 9\alpha\sin 2\alpha$$

$$x\sin\alpha(\sin 6\alpha-2\cos\alpha\sin 3\alpha)=\sin 3\alpha[\sin 6\alpha-(3-4\sin^2 3\alpha)\sin 2\alpha]$$

$$2x\sin\alpha\sin 3\alpha(\cos 3\alpha-\cos\alpha)=\sin 3\alpha\sin 2\alpha(3-4\sin^2 2\alpha-3+4\sin^2 3\alpha)$$

$$\begin{aligned}
-x\sin^2\alpha\sin 2\alpha\sin 3\alpha&=\sin 3\alpha\sin 2\alpha(\sin^2 3\alpha-\sin^2 2\alpha)\\
&=\sin 3\alpha\sin 2\alpha\quad\sin\alpha\sin 5\alpha
\end{aligned}$$

$$x\sin\alpha=-\sin 5\alpha$$

Substitute this in the first of the given equations,

$$y\sin 2\alpha=\sin 5\alpha+\sin 3\alpha,$$

$$2y\sin\alpha\cos\alpha=2\sin 4\alpha\cos\alpha,$$

$$y\sin\alpha=\sin 4\alpha$$

39 For $\sin\theta$ put x, for $\sin\phi$ put y; the given equations are

$$x^4+y^4=14x^2y^2, \quad x+y=\frac{1}{\sqrt{2}}.$$

From the first equation,

$$x^4+2x^2y^2+y^4=16x^2y^2,$$

$$\therefore\ x^2+y^2=\pm 4xy,$$

$$(x+y)^2=2xy\pm 4xy,$$

$$xy=-\frac{1}{4} \text{ or } \frac{1}{12}$$

$$x-y=\sqrt{(x+y)^2-4xy}=\pm\sqrt{\frac{3}{2}} \text{ or } \pm\sqrt{\frac{1}{6}}.$$

$$2x=\frac{1}{\sqrt{2}}\pm\frac{1}{\sqrt{6}} \text{ or } \frac{1}{\sqrt{2}}\pm\frac{\sqrt{3}}{\sqrt{2}}$$

$$2x=\cos\frac{\pi}{4}\pm\sin\frac{\pi}{4}\cot\frac{\pi}{3} \text{ or } \cos\frac{\pi}{4}\pm\sin\frac{\pi}{4}\tan\frac{\pi}{3},$$

$$\text{or } 2\sin\theta=\frac{\sin\left(\frac{1}{3}\pi\pm\frac{1}{4}\pi\right)}{\sin\frac{1}{3}\pi} \text{ or } \frac{\cos\left(\frac{1}{3}\pi\pm\frac{1}{4}\pi\right)}{\cos\frac{1}{3}\pi}$$

40 $$\tan\frac{1}{2}(\theta+\phi)=\frac{\tan\frac{1}{2}\theta+\tan\frac{1}{2}\phi}{1-\tan\frac{1}{2}\theta\tan\frac{1}{2}\phi}$$

$$=\frac{\tan^3\frac{1}{2}\phi+\tan\frac{1}{2}\phi}{1-\tan^4\frac{1}{2}\phi}, \text{ from tho first given equation,}$$

$$=\frac{\tan\frac{1}{2}\phi}{1-\tan^2\frac{1}{2}\phi}$$

$$=\frac{1}{2}\tan\phi=\tan\alpha.$$

Therefore $\theta+\phi=2\alpha$.

41 The first equation gives

$$2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}+\sin\alpha=2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}+\cos\alpha$$

$$\sin\alpha\left(2\cos\frac{\theta-\phi}{2}+1\right)=\cos\alpha\left(2\cos\frac{\theta-\phi}{2}+1\right)$$

Hence
$$\cos\frac{\theta-\phi}{2}=-\frac{1}{2},$$

$$\therefore\ \frac{\theta-\phi}{2}=2n\pi\pm\frac{2\pi}{3}.$$

But
$$\frac{\theta+\phi}{2}=\alpha$$

$$\theta=2n\pi\pm\frac{2\pi}{3}+\alpha,$$

$$\phi=-2n\pi\mp\frac{2\pi}{3}+\alpha.$$

42 From Ex. 6, Ch VIII, we have, $4\cos a\cos b\cos c=$
$$\cos(a+b+c)+\cos(-a+b+c)-\cos(a-b+c)-\cos(a-b-c)$$

Substitute this expression in the given equation, and transpose

$$\text{Now }\cos 2x-\cos(a+b+c)=-2\sin\left(x-\frac{a+b+c}{2}\right)\sin\left(x+\frac{a+b-c}{2}\right),$$

$$\cos(2x-2a)-\cos(-a+b-c)=-2\sin\left(x-\frac{a+b+c}{2}\right)\sin\left(x+\frac{-3a+b+c}{2}\right),$$

$$\cos(2x-2b)-\cos(a-b+c)=-2\sin\left(x-\frac{a+b+c}{2}\right)\sin\left(x+\frac{a-3b-c}{2}\right),$$

$$\cos(2x-2c)-\cos(a-b-c)=-2\sin\left(x-\frac{a+b+c}{2}\right)\sin\left(x+\frac{a+b-3c}{2}\right)$$

Hence one solution is given by $\sin\left(x-\dfrac{a+b+c}{2}\right)=0$

Therefore
$$x=n\pi+\frac{1}{2}(a+b+c)$$

The other solution is found from
$$\sin\theta+\sin(\theta-2a)+\sin(\theta-2b)+\sin(\theta-2c)=0,$$
where θ stands for $x+\frac{1}{2}(a+b+c)$.

Hence $\sin\theta\,(1+\cos 2a+\cos 2b-\cos 2c)=\cos\theta\,(\sin 2a+\sin 2b-\sin 2c)$,
$$\theta=\tan^{-1}\frac{\sin 2a+\sin 2b+\sin 2c}{1+\cos 2a+\cos 2b+\cos 2c},$$
$$x=-\frac{1}{2}(a+b+c)+\tan^{-1}\frac{\sin 2a+\sin 2b+\sin 2c}{1+\cos 2a+\cos 2b+\cos 2c}$$

43 From Ex 23, one solution is seen to be $x=a+b+c$ The general solution may however be found as follows

Let $\tan x=t,\ \tan a=\alpha,\ \tan b=\beta,\ \tan c=\gamma$

Expand the cosines and divide through by $\cos a \cos b \cos c \cos^3 x$

$$(1+\alpha t)(1+\beta t)(1+\gamma t)=\alpha\beta\gamma t \quad \frac{1}{\cos^2 x}+\frac{1}{\cos^2 x}=(1+t^2)(\alpha\beta\gamma t+1)$$

$$1+(\alpha+\beta+\gamma)t+(\alpha\beta+\alpha\gamma+\beta\gamma)t^2+\alpha\beta\gamma t^3=\alpha\beta\gamma t+1+\alpha\beta\gamma t^3+t^2$$

$$t=0,$$

or
$$t=\frac{\alpha+\beta+\gamma-\alpha\beta\gamma}{1-\alpha\beta-\alpha\gamma-\beta\gamma}=\tan(a+b+c). \qquad \text{(Art 113)}$$

44 Multiply out, therefore

$$\sin^2 x+\sin^2 a+\cos^2 a \sin^2 x+\sin^2 a\cos^2 a=\sin^2 x+2\sin x\sin a\cos a$$
$$+\sin^2 a\cos^2 a+\sin^2 a\cos^2 x-2\sin a\cos a\sin x\cos x+\cos^2 a\sin^2 x.$$
$$\sin^2 a\sin^2 x=2\sin a\cos a\sin x(1-\cos x),$$
$$\sin x=0, \text{ from which } x=\kappa\pi,$$

or
$$\sin a\sin x=2\cos a(1-\cos x),$$
$$2\sin a\sin\frac{x}{2}\cos\frac{x}{2}=4\cos a\sin^2\frac{x}{2},$$
$$\tan\frac{x}{2}=\frac{1}{2}\tan a,$$
$$\therefore x=2n\pi+2\tan^{-1}\left(\frac{1}{2}\tan a\right)$$

45 Since α and β are roots of the given equation, we have

$$a\cos\alpha+b\sin\alpha=c,$$

and
$$a\cos\beta+b\sin\beta=c$$

Subtract, therefore

$$-2a\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)+2b\sin\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\alpha+\beta)=0$$

Hence
$$\frac{\cos\frac{1}{2}(\alpha+\beta)}{a}=\frac{\sin\frac{1}{2}(\alpha+\beta)}{b}$$

Each of these ratios
$$=\frac{\cos\frac{1}{2}(\alpha+\beta)\cos\alpha+\sin\frac{1}{2}(\alpha+\beta)\sin\alpha}{a\cos\alpha+b\sin\alpha}$$
$$=\frac{\cos\frac{1}{2}(\alpha-\beta)}{c}.$$

46 From the two given equations we see that θ and ϕ are roots of the equation $a\cos x+b\sin x+c=0$

From this equation we can obtain an equation for either $\cos x$ or $\sin x$

(i) $$(a\cos x+c)^2=b^2\sin^2 x=b^2(1-\cos^2 x),$$

$$\therefore\ (a^2+b^2)\cos^2 x+2ac\cos x-c^2-b^2=0$$

$$\therefore\ \cos\theta+\cos\phi=-\frac{2ac}{a^2+b^2}.$$

(ii) $$(b\sin x+c)^2=a^2\cos^2 x=a^2(1-\sin^2 x),$$

$$(a^2+b^2)\sin^2 x+2bc\sin x+c^2-a^2=0$$

$$\therefore\ \sin\theta+\sin\phi=-\frac{2bc}{a^2+b^2}.$$

Square and add the results of (i) and (ii),

$$\therefore\ 2+2\cos(\theta-\phi)=\frac{4c^2}{a^2+b^2}.$$

$$\therefore\ \cos^2\frac{1}{2}(\theta-\phi)=\frac{c^2}{a^2+b^2}$$

This result is also obvious from the ratios found in Ex. 45

47 As in the preceding example we have,

$$\cos 2\alpha+\cos 2\beta=\frac{2ac}{a^2+b^2},$$

$$2\cos^2\alpha-1+2\cos^2\beta-1=\frac{2ac}{a^2+b^2}.$$

$$\therefore\ \cos^2\alpha+\cos^2\beta=1+\frac{ac}{a^2+b^2}=\frac{a^2+ac+b^2}{a^2+b^2}$$

48 The equation

$$a^2\cos\alpha\,.\cos\theta+a\sin\theta+1+a\sin\alpha=0 \qquad (1)$$

is satisfied by $\theta=\beta$ and $\theta=\gamma$, which are therefore the roots We can, as in Ex. 46, express this as an equation for $\cos\theta$ or $\sin\theta$

$$(a^2\cos\alpha\,.\cos\theta+1+a\sin\alpha)^2=a^2\sin^2\theta=a^2(1-\cos^2\theta)$$

$$\therefore\ a^2(1+a^2\cos^2\alpha)\cos^2\theta-2a^2\cos\alpha(1+a\sin\alpha)\cos\theta+(1+a\sin\alpha)^2-a^2=0$$

The roots of this equation in $\cos\theta$ are $\cos\beta$, $\cos\gamma$.

$$\cos\beta\cos\gamma=\frac{(1+a\sin\alpha)^2-a^2}{a^4(1+a^2\cos^2\alpha)}$$

Also, as in Ex. 46, we obtain

$$\sin\beta+\sin\gamma=\frac{-2a(1+a\sin\alpha)}{a^2(1+a^2\cos^2\alpha)}.$$

$$a^2\cos\beta\cos\gamma+a(\sin\beta+\sin\gamma)$$
$$=\frac{(1+a\sin\alpha)^2-a^2-2(1+a\sin\alpha)}{1+a^2\cos^2\alpha}$$
$$=\frac{-1-a^2\cos^2\alpha}{1+a^2\cos^2\alpha}=-1.$$

Again, subtracting the two given equations

$$a\cos\alpha(\cos\beta-\cos\gamma)+\sin\beta-\sin\gamma=0.$$
$$a\cos\alpha\sin\frac{1}{2}(\beta+\gamma)-\cos\frac{1}{2}(\beta+\gamma)=0 \quad \ldots\ldots \text{(ii)},$$

provided $\sin\frac{1}{2}(\beta-\gamma)$ is not zero. Since β and γ are not equal and cannot differ by a multiple of 2π, $\sin\frac{1}{2}(\beta-\gamma)$ cannot be zero

In the same way we can find that

$$a\cos\beta\sin\frac{1}{2}(\gamma+\alpha)=\cos\frac{1}{2}(\gamma+\alpha)$$

From these two equations we have

$$\frac{\cos\alpha}{\cos\beta}=\frac{\cos\frac{1}{2}(\gamma+\alpha)\sin\frac{1}{2}(\beta+\gamma)}{\sin\frac{1}{2}(\gamma+\alpha)\cos\frac{1}{2}(\beta+\gamma)}.$$

$$\frac{\cos\alpha-\cos\beta}{\cos\alpha+\cos\beta}=\frac{\sin\frac{1}{2}(\alpha-\beta)}{\sin\left(\frac{1}{2}\alpha+\frac{1}{2}\beta+\gamma\right)}.$$

$$\frac{-2\sin\frac{1}{2}(\alpha+\beta)}{\cos\alpha+\cos\beta}=\frac{1}{\sin\left(\frac{1}{2}\alpha+\frac{1}{2}\beta+\gamma\right)}.$$

$$\cos(\alpha+\beta+\gamma)-\cos\gamma=\cos\alpha+\cos\beta,$$

or $$\cos\alpha+\cos\beta+\cos\gamma=\cos(\alpha+\beta+\gamma).$$

49 The equation is the same as that in Ex 45, where

$$a=\frac{\cos\phi}{\cos^2\alpha},\quad b=\frac{\sin\phi}{\sin^2\alpha},\quad c=-1$$

$$\frac{\cos\frac{1}{2}(\theta_1+\theta_2)\cos^2\alpha}{\cos\phi}=\frac{\sin\frac{1}{2}(\theta_1+\theta_2)\sin^2\alpha}{\sin\phi}=-\cos\frac{1}{2}(\theta_1-\theta_2)$$

$$\therefore \cos^2\frac{1}{2}(\theta_1+\theta_2)\cos^4\alpha+\sin^2\frac{1}{2}(\theta_1+\theta_2)\sin^4\alpha=\cos^2\frac{1}{2}(\theta_1-\theta_2)$$

$$\cos^4\alpha\,\{1+\cos(\theta_1+\theta_2)\}+\sin^4\alpha\,\{1-\cos(\theta_1+\theta_2)\}=1+\cos(\theta_1-\theta_2)$$

$$\cos\theta_1\cos\theta_2(\cos^4\alpha-\sin^4\alpha-1)+\sin\theta_1\sin\theta_2(-\cos^4\alpha+\sin^4\alpha-1)$$
$$=1-\cos^4\alpha-\sin^4\alpha \qquad \text{(i)}$$

Now $\cos^4\alpha-\sin^4\alpha-1=(\cos^2\alpha+\sin^2\alpha)(\cos^2\alpha-\sin^2\alpha)-1$

$$=\cos^2\alpha-\sin^2\alpha-1=-2\sin^2\alpha,$$

$$-\cos^4\alpha+\sin^4\alpha-1=-\cos^2\alpha+\sin^2\alpha-1=-2\cos^2\alpha,$$

$$1-\cos^4\alpha-\sin^4\alpha=(\sin^2\alpha+\cos^2\alpha)^2-\cos^4\alpha-\sin^4\alpha=2\sin^2\alpha\cos^2\alpha$$

Substitute these in equation (i) and divide by $\sin^2\alpha\cos^2\alpha$,

$$\frac{\cos\theta_1\cos\theta_2}{\cos^2\alpha}+\frac{\sin\theta_1\sin\theta_2}{\sin^2\alpha}+1=0$$

Or thus From the given equation form an equation for $\cos\theta$,

$$\left(1+\frac{\cos\theta\cos\phi}{\cos^2\alpha}\right)^2=\frac{\sin^2\phi}{\sin^4\alpha}(1-\cos^2\theta).$$

The product of the roots is the last term of this equation divided by the coefficient of $\cos^2\theta$

$$\cos\theta_1\cos\theta_2=\frac{1-\dfrac{\sin^2\phi}{\sin^4\alpha}}{\dfrac{\cos^2\phi}{\cos^4\alpha}+\dfrac{\sin^2\phi}{\sin^4\alpha}}=\frac{\cos^4\alpha(\sin^4\alpha-\sin^2\phi)}{\cos^2\phi\sin^4\alpha+\sin^2\phi\cos^4\alpha}$$

Similarly

$$\sin\theta_1\sin\theta_2=\frac{1-\dfrac{\cos^2\phi}{\cos^4\alpha}}{\dfrac{\cos^2\phi}{\cos^4\alpha}+\dfrac{\sin^2\phi}{\sin^4\alpha}}=\frac{\sin^4\alpha(\cos^4\alpha-\cos^2\phi)}{\cos^2\phi\sin^4\alpha+\sin^2\phi\cos^4\alpha}$$

$$\frac{\cos\theta_1\cos\theta_2}{\cos^2\alpha}+\frac{\sin\theta_1\sin\theta_2}{\sin^2\alpha}=\frac{\sin^2\alpha\cos^2\alpha-\sin^2\phi\cos^2\alpha-\cos^2\phi\sin^2\alpha}{\cos^2\phi\sin^4\alpha+\sin^2\phi\cos^4\alpha} \qquad \text{(ii)}$$

But

$$\cos^2\phi\sin^4\alpha+\sin^2\phi\cos^4\alpha=\cos^2\phi\sin^2\alpha(1-\cos^2\alpha)+\sin^2\phi\cos^2\alpha(1-\sin^2\alpha)$$
$$=\cos^2\phi\sin^2\alpha+\sin^2\phi\cos^2\alpha-\sin^2\alpha\cos^2\alpha(\sin^2\phi+\cos^2\phi)$$

the right-hand side of equation (ii)$=-1$, and we obtain the required result

50 Let x_4 be the fourth root. From the given equation form an equation for $\sin x$

$$\left(1+\frac{b}{\sin x}\right)^2=\frac{a^2}{\cos^2 x}=\frac{a^2}{1-\sin^2 x}.$$

$$(b+\sin x)^2(1-\sin^2 x)=a^2\sin^2 x.$$

$$\sin^4 x+A\sin^3 x+B\sin^2 x+C\sin x-b^2=0.$$

$$\therefore \sin x_1\sin x_2\sin x_3\sin x_4=-b^2$$

$$\frac{\sin x_1\sin x_2\sin x_3}{b}=-\frac{b}{\sin x_4}$$

Similarly $$\cos x_1\cos x_2\cos x_3\cos x_4=-a^2.$$

Hence $$\frac{\cos x_1\cos x_2\cos x_3}{b}+\frac{\sin x_1\sin x_2\sin x_3}{a}$$

$$=-\left(\frac{a}{\cos x_4}+\frac{b}{\sin x_4}\right)=1.$$

Since x_4 is a root of the equation $\dfrac{a}{\cos x}+\dfrac{b}{\sin x}+1=0$

51 Let each fraction $=\kappa$. It will be possible to find angles λ, μ, ν such that

$$x=\kappa\cos\lambda,\quad y=\kappa\cos\mu,\quad z=\kappa\cos\nu,$$

if $$\cos^2\mu+\cos^2\nu-2\cos\alpha\cos\mu\cos\nu=\sin^2\alpha=1-\cos^2\alpha,$$

with two similar equations.

From the result of Art 115, by putting $\pi-\alpha$ for A, &c, we see that there must be some relation such as

and similarly $$\left.\begin{aligned}\pi-\alpha+\mu+\nu&=\pi,\\ \pi-\beta+\lambda+\nu&=\pi,\\ \pi-\gamma+\lambda+\mu&=\pi\end{aligned}\right\}\qquad (1)$$

$$\lambda=s-\alpha,\quad \mu=s-\beta,\quad \nu=s-\gamma,$$

$$x : y : z :: \cos(s-\alpha) : \cos(s-\beta) : \cos(s-\gamma)$$

Instead of the equations (1) we may (Art 115) take such a set as

$$\alpha+\pi-\mu+\nu=\pi,\quad \beta+\pi-\lambda+\nu=\pi,\quad \pi-\gamma+\mu+\lambda=\pi.$$

These equations give one of the other sets of ratios

If we take the equations (Art. 115)

$$\alpha-\mu+\nu=0,\quad \beta-\nu+\lambda=0,\quad \gamma-\lambda+\mu=0,$$

we should have in relation $\alpha+\beta+\gamma=0$, which is not given

52. $(l'+l\cos\omega)^2=l'^2+2ll'\cos\omega+l^2\cos^2\omega$

$$=1-l^2+l^2\cos^2\omega, \text{ from the first equation,}$$
$$=1-l^2\sin^2\omega$$

$$(m'+m\cos\omega)^2=m'^2+2mm'\cos\omega+m^2\cos^2\omega$$
$$=1-m^2+m^2\cos^2\omega, \text{ from the second equation,}$$
$$=1-m^2\sin^2\omega$$

$$(l'+l\cos\omega)(m'+m\cos\omega)=l'm'+(l'm+lm')\cos\omega+lm\cos^2\omega$$
$$=-lm+lm\cos^2\omega, \text{ from the third equation,}$$
$$=-lm\sin^2\omega$$

Hence $$(1-l^2\sin^2\omega)(1-m^2\sin^2\omega)=(-lm\sin^2\omega)^2.$$
$$1-(l^2+m^2)\sin^2\omega=0,$$
$$l^2+m^2=\operatorname{cosec}^2\omega.$$

53 Here $r=-4$, $q=6$; therefore

$$\cos 3a=-16\left(\frac{3}{24}\right)^{\frac{3}{2}}=-16\left(\frac{1}{8}\right)^{\frac{3}{2}}=-\frac{16}{8}\left(\frac{1}{8}\right)^{\frac{1}{2}}=-\frac{2}{2\sqrt{2}}=-\frac{1}{\sqrt{2}}.$$

Therefore $3a=\frac{3\pi}{4}$. Hence $a=\frac{\pi}{4}$ Therefore the roots are

$$2\left(\frac{6}{3}\right)^{\frac{1}{2}}\cos\frac{\pi}{4}, \text{ and } 2\left(\frac{6}{3}\right)^{\frac{1}{2}}\cos\left(\frac{2\pi}{3}\pm\frac{\pi}{4}\right).$$

Now $$2\left(\frac{6}{3}\right)^{\frac{1}{2}}\cos\frac{\pi}{4}=2\sqrt{2}\frac{1}{\sqrt{2}}=2,$$

$$2\left(\frac{6}{3}\right)^{\frac{1}{2}}\cos\left(\frac{2\pi}{3}+\frac{\pi}{4}\right)=2\sqrt{2}\cos\frac{11\pi}{12}=-2\sqrt{2}\cos\frac{\pi}{12}=-(\sqrt{3}+1),$$

$$2\left(\frac{6}{3}\right)^{\frac{1}{2}}\cos\left(\frac{2\pi}{3}-\frac{\pi}{4}\right)=2\sqrt{2}\cos\frac{5\pi}{12}=\sqrt{3}-1.$$

54 Here $r=1$, $q=3$; therefore $\cos 3a=4\left(\frac{3}{12}\right)^{\frac{3}{2}}=4\left(\frac{1}{4}\right)^{\frac{3}{2}}=\frac{1}{2}$; therefore $3a=60^\circ$, therefore $a=20^\circ$. Therefore the roots are $2\cos 20^\circ$, and $2\cos(120^\circ\pm 20)$

Also $$2\cos(120^\circ+20^\circ)=2\cos 140^\circ=-2\cos 40^\circ$$

And $$2\cos(120^\circ-20^\circ)=2\cos 100^\circ=-2\sin 10^\circ.$$

55 Take the equation $x^5-px^3+qx-r=0$

Put $x=ny$, thus $n^5y^5-pn^3y^3+qny-r=0$,

therefore $$y^5-\frac{p}{n^2}y^3+\frac{q}{n^4}y=\frac{r}{n^5}$$

Now by Example VIII 59, $\cos 5a=16\cos^5 a-20\cos^3 a+5\cos a$

Thus $$\cos^5 a-\frac{5}{4}\cos^3 a+\frac{5}{16}\cos a=\frac{\cos 5a}{16}.$$

Assume $y=\cos a$, $\frac{5}{4}=\frac{p}{n^2}$, $\frac{5}{16}=\frac{q}{n^4}$, then $\frac{r}{n^5}=\frac{\cos 5a}{16}$

Here $\left(\frac{5}{4}\right)^2=\frac{p^2}{n^4}$, and $\frac{5}{16}=\frac{q}{n^4}$, so that we have $n^4=\frac{16p^2}{25}$, and $n^4=\frac{16q}{5}$
Thus the process will not be admissible unless $p^2=5q$, and this condition is satisfied by hypothesis

Then a must be found from $\cos 5a=\frac{16r}{n^5}$, put for n its value $\left(\frac{4p}{5}\right)^{\frac{1}{2}}$
thus $\cos 5a=16r\times\left(\frac{5}{4p}\right)^{\frac{5}{2}}=\frac{r}{2}\left(\frac{5}{p}\right)^{\frac{5}{2}}$ The process then will not be admissible if this expression is numerically greater than unity. Hence $\left(\frac{r}{2}\right)^2\left(\frac{5}{p}\right)^5$ must not be greater than unity; that is $\left(\frac{r}{2}\right)^2$ must not be greater than $\left(\frac{p}{5}\right)^5$

Suppose this condition also to hold, then one root is $n\cos a$, that is $2\left(\frac{p}{5}\right)^{\frac{1}{2}}\cos a$

Moreover we might also suppose $y=\cos\left(\frac{2\pi}{5}\pm a\right)$ or $y=\cos\left(\frac{4\pi}{5}\pm a\right)$, and we shall arrive at the same value for $\cos 5a$, since

$$\cos 5\left(\frac{2\pi}{5}\pm a\right)=\cos 5a \text{ and } \cos 5\left(\frac{4\pi}{5}\pm a\right)=\cos 5a$$

Hence we see that the other roots of the equation are

$$2\left(\frac{p}{5}\right)^{\frac{1}{2}}\cos\left(\frac{2\pi}{5}\pm a\right) \text{ and } 2\left(\frac{p}{5}\right)^{\frac{1}{2}}\cos\left(\frac{4\pi}{5}\pm a\right).$$

56 Proceed as in Example 55

Here $r=8$, $q=20$, $p=10$

$$\cos 5a=4\left(\frac{1}{2}\right)^{\frac{5}{2}}=\frac{1}{\sqrt{2}}, \text{ therefore } 5a=45^0$$

Hence the roots are

$$2\sqrt{2}\cos 9^\circ,\quad 2\sqrt{2}\cos(72^\circ \pm 9^\circ),\quad 2\sqrt{2}\cos(144^\circ \pm 9^\circ),$$

that is

$$2\sqrt{2}\cos 9^\circ,\quad 2\sqrt{2}\cos 63^\circ,\quad 2\sqrt{2}\cos 81^\circ,\quad 2\sqrt{2}\cos 153^\circ,\quad 2\sqrt{2}\cos 135^\circ,$$

the last is equal to -2

57. Since $\cos\frac{\pi}{7} = -\cos\frac{6\pi}{7}$, $\cos\frac{3\pi}{7} = -\cos\frac{4\pi}{7}$, $\cos\frac{5\pi}{7} = -\cos\frac{2\pi}{7}$, the equation whose roots are $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$, $\cos\frac{5\pi}{7}$ will be found by putting $-x$ for x in the equation found in Art. 273

The required equation may be found independently by the method of Art 273 from considering the equation

$$\cos 4\theta + \cos 3\theta = 0 \qquad \text{(i)},$$

or,

$$\cos 4\theta = \cos(\pi - 3\theta).$$

This equation gives

$$4\theta = 2n\pi \pm (\pi - 3\theta),$$

$$7\theta = (2n+1)\pi, \text{ or } \theta = (2n-1)\pi.$$

If $\cos\theta = x$, equation (i) is

$$8x^4 - 8x^2 + 1 + 4x^3 - 3x = 0$$

Divide by $x+1$; therefore

$$8x^3 - 4x^2 - 4x - 1 = 0$$

is an equation whose roots are the values of $\cos\frac{2n+1}{7}\pi$, namely

$$\cos\frac{\pi}{7},\quad \cos\frac{3\pi}{7},\quad \cos\frac{5\pi}{7}.$$

58 In the equation of Art 273 put $1-2y$ for x

Thus $$y = \frac{1}{2}(1-x) = \frac{1}{2}\left(1 - \cos\frac{2n\pi}{7}\right) = \sin^2\frac{n\pi}{7}$$

The roots of the equation in y will therefore be $\sin^2\frac{\pi}{7}$, $\sin^2\frac{2\pi}{7}$, $\sin^2\frac{3\pi}{7}$.

This equation is

$$8(1-2y)^3 + 4(1-2y)^2 - 4(1-2y) - 1 = 0$$

Expanding this we obtain

$$64y^3 - 112y^2 + 56y - 7 = 0 \quad \ldots \qquad \text{(i)}$$

Similarly, in the equation of Ex 57, write $1-2y$ for x, so that

$$y=\frac{1}{2}(1-x)=\frac{1}{2}\left(1-\cos\frac{2n+1}{7}\pi\right)=\sin^2\frac{2n+1}{14}\pi$$

The roots of the corresponding equation in y will therefore be

$$\sin^2\frac{\pi}{14},\quad \sin^2\frac{3\pi}{14},\quad \sin^2\frac{5\pi}{14}.$$

This equation is

$$8(1-2y)^3-4(1-2y)^2-4(1-2y)+1=0,$$

or, $$64y^3-80y^2+24y-1=0. \qquad \text{(ii)}$$

59 (i) The sum of the roots of the equation found in Art 273 is $-\frac{1}{2}$;

that is, $$\cos a+\cos 2a+\cos 3a=-\frac{1}{2}.$$

Now $(\sin a+\sin 2a+\sin 4a)^2$

$$=\sin^2 a+\sin^2 2a+\sin^2 4a+2\sin a\sin 2a+2\sin 4a\sin a+2\sin 2a\sin 4a$$

$$=\frac{1}{2}(3-\cos 2a-\cos 4a-\cos 8a)+\cos a-\cos 3a+\cos 3a-\cos 5a+\cos 2a-\cos 6a$$

Now $\cos 6a=\cos a$, $\cos 5a=\cos 2a$, $\cos 4a=\cos 3a$, $\cos 8a=\cos a$,

$$(\sin a+\sin 2a+\sin 4a)^2=\frac{1}{2}(3-\cos a-\cos 2a-\cos 3a)=\frac{1}{2}\left(3+\frac{1}{2}\right)$$

$$\therefore \sin a+\sin 2a+\sin 4a=\frac{1}{2}\sqrt{7}$$

(ii) By Art 273 $\cos a$, $\cos 2a$, $\cos 3a$ are roots of the equation

$$8x^3+4x^2-4x-1=0,$$

$\sec a$, $\sec 2a$, $\sec 3a$ (or $\sec 4a$) are roots of the equation

$$8\left(\frac{1}{x}\right)^3+4\left(\frac{1}{x^2}\right)-4\left(\frac{1}{x}\right)-1=0,$$

or, $$x^3+4x^2-4x-8=0,$$

$$\sec a+\sec 2a+\sec 4a=-4,$$

$$\sec a\sec 2a+\sec a\sec 4a+\sec 2a\sec 4a=-4$$

Subtract twice the second of these from the square of the first;

$$\sec^2 a+\sec^2 2a+\sec^2 4a=24,$$

$$1-\tan^2 a+1-\tan^2 2a+1-\tan^2 4a=24,$$

$$\tan^2 a+\tan^2 2a+\tan^2 4a=21.$$

(iii) From Ex 58 the equation for $\sin^2\frac{1}{2}\alpha$, $\sin^2\alpha$, $\sin^2\frac{3}{2}\alpha$ is

$$64y^3-112y^2+56y-7=0,$$

$$\therefore \sin^2\frac{1}{2}\alpha\sin^2\alpha\sin^2\frac{3}{2}\alpha=\frac{7}{64};$$

$$\therefore \sin\frac{1}{2}\alpha\sin\alpha\sin\frac{3}{2}\alpha=\frac{1}{8}\sqrt{7}.$$

60 The side of a regular heptagon subtends at the circumference an angle $\frac{\pi}{7}$; therefore if x represent the side, then $x=2\sin\frac{\pi}{7}$.

From Art 273 we see that the equation

$$y^3+y^2-2y-1=0 \qquad \text{(i)}$$

is satisfied by $y=2\cos\frac{2\pi}{7}$.

Now $$2-x^2=2-4\sin^2\frac{\pi}{7}=2\cos\frac{2\pi}{7}=y.$$

Substituting in equation (i), we see that x is a root of

$$(2-x^2)^3+(2-x^2)^2-2(2-x^2)-1=0,$$

or, $$7-14x^2+7x^4-x^6=0 \qquad \text{(ii)}.$$

Now $y=2\cos\frac{4\pi}{7}$, or $2\cos\frac{6\pi}{7}$ are also roots of equation (i).

Hence, in the same way as before, $x=2\sin\frac{2\pi}{7}$, or $2\sin\frac{3\pi}{7}$ are also roots of equation (ii), and $2\sin\frac{2\pi}{7}$, $2\sin\frac{3\pi}{7}$ represent the lengths of the short and long diagonals of the heptagon

61. The given equation may be written

$$(2x+1)\left(4x^3-3x+\frac{1}{2}\right)=0 \qquad \ldots\ldots \qquad \text{(i)}$$

Now $$\frac{1}{2}=\sin\frac{\pi}{6}=3\sin\frac{\pi}{18}-4\sin^3\frac{\pi}{18};$$

$$4\sin^3\frac{\pi}{18}-3\sin\frac{\pi}{18}+\frac{1}{2}=0.$$

Therefore equation (i) is satisfied by $x=\sin\frac{\pi}{18}$.

62
$$\sin\frac{\pi}{16}\sin\frac{7\pi}{16}=\frac{1}{2}\left(\cos\frac{6\pi}{16}-\cos\frac{8\pi}{16}\right)=\frac{1}{2}\cos\frac{3\pi}{8},$$

$$\sin\frac{3\pi}{16}\sin\frac{5\pi}{16}=\frac{1}{2}\left(\cos\frac{2\pi}{16}-\cos\frac{8\pi}{16}\right)=\frac{1}{2}\cos\frac{\pi}{8},$$

$$\sin\frac{\pi}{16}\sin\frac{3\pi}{16}\sin\frac{5\pi}{16}\sin\frac{7\pi}{16}=\frac{1}{4}\cos\frac{\pi}{8}\cos\frac{3\pi}{8}=\frac{1}{8}\left(\cos\frac{2\pi}{8}-\cos\frac{4\pi}{8}\right)=\frac{1}{8\sqrt{2}}$$

63 The equation $\cos 3\theta=\frac{1}{2}$ is satisfied by $3\theta=\frac{\pi}{3},\ \frac{5\pi}{3},\ \frac{7\pi}{3}$

Hence putting $\cos\theta=x$ we see that the roots of

$$4x^3-3x=\frac{1}{2} \qquad \text{(i)}$$

are
$$\cos\frac{\pi}{9},\quad \cos\frac{5\pi}{9},\quad \cos\frac{7\pi}{9},$$

or
$$\cos\frac{\pi}{9},\ -\cos\frac{4\pi}{9},\ -\cos\frac{2\pi}{9}.$$

If these be called $x_1,\ x_2,\ x_3,$ then from the equation (i)

$$x_1+x_2+x_3=0,\ x_1x_2+x_1x_3+x_2x_3=-\frac{3}{4} \qquad \text{(ii)}$$

Since $4x_1^3-3x_1=\frac{1}{2}$, therefore $4x_1^4=3x_1^2+\frac{1}{2}x$, with similar relations for x_2 and x_3,

$$4\Sigma x_1^4=3\Sigma x_1^2+\frac{1}{2}\Sigma x_1=3\Sigma x_1^2$$

$$=3\left\{(x_1+x_2+x_3)^2-2\left(x_1x_2+x_1x_3+x_2x_3\right)\right\}$$

$$=+6\ \frac{3}{4}$$

from the equations (ii).

Therefore
$$\Sigma x_1^4=\frac{9}{8}$$

Since $\cos^4\frac{3\pi}{9}=\cos^4\frac{\pi}{3}=\left(\frac{1}{2}\right)^4=\frac{1}{16}$, we obtain

$$\Sigma x_1^4+\cos^4\frac{3\pi}{9}=\frac{9}{8}+\frac{1}{16},$$

or
$$\cos^4\frac{\pi}{9}+\cos^4\frac{2\pi}{9}+\cos^4\frac{3\pi}{9}+\cos\frac{4\pi}{9}=\frac{19}{16}.$$

From equation (1)

$$8-6\cdot\frac{1}{x^2}-\frac{1}{x^3}=0;$$

the roots of the equation

$$y^3+6y^2-8=0$$

are $$\sec\frac{\pi}{9},\ -\sec\frac{2\pi}{9},\ -\sec\frac{4\pi}{9}.$$

Calling these y_1, y_2, y_3 we have

$$y_1+y_2+y_3=-6,\ y_1y_2+y_1y_3+y_2y_3=0,\ y_1y_2y_3=8,$$
$$y_1^2+y_2^2+y_3^2=(y_1+y_2+y_3)^2-2(y_1y_2+y_1y_3+y_2y_3)$$
$$=36,$$
$$\therefore\ y_1^4+y_2^4+y_3^4+2(y_1^2y_2^2+y_1^2y_3^2+y_2^2y_3^2)=36^2=1296;$$

and $$y_1^2y_2^2+y_1^2y_3^2+y_2^2y_3^2=(y_1y_2+y_1y_3+y_2y_3)^2-2y_1y_2y_3(y_1+y_2+y_3)$$
$$=96;$$
$$y_1^4+y_2^4+y_3^4=1296-2\times96=1104,$$

and $$\sec^4\frac{3\pi}{9}=2^4=16;$$

$$\sec^4\frac{\pi}{9}+\sec^4\frac{2\pi}{9}+\sec^4\frac{3\pi}{9}+\sec^4\frac{4\pi}{9}=1104+16=1120$$

64. The equation $\cos 6\theta=\cos 5\theta$ is satisfied by

$$6\theta=2n\pi\pm5\theta,$$

i.e. by $$\theta=2n\pi\ \text{ or }\ \frac{2n\pi}{11}$$

Hence, as in Art. 273, if we put $x=\cos\theta$ and expand $\cos 6\theta$, $\cos 5\theta$ we get an equation whose roots are $x=\cos\frac{2\pi}{11}$, &c, and $x=1$

$$\cos 6\theta=2\cos^2 3\theta-1=2(4x^3-3x)^2-1$$
$$=32x^6-48x^4+18x^2-1,$$
$$\cos 5\theta=16x^5-20x^3+5x.\qquad\text{(Ch VIII. Ex 59)}$$

The required equation is therefore

$$32x^6-16x^5-48x^4+20x^3+18x^2-5x-1=0$$

Divide by $x-1$,

$$32x^5+16x^4-32x^3-12x^2+6x+1=0\qquad(1)$$

Put $2x=y$,

$$y^5+y^4-4y^3-3y^2+3y+1=0.$$

This is therefore an equation whose roots are

$$2\cos\frac{2\pi}{11},\quad 2\cos\frac{4\pi}{11},\ \&c$$

If in equation (1) we put $\frac{1}{z}$ for x the resulting equation in z has for roots $\sec\frac{2\pi}{11}$, . $\sec\frac{10\pi}{11}$, or since $\sec\frac{10\pi}{11}=-\sec\frac{\pi}{11}$, &c , the roots are $-\sec\frac{\pi}{11}$, $\sec\frac{2\pi}{11}$, &c The equation in z is

$$z^5+6z^4-12z^3-32z^2+16z+32=0$$

$$\sec^2\frac{\pi}{11}\sec^2\frac{2\pi}{11}\sec^2\frac{3\pi}{11}\sec^2\frac{4\pi}{11}\sec^2\frac{5\pi}{11}=(-32)^2=1024,$$

$$\sec^2\frac{2\pi}{11}+\sec^2\frac{4\pi}{11}+\quad=\left(\sec\frac{2\pi}{11}+\sec\frac{4\pi}{11}+\quad\right)^2$$

$$-2\left(\sec\frac{2\pi}{11}\sec\frac{4\pi}{11}+\quad\right)$$

$$=(-6)^2-2(-12)=60,$$

or
$$\sec^2\frac{\pi}{11}+\sec^2\frac{2\pi}{11}+\quad=60.$$

65 From Art. 271 (compare also Art 284) we have

$$\tan n\theta=\frac{n\tan\theta-\frac{n(n-1)(n-2)}{1\ 2\ 3}\tan^3\theta+}{1-\frac{n(n-1)}{1\ 2}\tan^2\theta+}.$$

Put $\tan n\theta=0$; then $n\theta=m\pi$, or $\theta=0$

Hence
$$n-\frac{n(n-1)(n-2)}{1\ 2\ 3}\tan^2\theta+\quad=0$$

is an equation satisfied by $\theta=\frac{m\pi}{n}$; put $x=\tan^2\theta$ and $n=11$. The required equation is therefore

$$1-15x+42x^2-30x^3+5x^4-\frac{1}{11}x^5=0.$$

From this equation we see that

$$\tan^2\frac{\pi}{11}+\tan^2\frac{2\pi}{11}+\tan^2\frac{3\pi}{11}+\tan^2\frac{4\pi}{11}+\tan^2\frac{5\pi}{11}=55;$$

from this we can derive the result (1) of Ex. 64.

66 If $\sin 3x = \frac{\sqrt{3}}{2}$, then

$$3x = 60^\circ \text{ or } 120^\circ \text{ or } -240^\circ, \text{ \&c}$$

Hence the different values of $\sin x$ which satisfy

$$\frac{\sqrt{3}}{2} = 3\sin x - 4\sin^3 x \quad . \qquad \text{(i)}$$

are
$$\sin 20^\circ, \ \sin 40^\circ, \ -\sin 80^\circ$$

The product of the roots of equation (i) is $-\frac{\sqrt{3}}{8}$.

Hence
$$\sin 20^\circ . \sin 40^\circ . \sin 80^\circ = \frac{\sqrt{3}}{8}.$$

$$\therefore \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}.$$

If $\cos 3x = \frac{1}{2}$,

$$3x = 60^\circ, \ 360^\circ - 60^\circ, \ 720^\circ - 60^\circ, \text{ \&c}$$

Hence the different values of $\cos x$ which satisfy

$$\frac{1}{2} = 4\cos^3 x - 3\cos x \quad . \quad . \qquad \text{(ii)}$$

are
$$\cos 20^\circ, \ \cos 100^\circ, \ \cos 220^\circ,$$

or
$$\cos 20^\circ, \ -\cos 80^\circ, \ -\cos 40^\circ$$

The product of the roots of equation (ii) is $\frac{1}{8}$

$$\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{16}.$$

Otherwise:

$$\sin 20^\circ \sin 40^\circ = \sin(30^\circ - 10^\circ)\sin(30^\circ + 10^\circ)$$
$$= \sin^2 30^\circ - \sin^2 10^\circ = \frac{1}{4} - \sin^2 10^\circ$$
$$= \frac{1}{4}(4\cos^2 10^\circ - 3)$$

$$\sin 20^\circ \sin 40^\circ \sin 80^\circ = \frac{1}{4}\cos 10^\circ (4\cos^2 10^\circ - 3)$$
$$= \frac{1}{4}(4\cos^3 10^\circ - 3\cos 10^\circ) = \frac{1}{4}\cos(3 . 10^\circ) = \frac{\sqrt{3}}{8},$$

which gives the first result.

Also $\quad 16 \sin 20^0 \cos 20^0 \cos 40^0 \cos 60^0 \cos 80^0$

$$=8 \sin 40^0 \cos 40^0 \cos 60^0 \cos 80^0 = 4 \sin 80^0 \cos 80^0 \cos 60^0$$
$$=2 \sin 160^0 \cos 60^0 = \sin 160^0 = \sin 20^0,$$

which gives the second result

67 The equation is

$$y^6 - 6y^4 + 8y^2 + 1 = y(y^4 - 6y^2 + 8),$$

or
$$y^2(y^2-2)(y^2-4)+1 = y(y^2-2)(y^2-4)$$

Put $x+2$ for y^2, and square, therefore

$$\{x(x+2)(x-2)+1\}^2 = (x+2) \quad x^2 \quad (x-2)^2,$$
$$(x^3-4x+1)^2 = x^2(x^2-4)(x-2),$$

i e
$$x^6 - 8x^4 + 2x^3 + 16x^2 - 8x + 1 = x^5 - 2x^4 - 4x^3 + 8x^2,$$

i e
$$x^6 - x^5 - 6x^4 + 6x^3 + 8x^2 - 8x + 1 = 0$$

This is the same equation as that given, which is therefore satisfied by x, where

$$x = \left(2 \cos \frac{2\pi}{21}\right)^2 - 2 = 2\left(2 \cos^2 \frac{2\pi}{21} - 1\right)$$
$$= 2 \cos \frac{4\pi}{21}$$

In the same way $2 \cos \frac{8\pi}{21}$, $2 \cos \frac{16\pi}{21}$, $2 \cos \frac{32\pi}{21}$, $2 \cos \frac{64\pi}{21}$ are roots.

Since
$$2 \cos \frac{32\pi}{21} = 2 \cos \frac{(42-10)\pi}{21} = 2 \cos \frac{10\pi}{21},$$

and
$$2 \cos \frac{64\pi}{21} = 2 \cos \frac{(84-20)\pi}{21} = 2 \cos \frac{20\pi}{21},$$

the other roots are

$$2 \cos \frac{4\pi}{21}, \quad 2 \cos \frac{8\pi}{21}, \quad 2 \cos \frac{10\pi}{21}, \quad 2 \cos \frac{16\pi}{21}, \quad 2 \cos \frac{20\pi}{21}.$$

The equation given may be found as in Art 273 by means of the equation $\cos 11\theta = \cos 10\theta$ The resulting equation for $2 \cos \theta$ will include roots $2 \cos \frac{6\pi}{21}$, $2 \cos \frac{12\pi}{21}$, $2 \cos \frac{18\pi}{21}$ and $2 \cos \frac{14\pi}{21}$, and $2 \cos 0$ It will therefore contain factors $x-2$, $x+1$, x^3+x^2-2x-1, the last factor corresponding to roots $2 \cos \frac{2\pi}{7}$, &c (Art 273)

The equation may also be more simply derived from considering the equation

$$\cos 7\theta = \frac{1}{2}$$

68 It is required to prove that the two given equations are satisfied by

$$\frac{x}{\sin 2\alpha}=\frac{y}{\sin 3\alpha}=\frac{a}{\sin\alpha}, \text{ where } \alpha=\frac{\pi}{7}$$

From these, $x=2a\cos\alpha$,

$$ay+a^2=a^2\,\frac{\sin\alpha+\sin 3\alpha}{\sin\alpha}=2a^2\,\frac{\sin 2\alpha\cos\alpha}{\sin\alpha}=4a^2\cos^2\alpha=x^2,$$

$$y^2-a^2=a^2\,\frac{\sin^2 3\alpha-\sin^2\alpha}{\sin^2\alpha}=a^2\,\frac{\sin 4\alpha\sin 2\alpha}{\sin\alpha\,.\sin\alpha}=xy, \text{ since } \sin 4\alpha=\sin 3\alpha$$

Hence both equations are satisfied

Put $$2\cos\theta=p,\qquad x=ap,$$

$$ay=x^2-a^2=a^2(p^2-1).$$

Hence from the equation $y^2=a^2+xy$,

$$(p^2-1)^2=1+p\,(p^2-1),$$

or $$p^4-p^3-2p^2+p=0,$$

$$p=0, \text{ which gives } x=0,\ y=-a;$$

or $$p^3-p^2-2p+1=0$$

The values of p satisfying this equation (Ex 57) are

$$2\cos\frac{\pi}{7},\ 2\cos\frac{3\pi}{7},\ 2\cos\frac{5\pi}{7}$$

The first solution corresponds with that given

If $$x=2\cos\frac{3\pi}{7},$$

then $$y=a\left(4\cos^2\frac{3\pi}{7}-1\right)=a\left(3-4\sin^2\frac{3\pi}{7}\right)$$

$$=a\frac{\sin\frac{9\pi}{7}}{\sin\frac{3\pi}{7}}=-a\,.\frac{\sin\frac{2\pi}{7}}{\sin\frac{3\pi}{7}}.$$

Hence $$\frac{x}{\sin\frac{\pi}{7}}=\frac{y}{-\sin\frac{2\pi}{7}}=\frac{a}{\sin\frac{3\pi}{7}}.$$

If $$x=2\cos\frac{5\pi}{7},$$

then $$y=a\left(4\cos^2\frac{5\pi}{7}-1\right)=a\left(3-4\sin^2\frac{5\pi}{7}\right)$$

$$=a.\frac{\sin\frac{15\pi}{7}}{\sin\frac{5\pi}{7}}=a\frac{\sin\frac{\pi}{7}}{\sin\frac{2\pi}{7}}$$

Hence $$\frac{x}{-\sin\frac{3\pi}{7}}=\frac{y}{\sin\frac{\pi}{7}}=\frac{a}{\sin\frac{2\pi}{7}}.$$

If a regular heptagon $ABCDE$ whose side is a be inscribed in a circle of radius R, and $AC=x$, $AD=y$, we obtain (by Euc VI. D) from the quadrilateral $ABCD$

$$x^2=a^2+ay,$$

and from the quadrilateral $ABDE$

$$y^2=a^2+xy$$

Hence since a subtends at the circumference an angle $\frac{\pi}{7}$, x an angle $\frac{2\pi}{7}$, y an angle $\frac{3\pi}{7}$ we see that

$$x=2R\sin\frac{2\pi}{7},\quad y=2R\sin\frac{3\pi}{7},\quad a=2R\sin\frac{\pi}{7}$$

69 The geometrical solution of the three given equations may be obtained as in Ex. 68 by considering a regular polygon of nine sides each equal to a inscribed in a circle, the diagonals being x, y, z

To solve the equations analytically, put $x=2a\cos\theta$

from the first equation

$$y=a(4\cos^2\theta-1)=a(3-4\sin^2\theta)=\frac{a\sin 3\theta}{\sin\theta}$$

from the second equation

$$2a\cos\theta\quad z=a^2\frac{\sin^3 3\theta-\sin^2\theta}{\sin^2\theta}=a^2\frac{\sin 4\theta\quad\sin 2\theta}{\sin^2\theta},$$

or $$z=\frac{a\sin 4\theta}{\sin\theta}$$

Since the third equation must be satisfied we must have

$$\frac{\sin 4\theta\sin 3\theta}{\sin^2\theta}=\frac{\sin^2 4\theta-\sin^2\theta}{\sin^2\theta}=\frac{\sin 5\theta\sin 3\theta}{\sin^2\theta}.$$

either $\sin 3\theta=0$, which gives

$$x=\pm a,\ y=0,\ z=\mp a,$$

or $$\sin 5\theta=\sin 4\theta$$

$$\theta=\frac{\pi}{9},\ \frac{3\pi}{9},\ \frac{5\pi}{9}\text{ or }\frac{7\pi}{9}.$$

70 Since (see Ex 65)

$$\tan n\phi = \frac{n\tan\phi - \frac{n(n-1)(n-2)}{1.2.3}\tan^3\phi + \ldots}{1 - \frac{n(n-1)}{1.2}\tan^2\phi + \ldots}$$

we have, putting $\tan\phi = \frac{1}{x}$,

$$\cot n\phi = \frac{x^n - \frac{n(n-1)}{1.2}x^{n-2} + \ldots}{nx^{n-1} - \frac{n(n-1)(n-2)}{1.2.3}x^{n-3} + \ldots}.$$

Hence the given equation becomes

$$\cot n\phi = \cot n\theta,$$

$$\therefore n\phi = r\pi + n\theta,$$

or

$$\phi = \theta + \frac{r\pi}{n}.$$

The values of x which satisfy the given equation are the values of $\cot\phi$,

$$\text{i e of } \cot\left(\theta + \frac{r\pi}{n}\right)$$

The sum of the roots of the given equation in x is clearly $n\cot n\theta$, and since the roots are given by

$$x = \cot\left(\theta + \frac{r\pi}{n}\right),$$

we have

$$\cot\theta + \cot\left(\theta + \frac{\pi}{n}\right) + \cot\left(\theta + \frac{2\pi}{n}\right) + \ldots + \cot\left(\theta + \frac{n-1}{n}\pi\right) = n\cot n\theta.$$

71 Let $u = 2^n\cos\theta\cos 2\theta\cos 2^2\theta \ldots \cos 2^{n-1}\theta$,

then $u\sin\theta = 2^n\sin\theta\cos\theta\cos 2\theta\cos 2^2\theta \ldots \cos 2^{n-1}\theta$

$$= 2^{n-1}\sin 2\theta\cos 2\theta\cos 2^2\theta \ldots \cos 2^{n-1}\theta$$

$$= 2^{n-2}\sin 2^2\theta\cos 2^2\theta \ldots \cos 2^{n-1}\theta$$

$$= \&c$$

$$= \sin 2^n\theta$$

$$= \sin\frac{2^n}{2^n+1}\pi = \sin\left(\pi - \frac{\pi}{2^n+1}\right)$$

$$= \sin\theta,$$

$$\therefore u = 1$$

72 $\tan(\tan^{-1}\alpha+\tan^{-1}\beta+\tan^{-1}\gamma)$

$$=\frac{\alpha+\beta+\gamma-\alpha\beta\gamma}{1-\alpha\beta-\alpha\gamma-\beta\gamma}=\frac{-p+p}{1-q}=0,$$

unless $q=1$

73 $$\tan(\theta+\phi+\psi)=\frac{\tan\theta+\tan\phi+\tan\psi-\tan\theta\tan\phi\tan\psi}{1-\tan\theta\tan\phi-\tan\theta\tan\psi-\tan\phi\tan\psi}$$

$$=\frac{a+1-c}{1-c+a}=1=\tan\frac{\pi}{4}$$

$$\theta+\phi+\psi=n\pi+\frac{\pi}{4}.$$

74 $\sin\theta\cos\beta+\cos\theta\sin\beta=2a\sin\theta\cos\theta+b,$

$$\left\{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\beta+\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)\sin\beta\right\}\left(\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\right)$$

$$=4a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)+b\left(\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\right)^2$$

Divide by $\cos^4\frac{\theta}{2}$ and put $\tan\frac{\theta}{2}=t$

$$2t(1+t^2)\cos\beta+(1-t^4)\sin\beta=4at(1-t^2)+b(1+t^2)^2$$

This equation gives four values of t, it may be written

$$t^4(b+\sin\beta)+At^3+2bt^2+Bt+b-\sin\beta=0 \qquad (1)$$

From Art 270 we may deduce (cf Art. 285) that if s_r denote the products of $\tan\frac{\theta_1}{2}$, $\tan\frac{\theta_2}{2}$, $\tan\frac{\theta_3}{2}$, $\tan\frac{\theta_4}{4}$ taken r at a time,

$$\tan\frac{1}{2}(\theta_1+\theta_2+\theta_3+\theta_4)=\frac{s_1-s_3}{1-s_2+s_4}$$

From equation (1) we have

$$s_2=\frac{2b}{b+\sin\beta},\quad s_4=\frac{b-\sin\beta}{b+\sin\beta},$$

$$1-s_2+s_4=1-\frac{2b}{b+\sin\beta}+\frac{b-\sin\beta}{b+\sin\beta}=0$$

$$\tan\frac{1}{2}(\theta_1+\theta_2+\theta_3+\theta_4)=\infty,$$

$$\frac{1}{2}(\theta_1+\theta_2+\theta_3+\theta_4)=(2n+1)\frac{\pi}{2},$$

$$\theta_1+\theta_2+\theta_3+\theta_4=(2n+1)\pi$$

75 Let $\tan\theta=t$, $\tan\alpha=a$; the given equation is

$$\frac{t+a}{1-at}=\frac{-2\lambda t}{t^2-1},$$

$$t^3+at^2-t-a=-2\lambda t+2\lambda at^2$$

Therefore with the notation of the last example

$$s_1=-a+2\lambda a,\ s_2=-1+2\lambda,\ s_3=a;$$

$$\tan(\theta_1+\theta_2+\theta_3)=\frac{s_1-s_3}{1-s_2}=\frac{-a+2\lambda a-a}{2-2\lambda}$$

$$=-a=-\tan\alpha=\tan(n\pi-\alpha),$$

$$\theta_1+\theta_2+\theta_3+\alpha=n\pi$$

76 From the first equation we find

$$x+2y\cos\theta+z(4\cos^2\theta-1)=4\cos\theta(2\cos^2\theta-1),$$

similarly for the other equations

Hence $\cos\theta$, $\cos\phi$, $\cos\psi$ are the values of ξ given by the equation

$$x+2y\xi+z(4\xi^2-1)=4\xi(2\xi^2-1),$$

or,
$$8\xi^3-4z\xi^2-(4+2y)\xi+z-x=0,$$

$$P_1=\frac{z}{2},\ 8P_2=-4-2y,\ 8P_3=x-z,$$

$$x=8P_3+2P_1,\ y=-4P_2-2,\ z=2P_1.$$

77 Suppose $\cos\alpha$ to be one value of x, $\tan\beta$ one value of y, then

$$\cos^{-1}x=2n\pi\pm\alpha,\ \tan^{-1}y=m\pi+\beta,$$

$$\sin(\cos^{-1}x+\tan^{-1}y)=\sin(\overline{2n+m}\pi+\beta\pm\alpha).$$

$2n+m$ may have any value, therefore the given expression may have any of the four values

$$\sin(\beta+\alpha),\ \sin(\beta-\alpha),\ -\sin(\beta+\alpha),\ -\sin(\beta-\alpha).$$

78. Let
$$\tan^{-1}\left\{\sin\left(\cos^{-1}\sqrt{\frac{2}{3}}\right)\right\}=\theta,$$

$$\sin\left(\cos^{-1}\sqrt{\frac{2}{3}}\right)=\tan\theta$$

If
$$\cos^{-1}\sqrt{\frac{2}{3}}=\phi,$$

$$\cos\phi=\sqrt{\frac{2}{3}},$$

$$\sin\phi=\pm\sqrt{\frac{1}{3}}.$$

Hence $$\tan\theta = \pm\sqrt{\frac{1}{3}},$$

$$\theta = n\pi \pm \frac{\pi}{6}$$

79 We have $$a\left(\sin\theta - \frac{1}{\sin\theta}\right) + b\left(\cos\theta - \frac{1}{\cos\theta}\right) = 0,$$

i.e. $$\frac{a\cos^2\theta}{\sin\theta} + \frac{b\sin^2\theta}{\cos\theta} = 0$$

$$a\cos^3\theta = -b\sin^3\theta,$$

$$\frac{\sin\theta}{a^{\frac{1}{3}}} = \frac{\cos\theta}{-b^{\frac{1}{3}}}.$$

Each of these $$= \frac{(\sin^2\theta + \cos^2\theta)^{\frac{1}{2}}}{(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}}} = \frac{1}{(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}}},$$

$$a\sin\theta + b\cos\theta = \frac{a\ \ a^{\frac{1}{3}} - b\ .\ b^{\frac{1}{3}}}{(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}}}$$

$$= (a^{\frac{2}{3}} - b^{\frac{2}{3}})\,(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}}$$

80 $\sin(n+1)\theta + \sin(n-1)\theta = 2\cos\theta\sin n\theta$

$$\frac{\sin(n+1)\theta}{\sin n\theta} = 2\cos\theta - \frac{\sin(n-1)\theta}{\sin n\theta} = 2\cos\theta - \cfrac{1}{\cfrac{\sin n\theta}{\sin(n-1)\theta}}$$

$$= 2\cos\theta - \cfrac{1}{2\cos\theta - \cfrac{1}{\cfrac{\sin(n-1)\theta}{\sin(n-2)\theta}}}$$

$$= \qquad .$$

$$= 2\cos\theta - \frac{1}{2\cos\theta -}\,\frac{1}{2\cos\theta -}\qquad - \cfrac{1}{\cfrac{\sin 2\theta}{\sin\theta}},$$

which is the required result

81. $$\frac{1}{\cos 2\theta} = \frac{1}{2\cos^2\theta - 1} = \frac{\sec\theta}{2\cos\theta - \cfrac{1}{\cos\theta}}$$

$$= \cfrac{\sec\theta}{2\cos\theta - \cfrac{\sec\frac{\theta}{2}}{2\cos\frac{\theta}{2} - \cfrac{1}{\cos\frac{\theta}{2}}}}$$

$$= \qquad .$$

$$= \frac{\sec\theta}{2\cos\theta -}\;\frac{\sec\frac{\theta}{2}}{2\cos\frac{\theta}{2} -}\;\frac{\sec\frac{\theta}{2^2}}{2\cos\frac{\theta}{2^2} -}$$

82 $$\frac{\cos\theta}{\cos^2\frac{\theta}{2}} = \frac{2\cos^2\frac{\theta}{2} - 1}{\cos^2\frac{\theta}{2}} = 2 - \frac{1}{\cos^2\frac{\theta}{2}};$$

$$2 = \frac{1}{\cos^2\frac{\theta}{2}} + \frac{1}{2}\cdot\frac{\cos\theta}{\cos^2\frac{\theta}{2}}\ 2$$

$$= \frac{1}{\cos^2\frac{\theta}{2}} + \frac{1}{2}\,\frac{\cos\theta}{\cos^2\frac{\theta}{2}}\left(\frac{1}{\cos^2\frac{\theta}{4}} + \frac{1}{2}\,\frac{\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{4}}\ 2\right)$$

$$= \frac{1}{\cos^2\frac{\theta}{2}} + \frac{1}{2}\,\frac{\cos\theta}{\cos^2\frac{\theta}{2}\cos^2\frac{\theta}{2^2}} + \frac{1}{2^2}\cdot\frac{\cos\theta\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}\cos^2\frac{\theta}{2^2}}\ 2,$$

and so on, continually putting instead of the final 2 an expression of the form

$$\frac{1}{\cos^2\frac{\theta}{2^r}} + \frac{1}{2}\cdot\frac{\cos\frac{\theta}{2^{r-1}}}{\cos^2\frac{\theta}{2^r}}.$$

83 See Miscellaneous Examples 37, in which put $\frac{\theta}{2^{n-1}}$ instead of θ.

84. $$\tan^2\theta = \frac{\sin^2\theta}{1+\sin\alpha}\cdot\frac{1+\sin\alpha}{\cos^2\theta} = \frac{\sin^2\theta}{1+\sin\alpha}\cdot\frac{\sin^2\theta+\cos^2\theta+\sin\alpha}{\cos^2\theta}$$

$$= \frac{\sin^2\theta}{1+\sin\alpha}\left(1+\frac{\sin^2\theta+\sin\alpha}{\cos^2\theta}\right)$$

$$= \frac{\sin^2\theta}{1+\sin\alpha}\left(1+\frac{\sin^2\theta+\sin\alpha}{1+\sin\beta}\quad\frac{\sin^2\theta+\cos^2\theta+\sin\beta}{\cos^2\theta}\right)$$

$$= \frac{\sin^2\theta}{1+\sin\alpha} + \frac{\sin^2\theta}{1+\sin\alpha}\cdot\frac{\sin^2\theta+\sin\alpha}{1+\sin\beta}\left(1+\frac{\sin^2\theta+\sin\beta}{\cos^2\theta}\right)$$

$$= \frac{\sin^2\theta}{1+\sin\alpha} + \frac{\sin^2\theta}{1+\sin\alpha}\cdot\frac{\sin^2\theta+\sin\alpha}{1+\sin\beta}$$

$$+ \frac{\sin^2\theta}{1+\sin\alpha}\cdot\frac{\sin^2\theta+\sin\alpha}{1+\sin\beta}\left(\frac{\sin^2\theta+\sin\beta}{1+\sin\gamma}\cdot\frac{\sin^2\theta+\cos^2\theta+\sin\gamma}{\cos^2\theta}\right),$$

and so on.

85
$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} = \frac{2}{\cot x - \tan x},$$

$$\tan x = \frac{2}{\cot\frac{x}{2} - \tan\frac{x}{x}},$$

$$\tan 2x = \cfrac{2}{\cot x - \cfrac{2}{\cot\frac{x}{2} - \tan\frac{x}{2}}}$$

$$= \cfrac{2}{\cot x - \cfrac{2}{\cot\frac{x}{2} - \cfrac{2}{\cot\frac{x}{2^2} - \tan\frac{x}{2^2}}}},$$

and so on

Hence since $\tan\frac{x}{2^n}$ is ultimately zero, the value of the continued fraction is $\tan 2x$

It is assumed that the continued fraction is convergent; this may be easily proved

86
$$\frac{\left(2\sin\frac{x}{2}\right)^2}{2\sin x} = \frac{4\sin^2\frac{x}{2}}{4\sin\frac{x}{2}\cos\frac{x}{2}} = \tan\frac{x}{2}$$

Similarly
$$\frac{\left(2\sin\frac{x}{4}\right)^4}{\left(2\sin\frac{x}{2}\right)^2} = \tan^2\frac{x}{4},$$

$$\frac{\left(2\sin\frac{x}{8}\right)^8}{\left(2\sin\frac{x}{4}\right)^4} = \tan^4\frac{x}{8},$$

$$\frac{\left(2\sin\frac{x}{2^n}\right)^{2^n}}{\left(2\sin\frac{x}{2^{n-1}}\right)^{2^{n-1}}} = \left(\tan\frac{x}{2^n}\right)^{2^{n-1}}$$

Multiplying we obtain the required result

87.
$$\frac{1-e^{-2x}}{2} = 2\cdot\frac{1+e^{-x}}{2}\cdot\frac{1-e^{-x}}{2}$$

$$= 2\cdot\frac{1+e^{-x}}{2}\; 2\cdot\frac{1+e^{-\frac{x}{2}}}{2}\;\frac{1-e^{-\frac{x}{2}}}{2}$$

$$=2^2 \cdot \frac{1+e^{-x}}{2} \cdot \frac{1+e^{-\frac{x}{2}}}{2} \cdot 2 \cdot \frac{1+e^{-\frac{x}{2^2}}}{2} \cdot \frac{1-e^{-\frac{x}{2^2}}}{2}$$

$$= \qquad \cdot \qquad \cdots\cdot$$

$$=2^{n+1} \cdot \frac{1+e^{-x}}{2} \cdot \frac{1+e^{-\frac{x}{2}}}{2} \cdot \frac{1+e^{-\frac{x}{2^2}}}{2} \qquad \frac{1+e^{-\frac{x}{2^n}}}{2} \times \frac{1-e^{-\frac{x}{2^n}}}{2}.$$

Now $2^n\left(1-e^{-\frac{x}{2^n}}\right)=2^n\left(\frac{x}{2^n}-\frac{1}{2}\frac{x^2}{2^{2n}}+\ldots\right)$

$$=x-\frac{1}{2}\frac{x^2}{2^n}+ \ .$$

$$=x,$$

when n is indefinitely increased.

Hence dividing both sides by x we obtain the required result

88 $\cos^{-1}x+\cos^{-1}y=\pi-\cos^{-1}z$,

$$\cdot \quad \cos(\cos^{-1}x+\cos^{-1}y)=\cos(\pi-\cos^{-1}z)=-\cos(\cos^{-1}z),$$

$$xy-\sin(\cos^{-1}x)\sin^{-1}(\cos^{-1}y)=-z,$$

if $\cos^{-1}x=\alpha$,

$$x=\cos\alpha,$$

$$\sqrt{1-x^2}=\sin\alpha=\sin(\cos^{-1}x);$$

$$xy+z=\sqrt{(1-x^2)(1-y^2)};$$

$$\cdot \quad x^2y^2+2xyz+z^2=1-x^2-y^2+x^2y^2,$$

$$x^2+y^2+z^2+2xyz=1.$$

89 $\sin(\pm A)=\sin(n\pi\pm B\pm C)$,

$$\pm x=\pm\sin B\cos C\pm\cos B\sin C$$

$$=\pm y\cos C\pm z\cos B,$$

$$(\pm x\pm y\cos C)^2=z^2\cos^2 B,$$

or $x^2+y^2(1-z^2)\pm 2xy\cos C=z^2(1-y^2)$,

$$(x^2+y^2-z^2)^2=(2xy\cos C)^2=4x^2y^2(1-z^2),$$

$$x^4+y^4+z^4+2x^2y^2-2y^2z^2-2x^2z^2=4x^2y^2-4x^2y^2z^2,$$

$$x^4+y^4+z^4-2x^2y^2-2y^2z^2-2z^2x^2+4x^2y^2z^2=0$$

90. Put $n=1$ and multiply up, therefore

$$\sin^2 \alpha \cos^4 \theta + \cos^2 \alpha \sin^4 \theta = \sin^2 \alpha \cos^2 \alpha,$$

$$\sin^2 \alpha (1 - 2\sin^2 \theta + \sin^4 \theta) + \cos^2 \alpha \sin^4 \theta = \sin^2 \alpha - \sin^4 \alpha,$$

$$\sin^4 \theta - 2\sin^2 \theta \sin^2 \alpha + \sin^4 \alpha = 0;$$

$$\sin^2 \theta = \sin^2 \alpha,$$

also $$\cos^2 \theta = \cos^2 \alpha.$$

Hence $$\left(\frac{\sin^{2n} \theta}{\sin^{2n} \alpha}\right) \sin^2 \theta + \left(\frac{\cos^{2n} \theta}{\cos^{2n} \alpha}\right) \cos^2 \theta$$

$$= \sin^2 \theta + \cos^2 \theta = 1$$

91. From Art. 230 we have

$$c \cos \frac{1}{2}(A - B) = (a + b) \sin \frac{1}{2} C,$$

$$c \sin \frac{1}{2}(A - B) = (a - b) \cos \frac{1}{2} C,$$

if $$A - B = \frac{1}{2}\pi, \quad \cos \frac{1}{2}(A - B) = \sin \frac{1}{2}(A - B) = \frac{1}{\sqrt{2}};$$

$$\frac{c^2}{2(a+b)^2} + \frac{c^2}{2(a-b)^2} = \sin^2 \frac{1}{2} c + \cos^2 \frac{1}{2} c = 1,$$

or $$(a+b)^{-2} + (a-b)^{-2} = 2c^{-2}$$

If $A + B = \frac{1}{2}\pi$, then $C = \frac{1}{2}\pi$,

$$a^2 + b^2 = c^2,$$

or $$(a+b)^2 + (a-b)^2 = 2c^2.$$

92 $$\sin A = \sqrt{(1 - \cos^2 A)} = \sqrt{\left\{1 - \frac{(b^2 + c^2 - a^2)^2}{4b^2c^2}\right\}}$$

$$= \frac{1}{2bc}\sqrt{(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4)},$$

$$\sin B = \frac{1}{2ac}\sqrt{(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4)}$$

$$-\cos C = \cos (A + B) = \cos A \cos B - \sin A \sin B$$

$$= \frac{1}{4abc^2}\{(b^2+c^2-a^2)(c^2+a^2-b^2) - 2b^2c^2 - 2c^2a^2 - 2a^2b^2 + a^4 + b^4 + c^4\}$$

$$= \frac{1}{4abc^2}\{2c^4 - 2b^2c^2 - 2c^2a^2\},$$

or $$2ab \cos C = a^2 + b^2 - c^2$$

93 Multiply the equations by a, b, c respectively and subtract the third from the sum of the first two, therefore

$$a^2 + b^2 - c^2 = 2ab \cos C.$$

94 Each fraction $=\dfrac{a\cos B+b\cos A}{\sin A\cos B+\sin B\cos A}=\dfrac{a\cos B+b\cos A}{\sin C}$,

$$c=a\cos B+b\cos A$$

Similarly $\qquad b=c\cos A+a\cos C,$

and $\qquad a=b\cos C+c\cos B$

Hence the result follows from Ex 93

95. $\dfrac{1}{a^2}\left\{\dfrac{y^2}{b^2}+\dfrac{2yz}{bc}\cos A+\dfrac{z^2}{c^2}\right\}=\dfrac{1}{a^2}\left\{\dfrac{y^2}{b^2}+\dfrac{yz}{b^2c^2}(b^2+c^2-a^2)+\dfrac{z^2}{c^2}\right\}$

$$=\frac{1}{a^2}\left\{\frac{y(y+z)}{b^2}+\frac{z(y+z)}{c^2}-\frac{a^2}{b^2c^2}yz\right\}$$

$$=\frac{1}{a^2}\left\{-\frac{xy}{b^2}-\frac{xz}{c^2}-\frac{a^2yz}{b^2c^2}\right\}$$

$$=-\frac{xy}{a^2b^2}-\frac{xz}{a^2c^2}-\frac{yz}{b^2c^2},$$

a symmetrical expression

96 Take straight lines

$$OA=a,\ OB=b,\ OC=c,\ OD=d,$$

so that $\qquad \angle AOB=\theta,\ \angle BOC=\phi,\ COD=\psi$

The given equations are therefore equivalent to

$$AB^2=BC^2=CD^2=DA^2=\frac{1}{2}BD^2$$

Hence $ABCD$ is a square.

$$\frac{1}{2}AC^2=AB^2$$

But $\qquad \dfrac{1}{2}AC^2=\dfrac{1}{2}\{a^2+c^2-2ac\cos(\theta+\phi)\}$

97 The continued product

$$=\frac{2\Delta}{a+b+c}\ \frac{2\Delta'}{a+b+c'}\ \frac{2\Delta}{a-b+c}\cdot\frac{2\Delta}{a-b+c'}. \qquad \ldots\ldots\ldots (1),$$

dashed letters referring to the triangle ABC', (Art 231)

$$(a+b+c)(a-b+c)=a^2+c^2-b^2+2ac=2ac(1+\cos B),$$

$$(a+b+c')(a-b+c')=2ac'(1+\cos B'),$$

and $\qquad B'=\pi-B,\ \cos B'=-\cos B,$

product of the denominators in (1) is

$$4a^2cc'(1-\cos^2 B)=4a^2cc'\sin^2 B$$

$$=16\ \frac{1}{2}ac\sin B\cdot\frac{1}{2}ac'\sin B'$$

$$=16\Delta\Delta';$$

the expression (1) $\qquad =\Delta\Delta'.$

98 Put $x=\cos\theta$, then $x_1=\sqrt{\frac{1+\cos\theta}{2}}=\cos\frac{\theta}{2}$;

$$x_2=\sqrt{\frac{1}{2}\left(1+\cos\frac{\theta}{2}\right)}=\cos\frac{\theta}{2^2}, \text{ \&c}$$

$$\frac{\sqrt{1-x^2}}{x_1\ x_2\ x_3\ \ldots}=\frac{\sin\theta}{\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\ldots}=\theta \qquad \text{(Art 129)}$$

$$=\cos^{-1}x$$

99 Adding successive pairs of terms in the numerator the fraction becomes

$$\frac{2\sin x\,(\sin x+\sin 5x+\sin 9x-\sin 15x)}{4\,(1-\cos 2x)-(1-\cos 4x)}$$

$$=\frac{4\sin x\ \sin 3x\,(\cos 2x-\cos 12x)}{8\sin^2 x-2\sin^2 2x}$$

$$=\frac{8\sin x\sin 3x\sin 5x\sin 7x}{8\sin^2 x\,(1-\cos^2 x)}=\frac{\sin 3x}{\sin x}\cdot\frac{\sin 5x}{\sin x}\cdot\frac{\sin 7x}{\sin x}$$

The limit of this $=\frac{3x}{x}\cdot\frac{5x}{x}\cdot\frac{7x}{x}=3\cdot 5\cdot 7$.

100 $\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)=\sin^2\frac{\alpha}{2}-\sin^2\frac{\beta}{2}=-\frac{1}{2}(\cos\alpha-\cos\beta)$

Put $\cos\alpha=a$, $\cos\beta=b$, &c , hence the proposition to be proved is that

$$\Sigma\frac{a^m}{(a-b)(a-c)\ldots}=0,$$

where m is $< n-1$

By the method of partial fractions we have

$$\frac{x^{n-1}}{(1-ax)(1-bx)\ldots}=\Sigma\frac{1}{(a-b)(a-c)\ldots}\cdot\frac{1}{1-ax}$$

Expand both sides in powers of x by the binomial theorem, the coefficient of x^m on the right-hand side is

$$\Sigma\frac{a^m}{(a-b)(a-c)\ldots}.$$

On the left-hand side the coefficient of x^m is zero when m is $<n-1$, hence for such values of m

$$\Sigma\frac{a^m}{(a-b)(a-c)\ldots}=0$$

101 For A, B, C write $\frac{1}{2}\pi-\theta$, $\frac{1}{2}\pi-\phi$, $\frac{1}{2}\pi-\psi$ respectively, then

$$\cos A\cos B\cos C=\sin\theta\sin\phi\sin\psi,$$

where
$$\theta+\phi+\psi=\frac{3\pi}{2}-A-B-C=\frac{\pi}{2}.$$

By Art 272 (2) the greatest value of $\sin\theta\sin\phi\sin\psi$ is given by

$$\theta=\phi=\psi=\frac{1}{3}\cdot\frac{\pi}{2},$$

and is therefore $\left(\frac{1}{2}\right)^3$

Hence the least value of $1-8\sin\theta\sin\phi\sin\psi$ is $1-8\cdot\frac{1}{8}$ or zero

Hence $1-8\cos A\cos B\cos C$ cannot be negative

102 By Art. 252,

$$S=\frac{abc}{4R}=2R^2\sin A\sin B\sin C,$$

where
$$A+B+C=\pi$$

The maximum value of this is given by

$$A=B=C=\frac{\pi}{3} \qquad \text{(Art 272, 2)}$$

the area of the greatest triangle which can be inscribed in the circle

$$=2R^2\left(\sin\frac{\pi}{3}\right)^3=2R^2\left(\frac{\sqrt{3}}{2}\right)^3=\frac{3\sqrt{3}}{4}R^2$$

103 By the exponential theorem (Art 144)

$$(1+x)^n=1+n\log_e(1+x)+\frac{n^2}{2}\{\log_e(1+x)\}^2+\frac{n^3}{6}\{\log_e(1+x)\}^3$$

By the Binomial theorem

$$(1+x)^n=1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3$$
$$+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}x^4+ \qquad (1)$$

The coefficient of n^2 in this expansion is

$$\frac{x^2}{1.2}-\frac{1+2}{1.2.3}x^3+\frac{1.2+1.3+2.3}{1.2.3.4}x^4-\frac{1.2.3+2.3.4+}{1.2.3.4}\cdot\frac{x^5}{5}+ \ldots,$$

this series must therefore be equal to

$$\frac{1}{2}\{\log_e(1+x)\}^2$$

The coefficient of n^3 in (1) is

$$\frac{1}{1\,2\,3}x^3-\frac{1+2+3}{1\,2.3}\,\frac{x^4}{4}+\frac{1\,2+1\,3+1\,4+2.4+\ldots}{1\,2\,3\,4}\cdot\frac{x^5}{5}-\ldots$$

$$=\frac{1}{1\,2}\cdot\frac{x^3}{3}-\left(\frac{1}{1\,2}+\frac{1}{1.3}+\frac{1}{2\,3}\right)\frac{x^4}{4}+\left(\frac{1}{1.2}+\frac{1}{3.4}+\ldots\right)\frac{x^5}{5}-\ldots\;;$$

this series must therefore be

$$\frac{1}{6}\{\log_e(1+x)\}^3$$

104 Put $u=\tan 3A\cot A=\dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}\cdot\dfrac{1}{\tan A}=\dfrac{3-\tan^2 A}{1-3\tan^2 A}$.

$$\tan^2 A=\frac{3-u}{1-3u}=\frac{1}{3}\;\frac{3-u}{\frac{1}{3}-u},$$

if A is real $\tan^2 A$ must be positive, hence $3-u$ and $\frac{1}{3}-u$ must have the same sign, this will not be the case when u lies between 3 and $\frac{1}{3}$, therefore $\tan 3A\cot A$ cannot lie between 3 and $\frac{1}{3}$.

105 Let $x=\tan\alpha$ and

$$\frac{1}{u}=\frac{\tan 3\alpha}{\tan^3\alpha}=\frac{3-x^2}{x^2(1-3x^2)}.$$

Then $$3x^4-(u+1)x^2+3u=0$$

Solving this equation we find

$$x^2=\frac{u+1}{6}\pm\frac{\sqrt{(u^2-34u+1)}}{36}.$$

Since x is to be real $u^2-34u+1$ must be positive

Now $$u^2-34u+1=(u-17)^2-288$$
$$=\{u-(17+12\sqrt{2})\}\{u-(17-12\sqrt{2})\},$$

u must not lie between $17+12\sqrt{2}$ and $17-12\sqrt{2}$,

$17+12\sqrt{2}$ is a minimum value of u,

and $17-12\sqrt{2}$ is a maximum value of u

$\dfrac{1}{17+12\sqrt{2}}$ or $17-12\sqrt{2}$ is a maximum value of $\dfrac{1}{u}$,

and $\dfrac{1}{17-12\sqrt{2}}$ or $17+12\sqrt{2}$ is a minimum value of $\dfrac{1}{u}$

106 Let the given value of $\theta+\phi$ be ω, then

$$(a\cos\theta+b\cos\phi)^2+(a\sin\theta-b\sin\phi)^2$$

$$=a^2(\cos^2\theta+\sin^2\theta)+b^2(\cos^2\phi+\sin^2\phi)+2ab(\cos\theta\cos\phi-\sin\theta\sin\phi)$$

$$=a^2+b^2+2ab\cos\omega=\text{a constant.}$$

$(a\cos\theta+b\cos\phi)^2$ is greatest when $a\sin\theta-b\sin\phi=0$

the maximum value of $a\cos\theta+b\cos\phi$ is

$$\sqrt{(a^2+b^2+2ab\cos\omega)}$$

107. $(a^2+b^2+c^2)(\sin^2\theta+\sin^2\phi+\sin^2\psi)=(a\sin\theta+b\sin\phi+c\sin\psi)^2$

$$+(b\sin\psi-c\sin\phi)^2+(c\sin\theta-a\sin\psi)^2+(a\sin\phi-b\sin\theta)^2.$$

The right-hand side is least when

$$b\sin\psi-c\sin\phi=0,$$

$$c\sin\theta-a\sin\psi=0,$$

$$a\sin\phi-b\sin\theta=0,$$

for $a\sin\theta+b\sin\phi+c\sin\psi$ is a given quantity

the minimum value of $\sin^2\theta+\sin^2\phi+\sin^2\psi$ is

$$\frac{a\sin\theta+b\sin\phi+c\sin\psi}{\sqrt{(a^2+b^2+c^2)}}.$$

108. $\cos(\alpha+\theta)+m\cos\theta=(\cos\alpha+m)\cos\theta+\sin\alpha\sin\theta$

By Art 272 (3) the greatest possible value of this is

$$\sqrt{\{(\cos\alpha+m)^2+\sin^2\alpha\}};$$

n^2 is not greater than $\quad(\cos\alpha+m)^2+\sin^2\alpha,$

or $\quad 1+2m\cos\alpha+m^2$

109. $u=\frac{1}{2}a(1+\cos 2\theta)+b\sin 2\theta+\frac{1}{2}c(1-2\cos 2\theta)$

$$=\frac{1}{2}(a+c)+\frac{1}{2}(a-c)\cos 2\theta+b\sin 2\theta$$

$$=\frac{1}{2}(a+c)+m\cos 2\theta+b\sin 2\theta$$

Since $\quad \tan\phi=\frac{m}{b},\ \sin\phi=\frac{m}{\sqrt{m^2+b^2}},\ \cos\phi=\frac{b}{\sqrt{m^2+b^2}},$

and therefore

$$u=\frac{1}{2}(a+c)+\sqrt{(m^2+b^2)}(\sin\phi\cos 2\theta+\cos\phi\sin 2\theta)$$

$$=\frac{1}{2}(a+c)+\sqrt{(m^2+b^2)}\sin(2\theta+\phi).$$

The greatest and least values of u are given by $\sin(2\theta+\phi)=\pm 1$, if these be u_1 and u_2,

$$u_1=\frac{1}{2}(a+c)+\sqrt{(m^2+b^2)},$$

$$u_2=\frac{1}{2}(a+c)-\sqrt{(m^2+b^2)},$$

$$u_1+u_2=a+c,$$

$$u_1u_2=\frac{1}{4}(a+c)^2-m^2-b^2=ac-b^2,$$

u_1, u_2 are roots of the equation

$$x^2-(a+c)x+ac-b^2=0,$$

or

$$(x-a)(x-c)=b^2$$

110 Put $y=\dfrac{a-b\cos\theta}{\sin\theta}=\dfrac{(a-b)\cos^2\frac{\theta}{2}+(a+b)\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$.

Multiply up and divide by $\cos^2\frac{\theta}{2}$,

hence

$$(a+b)\tan^2\frac{\theta}{2}-2y\tan\frac{\theta}{2}+a-b=0.$$

If the roots of this equation for $\tan\frac{\theta}{2}$ are real we must have y^2 greater than $(a-b)(a+b)$

the numerically least value of y is $\sqrt{(a^2-b^2)}$

111 By Art 272 (2) the greatest value of

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

is given by

$$\frac{A}{2}=\frac{B}{2}=\frac{C}{2}=30^\circ,$$

and is therefore $(\sin 30^\circ)^3$, or $\frac{1}{8}$. If all the angles be positive the least value is 0.

112 Let θ have any value between 0 and $\frac{\pi}{2}$; let h be a small positive quantity. We have then to shew that

$$\frac{\theta+h}{\sin(\theta+h)}-\frac{\theta}{\sin\theta}\text{ is positive}$$

The sign of this expression is the same as the sign of

$$(\theta+h)\sin\theta-\theta\sin(\theta+h),$$

i e. of $$\theta\sin\theta(1-\cos h)+h\sin\theta-\theta\cos\theta\sin h,$$

i e of $$\theta\sin\theta(1-\cos h)+\sin\theta\sin h\left(\frac{h}{\sin h}-\frac{\theta}{\tan\theta}\right)$$

Now $1-\cos h$ is positive; and $\frac{h}{\sin h}$ is greater than unity while $\frac{\theta}{\tan\theta}$ is less than unity, by Art 118, thus the expression is positive

Hence since $\frac{\theta}{\sin\theta}$ increases as θ increases from 0 to $\frac{\pi}{2}$ its greatest value is when $\theta=\frac{\pi}{2}$, namely $\frac{\pi}{2}$. Thus

$$\frac{\pi}{2}\text{ is greater than }\frac{\theta}{\sin\theta},$$

i e $$\frac{1}{2}\pi\sin\theta>\theta.$$

113 Let θ have any value between 0 and $\frac{1}{2}\pi$, let h be a small positive quantity We have then to shew that

$$\frac{\theta}{\tan\theta}-\frac{\theta+h}{\tan(\theta+h)}\text{ is positive.}$$

The sign of this expression is the same as the sign of

$$\theta\cos\theta\sin(\theta+h)-(\theta+h)\cos(\theta+h)\sin\theta,$$

i.e of $$\theta\sin h-h\cos(\theta+h)\sin\theta,$$

i e of $$\frac{\theta}{\sin\theta}-\frac{h}{\sin h}\cos(\theta+h)$$

But as we may suppose h less than θ, we know by Ex 112 that $\frac{\theta}{\sin\theta}$ is greater than $\frac{h}{\sin h}$, and therefore greater than $\frac{h}{\sin h}\cos(\theta+h)$

114 Let $AP=x=BQ=CR$ The sum of the areas APR, BPQ, CQR is

$$=\frac{1}{2}[x(b-x)\sin A+x(c-x)\sin B+x(a-x)\sin c]$$

$$=\frac{1}{4R}[ax(b-x)+bx(c-x)+cx(a-x)]$$

$$=\frac{a+b+c}{4R}\left[\frac{ab+bc+ca}{a+b+c}\;x-x^2\right]$$

$$=\frac{(a+b+c)\,\Delta}{abc}\left[\frac{(ab+bc+ca)^2}{4(a+b+c)^2}-\left(x-\frac{1}{2}\cdot\frac{ab+bc+ca}{a+b+c}\right)^2\right]$$

The greatest value of this expression occurs when

$$x=\frac{1}{2}\,\frac{ab+bc+ca}{a+b+c},$$

and is therefore

$$\frac{(ab+bc+ca)^2}{abc\,(a+b+c)}\cdot\frac{\Delta}{4}.$$

the least value of the area of the triangle PQR is

$$\Delta\left\{1-\frac{1}{4}\,\frac{(ab+bc+ca)^2}{abc\,(a+b+c)}\right\}$$

115
$$\sin\omega\cos\theta-\cos\omega\sin\theta=\sin\omega\cos\alpha,$$
$$\therefore\ \tan\omega\cos\theta-\sin\theta=\tan\omega\cos\alpha$$

If θ is so small that we may neglect powers of θ above the second, we obtain from this equation (Arts 120, 121)

$$\tan\omega\left(1-\frac{1}{2}\theta^2\right)-\theta=\tan\omega\cos\alpha,$$

$$\theta=\tan\omega-\tan\omega\cos\alpha-\frac{1}{2}\tan\omega\,.\,\theta^2$$

$$=2\tan\omega\sin^2\frac{\alpha}{2}-\frac{1}{2}\tan\omega\,.\,\theta^2$$

In the term involving θ^2 substitute for θ

$$2\tan\omega\sin^2\frac{\alpha}{2}-\frac{1}{2}\tan\omega\,.\,\theta^2,$$

$$\theta=2\tan\omega\sin^2\frac{\alpha}{2}-\frac{1}{2}\tan\omega\left\{2\tan\omega\sin^2\frac{\alpha}{2}-\frac{1}{2}\tan\omega\ \theta^2\right\}^2$$

$$=2\tan\omega\sin^2\frac{\alpha}{2}-2\tan^3\omega\sin^4\frac{\alpha}{2}+\text{other terms}$$

Since θ is small it follows from the given equation that α is small, therefore the other terms are small compared with those given Hence approximately

$$\theta=2\tan\omega\sin^2\frac{\alpha}{2}\left(1-\tan^2\omega\sin^2\frac{\alpha}{2}\right)$$

116 Since $y=x-e\sin x$,

$$\tan\frac{1}{2}y=\tan\left(\frac{1}{2}x-\frac{1}{2}e\sin x\right)=\frac{\tan\frac{1}{2}x-\tan\left(\frac{1}{2}e\sin x\right)}{1+\tan\frac{1}{2}x\tan\left(\frac{1}{2}e\sin x\right)}$$

Since higher powers of e than the second are not required we may write

$$\frac{1}{2}e\sin x\ \text{ for }\ \tan\left(\frac{1}{2}e\sin x\right),$$

$$\therefore\ \tan\frac{1}{2}y=\frac{\tan\frac{1}{2}x-\frac{1}{2}e\sin x}{1+\frac{1}{2}e\sin x\tan\frac{x}{2}}$$

$$=\frac{\tan\frac{1}{2}x\left(1-e\cos^2\frac{x}{2}\right)}{1+e\sin^2\frac{x}{2}}$$

$$=\tan\frac{1}{2}x\left(1-e\cos^2\frac{x}{2}\right)\left(1+e\sin^2\frac{x}{2}\right)^{-1}$$

$$=\tan\frac{1}{2}x\left(1-e\cos^2\frac{x}{2}\right)\left(1-e\sin^2\frac{x}{2}+e^2\sin^4\frac{x}{2}-\quad\right)$$

$$=\tan\frac{1}{2}x\left(1-e\cos^2\frac{x}{2}-e\sin^2\frac{x}{2}+e^2\sin^2\frac{x}{2}\cos^2\frac{x}{2}+e^2\sin^4\frac{x}{2}+\quad\right)$$

$$=\tan\frac{1}{2}x\left(1-e+e^2\sin^2\frac{x}{2}\right),$$

neglecting higher powers of e.

117 Let D denote the point of contact of the circle with BC. Let AC intersect the circumference of the circle at E, and let AB intersect the circumference at F. Then the four straight lines AE, ED, DF, FA can be measured. Then, by Art 255, the diagonal AD can be determined

Then all the angles of the triangles ADE and ADF can be found, and thus the angles of the triangles ADC and ADB are known Thus DC and BD can be found See Euclid III 32

118 Let D be the point on AC produced through C such that the angle ADB is half the angle ACB, then $CD=CB$. Thus CB is known Again, let E be the point on BC produced through C such that the angle AEB is half the angle ACB, then $CE=CA$ Thus CA is known Then in the triangle ACB we know AC, and CB, and the angle ACB, thus AB can be found by Art 215

119 Let x denote the height of the balloon, and a, b, c the sides of the triangle ABC Let O be the point in the plane of ABC which is vertically under the balloon Then

$AO=x\cot 45^\circ=x$, $BO=x\cot 45^\circ=x$, $CO=x\cot 60^\circ=\frac{x}{\sqrt{3}}$ Therefore

$$\cos ACO=\frac{b^2+\frac{x^2}{3}-x^2}{\frac{2bx}{\sqrt{3}}}=\frac{3b^2-2x^2}{2bx\sqrt{3}},\quad \cos BCO=\frac{a^2+\frac{x^2}{3}-x^2}{\frac{2ax}{\sqrt{3}}}=\frac{3a^2-2x^2}{2ax\sqrt{3}}.$$

But ACB is a right angle, and therefore $\cos BCO=\sin ACO$; thus

$$\left(\frac{3b^2-2x^2}{2bx\sqrt{3}}\right)^2+\left(\frac{3a^2-2x^2}{2ax\sqrt{3}}\right)^2=1;$$

therefore $\qquad a^2(3b^2-2x^2)^2+b^2(3a^2-2x^2)^2=12a^2b^2x^2;$

therefore $\qquad 4x^4(a^2+b^2)-36a^2b^2x^2+9a^2b^2(a^2+b^2)=0,$

therefore $\qquad 4c^2x^4-36a^2b^2x^2+9a^2b^2c^2=0.$

120 Here the angle BAC – the angle BOC = the sum of the angles ABO and ACO

Now $\dfrac{\sin ACO}{\sin AOC} = \dfrac{AO}{AC}$, therefore $\sin ACO = \dfrac{n \sin\beta}{c}$, and since ACO is very small the circular measure of it is nearly equal to the sine, so that it is nearly equal to $\dfrac{n\sin\beta}{c}$

Again $\dfrac{\sin ABO}{\sin AOB} = \dfrac{AO}{AB}$, therefore $\sin ABO = \dfrac{n\sin(\alpha-\beta)}{b}$, therefore the circular measure of ABO is nearly equal to $\dfrac{n\sin(\alpha-\beta)}{b}$

Thus the circular measure of $BAC - BOC$ is nearly $n\left\{\dfrac{\sin(\alpha-\beta)}{b} + \dfrac{\sin\beta}{c}\right\}$

121 If the distance is 50 feet and the elevation is $\dfrac{\pi}{4}$, the height in feet is $50\tan\dfrac{\pi}{4}$, that is 50

But suppose the distance to be $50+h$, and the elevation to be $\dfrac{\pi}{4}+\alpha$ Then the height is $(50+h)\tan\left(\dfrac{\pi}{4}+\alpha\right)$ If α is very small this is very nearly equal to $(50+h)\left(\tan\dfrac{\pi}{4}+\alpha\sec^2\dfrac{\pi}{4}\right)$, by Art 188, that is $(50+h)(1+2\alpha)$

If h is also very small this is very nearly $50+h+100\alpha$ Now suppose $h=\dfrac{1}{12}$ and $\alpha=\dfrac{\pi}{180\times 60}$, then we obtain $50+\dfrac{1}{12}+\dfrac{\pi}{108}$. Thus the difference between this and the former value is $\dfrac{1}{12}+\dfrac{\pi}{108}$, that is about $\dfrac{1}{12}+\dfrac{1}{36}$, that is, $1\frac{1}{3}$ inches

122 Suppose that the tower and the spire each subtend the angle α

Then $$\tan\alpha = \frac{b}{a}, \quad \text{and } \tan 2\alpha = \frac{b+c}{a}$$

Therefore $$\frac{b+c}{a} = \frac{\dfrac{2b}{a}}{1-\dfrac{b^2}{a^2}} = \frac{2ab}{a^2-b^2};$$

therefore $b+c=\dfrac{2a^2b}{a^2-b^2}$, therefore $c=\dfrac{2a^2b}{a^2-b^2}-b=\dfrac{(a^2+b^2)b}{a^2-b^2}$

If however the height of the tower is $b+\beta$, and the height of the spire is $c+\gamma$, we have

$$c+\gamma=\frac{a^2(b+\beta)+(b+\beta)^3}{a^2-(b+\beta)^2}.$$

Hence, by subtraction,

$$\gamma=\frac{a^2(b+\beta)+(b+\beta)^3}{a^2-(b+\beta)^2}-\frac{(a^2+b^2)b}{a^2-b^2}.$$

Now $$(b+\beta)^2=b^2+2b\beta+\beta^2,$$
and if β is very small this is very nearly $b^2+2b\beta$

And $$(b+\beta)^3=b^3+3b^2\beta+3b\beta^2+\beta^3,$$
and if β is very small this is very nearly $b^3+3b^2\beta$

Thus $$\gamma=\frac{a^2b+b^3+(a^2+3b^2)\beta}{a^2-b^2-2b\beta}-\frac{a^2b+b^3}{a^2-b^2}$$

$$=\frac{(a^2+3b^2)(a^2-b^2)+2(a^2+b^2)b^2}{(a^2-b^2-2b\beta)(a^2-b^2)}\beta$$

$$=\frac{a^4+4a^2b^2-b^4}{(a^2-b^2)(a^2-b^2-2b\beta)}\beta$$

Therefore $$\frac{\gamma}{c}=\frac{(a^4+4a^2b^2-b^4)\beta}{(a^2-b^2)(a^2-b^2-2b\beta)}\cdot\frac{a^2-b^2}{(a^2+b^2)b}$$

$$=\frac{\beta}{b}\cdot\frac{a^4+4a^2b^2-b^4}{(a^2+b^2)(a^2-b^2-2b\beta)}$$

But when β is very small we may put a^2-b^2 for $a^2-b^2-2b\beta$; and thus

$$\frac{\gamma}{c}=\frac{\beta}{b}\cdot\frac{a^4+4a^2b^2-b^4}{a^4-b^4}.$$

123 We have $$a^2=b^2+c^2-2bc\cos A;$$
suppose that b is changed to $b+\beta$, and c to $c+\gamma$, thus

$$a^2=(b+\beta)^2+(c+\gamma)^2-2(b+\beta)(c+\gamma)\cos A.$$

Therefore, by subtraction,

$$2b\beta+\beta^2+2c\gamma+\gamma^2-2(b\gamma+c\beta+\beta\gamma)\cos A=0$$

If β and γ are very small this becomes very nearly

$$2b\beta+2c\gamma-2(b\gamma+c\beta)\cos A=0;$$

therefore $$\beta(b-c\cos A)+\gamma(c-b\cos A)=0,$$

therefore $$\beta a\cos C+\gamma a\cos B=0, \text{ by Art. 216}$$

Therefore $$\frac{\beta}{\cos B}+\frac{\gamma}{\cos C}=0,$$

therefore $$\beta\sec B+\gamma\sec C=0$$

124 Suppose h the height of the tower, r the radius, x the distance of the first place of observation from the centre Then

$$\frac{x}{r}=\operatorname{cosec}\frac{\beta}{2},\qquad \frac{x-a}{r}=\operatorname{cosec}\frac{\beta'}{2},$$

$$h=x\tan\alpha,\qquad h=(x-a)\tan\alpha'.$$

Hence $$\frac{a}{r} = \operatorname{cosec}\frac{\beta}{2} - \operatorname{cosec}\frac{\beta'}{2}$$

This finds r

Also $$h = x \tan\alpha' - a \tan\alpha' = \frac{h \tan\alpha'}{\tan\alpha} - a \tan\alpha',$$

therefore $$h = \frac{a \tan\alpha \tan\alpha'}{\tan\alpha' - \tan\alpha}.$$

This finds h

Again, from the first and second equations,

$$\frac{x-a}{x} = \frac{\operatorname{cosec}\frac{\beta'}{2}}{\operatorname{cosec}\frac{\beta}{2}}.$$

And from the third and fourth equations,

$$\frac{x-a}{x} = \frac{\cot\alpha'}{\cot\alpha}.$$

Therefore $$\frac{\operatorname{cosec}\frac{\beta'}{2}}{\operatorname{cosec}\frac{\beta}{2}} = \frac{\cot\alpha'}{\cot\alpha}.$$

125 We have $$\frac{a}{r} = \operatorname{cosec}\frac{\beta}{2} - \operatorname{cosec}\frac{\beta'}{2} \qquad (1)$$

If we suppose an error δ of the *same* sign to be made in β and β' these errors will tend to compensate each other, the greatest possible error in r will be determined by supposing that errors of *opposite* signs are made in β and β' Suppose then that instead of β we ought to have $\beta - \delta$, and instead of β' we ought to have $\beta' + \delta$ Then we have

$$\frac{a}{r-\rho} = \operatorname{cosec}\frac{\beta-\delta}{2} - \operatorname{cosec}\frac{\beta'+\delta}{2}$$

Hence, by subtraction, $\frac{a}{r-\rho} - \frac{a}{r}$, that is $\frac{a\rho}{r(r-\rho)}$

$$= \operatorname{cosec}\frac{\beta-\delta}{2} - \operatorname{cosec}\frac{\beta}{2} - \left\{\operatorname{cosec}\frac{\beta'+\delta}{2} - \operatorname{cosec}\frac{\beta'}{2}\right\}$$

Therefore, if δ and ρ be very small, we obtain

$$\frac{a\rho}{r^2} = \frac{\delta}{2}\left\{\frac{\cos\frac{\beta}{2}}{\sin^2\frac{\beta}{2}} + \frac{\cos\frac{\beta'}{2}}{\sin^2\frac{\beta'}{2}}\right\},$$ see Art 194

Thus
$$\frac{a\rho}{r^2}=\frac{\delta}{2}\,\frac{\cos\frac{\beta}{2}\left(1-\cos^2\frac{\beta'}{2}\right)+\cos\frac{\beta'}{2}\left(1-\cos^2\frac{\beta}{2}\right)}{\sin^2\frac{\beta'}{2}\sin^2\frac{\beta}{2}}$$

$$=\frac{\delta}{2}\,\frac{\left(\cos\frac{\beta}{2}+\cos\frac{\beta'}{2}\right)\left(1-\cos\frac{\beta}{2}\cos\frac{\beta'}{2}\right)}{\sin^2\frac{\beta}{2}\sin^2\frac{\beta'}{2}}$$

$$=\frac{\delta\cos\frac{\beta+\beta'}{4}\cos\frac{\beta'-\beta}{4}\left(1-\cos\frac{\beta}{2}\cos\frac{\beta'}{2}\right)}{\sin^2\frac{\beta}{2}\sin^2\frac{\beta'}{2}} \qquad (2)$$

Now (1) may be put in the form

$$\frac{a}{r}=\frac{\sin\frac{\beta'}{2}-\sin\frac{\beta}{2}}{\sin\frac{\beta'}{2}\sin\frac{\beta}{2}}=\frac{2\sin\frac{\beta'-\beta}{4}\cos\frac{\beta'+\beta}{4}}{\sin\frac{\beta'}{2}\sin\frac{\beta}{2}} \qquad (3)$$

Divide (2) by (3), then

$$\frac{\rho}{r}=\frac{\delta}{2}\cot\frac{1}{4}(\beta'-\beta)\,.\,\frac{1-\cos\frac{\beta}{2}\cos\frac{\beta'}{2}}{\sin\frac{\beta'}{2}\sin\frac{\beta}{2}}$$

$$=\frac{\delta}{2}\cot\frac{1}{4}(\beta'-\beta)\left\{\operatorname{cosec}\frac{\beta'}{2}\operatorname{cosec}\frac{\beta}{2}-\cot\frac{\beta'}{2}\cot\frac{\beta}{2}\right\}$$

If $\beta=60^\circ$ and $\beta'=120^\circ$, we obtain for $\frac{2\rho}{r}$ the value

$$\cot 15^\circ\{\operatorname{cosec}30^\circ\operatorname{cosec}60^\circ-\cot 30^\circ\cot 60^\circ\}\,\delta,$$

that is
$$(2+\sqrt{3})\left(\frac{4}{\sqrt{3}}-1\right)\delta, \quad \text{that is } \frac{5+2\sqrt{3}}{\sqrt{3}}\,\delta.$$

Put for δ the circular measure of $6'$, that is $\frac{\pi}{1800}$

Hence $\frac{2\rho}{r}=\frac{5+2\sqrt{3}}{\sqrt{3}}\times\frac{\pi}{1800}$, therefore $\frac{\rho}{r}=\frac{5+2\sqrt{3}}{\sqrt{3}}\times\frac{\pi}{3600}$

126 Let β denote the angle PSQ, and the equal angle QSR, and let ϕ denote the angle SQR

Then
$$\frac{PQ}{SQ}=\frac{\sin PSQ}{\sin SPQ}=\frac{\sin\beta}{\sin(\phi-\beta)},$$

and
$$\frac{QR}{SQ}=\frac{\sin QSR}{\sin SRQ}=\frac{\sin\beta}{\sin(\phi+\beta)},$$

therefore
$$\frac{PQ}{QR}=\frac{\sin(\phi+\beta)}{\sin(\phi-\beta)}$$

Let $PQ=a$, and $QR=b$, thus

$$a\sin(\phi-\beta)=b\sin(\phi+\beta),$$

therefore $a(\sin\phi\cos\beta-\cos\phi\sin\beta)=b(\sin\phi\cos\beta+\cos\phi\sin\beta)$;

therefore
$$\tan\phi=\frac{(a+b)\sin\beta}{(a-b)\cos\beta}=\frac{a+b}{a-b}\tan\beta$$

Also $\dfrac{1}{SQ}=\dfrac{\sin\beta}{a\sin(\phi-\beta)}$, and $\dfrac{1}{SQ}=\dfrac{\sin\beta}{b\sin(\phi+\beta)}$,

therefore
$$\frac{1}{SQ^2}=\frac{\sin^2\beta}{ab\sin(\phi-\beta)\sin(\phi+\beta)}=\frac{\sin^2\beta}{ab(\sin^2\phi-\sin^2\beta)}$$
$$=\frac{\tan^2\beta}{ab\{\sin^2\phi(1+\tan^2\beta)-\tan^2\beta\}}$$

But $\sin^2\phi=\dfrac{(a+b)^2\tan^2\beta}{(a-b)^2+(a+b)^2\tan^2\beta}$,

thus
$$\frac{1}{SQ^2}=\frac{(a-b)^2+(a+b)^2\tan^2\beta}{ab\{(a+b)^2-(a-b)^2\}}=\frac{(a-b)^2+(a+b)^2\tan^2\beta}{4a^2b^2}$$

Suppose that instead of β we ought to have $\beta+\alpha$, and instead of SQ we ought to have $SQ+c$, where α and c are very small Then

$$\frac{1}{(SQ+c)^2}=\frac{(a-b)^2}{4a^2b^2}+\frac{(a+b)^2}{4a^2b^2}\tan^2(\beta+\alpha)$$

Hence, by subtraction,

$$\frac{1}{(SQ+c)^2}-\frac{1}{SQ^2}=\frac{(a+b)^2}{4a^2b^2}\{\tan^2(\beta+\alpha)-\tan^2\beta\},$$

therefore
$$\frac{SQ^2-(SQ+c)^2}{SQ^2(SQ+c)^2}=\frac{(a+b)^2}{4a^2b^2}\{(\tan\beta+\alpha\sec^2\beta)^2-\tan^2\beta\},$$

approximately, by Art 188

Thus
$$-\frac{2c}{SQ^3}=\frac{(a+b)^2}{4a^2b^2}\,2\tan\beta\sec^2\beta\,\alpha,\text{ nearly;}$$

therefore
$$\frac{c}{SQ^3}=-\frac{(a+b)^2}{4a^2b^2}\frac{\sin\beta}{\cos^3\beta}\alpha,\text{ nearly.}$$

127 Let P be the top of the tower, PN the perpendicular to the ground, F the starting point, EFN the perpendicular to the course of the river, F and E being on opposite banks, D the point that he reaches, near E. Let $PN=2x$, then since $\tan PFN=2$, $NF=x$; hence $NE=a+x$. Since PN is perpendicular to ND and the angle

$$PDN=\cot^{-1}\frac{3}{2},$$

we have
$$ND=PN\cot PDN=3x.$$

Now
$$ND^2=NE^2+ED^2.$$

$$9x^2=(x+a)^2+c^2$$

$$\therefore\quad x-a=(9x^2-c^2)^{\frac{1}{2}}=3x\left(1-\frac{c^2}{9x^2}\right)^{\frac{1}{2}}$$

$$=3x\left(1-\frac{c^2}{18x^2}+\quad\right)$$

$$=3x-\frac{c^2}{6x}+\ldots;$$

$$2x=a+\frac{c^2}{6}\cdot\frac{1}{x}+$$

$$=a+\frac{c^2}{6}\left(\frac{a}{2}+\frac{c^2}{6}\cdot\frac{1}{x}+\quad\right)^{-1}$$

$$=a+\frac{c^2}{6}\cdot\frac{2}{a}\left(1-\frac{c^2}{3ax}-.\quad\right)$$

$$=a+\frac{c^2}{3a},\text{ approximately}$$

128 Let N be the foot of the tower, h the height. Then

$$BN=h\cot\beta,\quad AN=h\cot\alpha,$$

$$c=BN-AN=h(\cot\beta-\cot\alpha)=h\,\frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta},$$

$$h=\frac{c\sin\alpha\sin\beta}{\sin(\alpha-\beta)}$$

If BA is not in the same straight line as N, produce BA to X; let

$$\angle NAX=\theta,\quad NBX=\phi$$

Then
$$h\cot\beta=BN=c\cos\phi+AN\cos(\theta-\phi),$$

$$h\cot\alpha=AN;$$

$$h(\cot\beta-\cot\alpha)=c\cos\phi-AN\{1-\cos(\theta-\phi)\}$$

$$=c-\frac{1}{2}c\phi^2-\frac{1}{2}AN(\theta-\phi)^2\quad\ldots\quad(1),$$

approximately

Now from the triangle BAN,

$$\sin\phi=\frac{AN}{BN}\sin\theta,\quad \sin(\theta-\phi)=\frac{c}{BN}\cdot\sin\theta,$$

or since the angles are small,

$$\phi=\frac{AN}{BN}\ \theta,\quad \theta-\phi=\frac{c}{BN}\ \theta$$

Equation (i) becomes therefore

$$h\frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta}=c-\frac{1}{2}c\ \frac{AN^2}{BN^2}\ \theta^2-\frac{1}{2}AN\cdot\frac{c^2}{BN^2}\cdot\theta^2$$

$$=c-\frac{AN\ c}{2}\ \theta^2\left\{\frac{AN+c}{BN^2}\right\}$$

$$=c-\frac{c}{2}\cdot\frac{AN}{BN}\ \theta^2$$

$$=c\left\{1-\frac{1}{2}\frac{\cot\alpha}{\cot\beta}\theta^2\right\},$$

$$h=c\frac{\sin\alpha\sin\beta}{\sin(\alpha-\beta)}\left\{1-\frac{1}{2}\frac{\cot\alpha}{\cot\beta}\cdot\theta^2\right\}$$

the quantity to be subtracted is

$$c\ \frac{\cot\alpha}{\cot\beta}\ \frac{\theta^2}{2}\times\frac{\sin\alpha\sin\beta}{\sin(\alpha-\beta)},\ \text{or}\ \frac{c\cos\alpha\sin^2\beta}{\cos\beta\sin(\alpha-\beta)}\cdot\frac{\theta^2}{2}$$

129 Let P be the foot of the mountain; h the height Then

$$h=PA\tan\alpha=PB\tan\alpha=PC\tan\alpha$$

P is the centre of the circumcircle of the triangle ABC

$$PA=R=\frac{1}{2}a\operatorname{cosec}A,$$

$$h=\frac{1}{2}a\operatorname{cosec}A\tan\alpha$$

If there be an error then P is near the centre of the circumcircle, let this be O Then since there is no error in the altitude from A and B, PA is still $=PB$ PO is perpendicular to AB, let $PO=x$

$$h=PA\tan\alpha=PB\tan\alpha=PC\tan(\alpha+n'') \qquad \text{(i)}$$

Draw Pn perpendicular to CO. Then

$$CP=CO-On=R-x\cos POn=R-x\cos(B-A) \qquad \text{(ii)}$$

And $\quad AP=AO+x\cos APO=R+x\cos C$, approximately (iii).

Hence to determine x we have from (i), (ii) and (iii)

$$(R+x\cos C)\tan\alpha=(R-x\cos\overline{B-A})\tan(\alpha+n'')$$

$$R\{\tan(\alpha+n'')-\tan\alpha\}=x\{\cos C\tan\alpha+\cos(B-A)\tan(\alpha+n'')\},$$

$$R\frac{\sin n''}{\cos\alpha\cos(\alpha+n'')}=x\tan\alpha\{\cos C+\cos(B-A)\},\text{ nearly;}$$

$$=2x\tan\alpha\sin A\sin B$$

$$x=R\frac{\sin n''}{2\sin\alpha\cos(\alpha+n'')\sin A\sin B}$$

$$=R\cdot\frac{\sin n''}{\sin A\sin B\sin 2\alpha}$$

∴ from (i) and (iii) the true height

$$=R\tan\alpha\left\{1+\frac{\sin n''\cos C}{\sin A\sin B\sin 2\alpha}\right\}$$

130 Let the true value of A be $A+x$; of a, $a+\alpha$, of R, $R+\rho$

The area of the circle$=\pi(R+\rho)^2=\pi R^2+2\pi R\rho$

∴ the error in the area$=2\pi R\rho$,

$$(a+\alpha)^2=b^2+c^2-2bc\cos(A+x)$$

∴ approximately

$$a^2+2a\alpha=b^2+c^2-2bc\cos A\cos x+2bc\sin A\sin x$$

$$=b^2-c^2-2bc\cos A+2bc\,x\sin A$$

$$=a^2+2bcx\sin A$$

$$2a\alpha=2bcx\sin A$$

$$R+\rho=\frac{1}{2}\cdot\frac{a+\alpha}{\sin(A+x)}=\frac{1}{2}\frac{a+\alpha}{\sin A+x\cos A}$$

$$=\frac{1}{2}\frac{a}{\sin A}\left(1+\frac{\alpha}{a}\right)(1+x\cot A)^{-1}$$

$$=R\left(1+\frac{\alpha}{a}\right)(1-x\cot A)$$

$$=R\left(1+\frac{\alpha}{a}-x\cot A\right)$$

$$\rho=R\left(\frac{\alpha}{a}-x\cot A\right)$$

the error in the area $=2\pi R\rho$

$$=2\pi R^2\left(\frac{\alpha}{a}-x\cot A\right)$$

$$=2\pi R^2\left(\frac{bc\sin A}{a^2}-\cot A\right)x$$

$$=2\pi R^2\,\frac{1}{\sin A}(\sin B\sin C-\cos A)\,x$$

$$=2\pi\;\frac{1}{4}\cdot\frac{bc}{\sin B\sin C}\quad\frac{1}{\sin A}\cdot\cos B\cos C\,.\,x$$

$$=\frac{1}{2}\pi bcx\cot B\cot C\operatorname{cosec} A$$

131 Let A be the point of observation, C the cloud, C' its image, B the point where CC' cuts the surface of the lake, AN the perpendicular from A on CC' Let $CN=x$, $NB=h$, then $BC'=x+h$ Now

$$\frac{x}{\tan\alpha}=AN=\frac{x+2h}{\tan\beta},$$

$$x=\frac{2h\tan\alpha}{\tan\beta-\tan\alpha}=\frac{2h\sin\alpha\cos\beta}{\sin(\beta-\alpha)}$$

If $x+X$ be the real value of CN, $\alpha+\delta$ the real value of CAN, then

$$x+X=\frac{2h\cos\beta\sin(\alpha+\delta)}{\sin(\beta-\alpha-\delta)}$$

$$=\frac{2h\cos\beta(\sin\alpha+\delta\cos\alpha)}{\sin(\beta-\alpha)-\delta\cos(\beta-\alpha)}$$

$$=\frac{2h\cos\beta\sin\alpha}{\sin(\beta-\alpha)}(1+\delta\cot\alpha)(1-\delta\cot\overline{\beta-\alpha})^{-1},$$

$$X=\frac{2h\cos\beta\sin\alpha}{\sin(\beta-\alpha)}\quad\delta\{\cot\alpha+\cot(\beta-\alpha)\}$$

$$=\frac{2h\cos\beta\sin\alpha}{\sin(\beta-\alpha)}\quad\frac{\delta\sin\beta}{\sin\alpha\sin(\beta-\alpha)}$$

$$=h\delta\sin 2\beta\operatorname{cosec}^2(\beta-\alpha)$$

132 Let a, b, c, Δ be the real values of the sides, &c. of the triangle, and let $a+u$, $b+v$, $c+w$ be the measured lengths, $\Delta+\delta$ the area calculated therefrom Then

$$16\Delta^2=2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4,$$

and $$16(\Delta+\delta)^2=2(b+v)^2(c+w)^2+\;\ldots-(a+u)^4-$$

Multiply out omitting all terms containing u, v, w of degree higher than the first

$$\therefore\ 16(\Delta^2+2\Delta\delta)=2(b^2+2bv)(c^2+2cw)+\ \ -(a^4+4a^3u)-$$
$$=2b^2c^2+4bc^2v+4b^2cw+\ \ -a^4-4a^3u-$$
$$=16\Delta^2+4au(b^2+c^2-a^2)+4bv(a^2-b^2+c^2)+4cw(a^2+b^2-c^2)$$
$$\therefore\ 32\Delta\delta=8abc(u\cos A+v\cos B+w\cos C),$$
$$\delta=R(u\cos A+v\cos B+w\cos C)$$

If the triangle is acute-angled $\cos A$, $\cos B$, $\cos C$ are positive, and the greatest positive value of δ is found by taking the greatest positive values of u, v, w, namely x In this case therefore

$$\delta=Rx(\cos A+\cos B+\cos C)$$
$$=(R+r)x \qquad \text{[Ch. XVI Ex 27]}$$

If the triangle is obtuse-angled, A being the obtuse angle, then $\cos A$ is negative The greatest positive value of δ will therefore be given by

$$u=-x,\ v=x,\ w=x$$

In this case
$$\delta=Rx(-\cos A+\cos B+\cos C)$$
$$=Rx\left(4\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}-1\right)$$
$$=Rx\left(\frac{r_1}{R}-1\right) \qquad \text{[Ch. XVI Ex. 21]}$$
$$=x(r_1-R).$$

XIX

1. $\{\cos 4A+\sqrt{(-1)}\sin 4A\}^{\frac{1}{2}}=\pm\{\cos 2A+\sqrt{(-1)}\sin 2A\}$ by Art 267.

2 $-1=\cos\pi=\cos\pi+\sqrt{(-1)}\sin\pi$,

therefore one value of $(-1)^{\frac{1}{3}}=\cos\frac{\pi}{3}+\sqrt{(-1)}\sin\frac{\pi}{3}$,

so we may put $-1=\cos 3\pi$, or $\cos 5\pi$, and thus we obtain two other values for $(-1)^{\frac{1}{3}}$, namely,

$$\cos\frac{3\pi}{3}+\sqrt{(-1)}\sin\frac{3\pi}{3}, \quad \text{that is } -1,$$

and
$$\cos\frac{5\pi}{3}+\sqrt{(-1)}\sin\frac{5\pi}{3}.$$

3 We may put $-1 = \cos\pi$, or $\cos 3\pi$, or $\cos 5\pi$, or $\cos 7\pi$, or $\cos 9\pi$, or $\cos 11\pi$, and thus

$$-1 = \cos\theta + \sqrt{(-1)}\sin\theta$$

where $\theta = \pi$, or 3π, or 5π, or 7π, or 9π, or 11π

Hence the six values of $(-1)^{\frac{1}{6}}$ are contained in

$$\cos\frac{\theta}{6} + \sqrt{(-1)}\sin\frac{\theta}{6},$$

where θ has any of the six values just specified

4
$$1 + \sqrt{(-1)} = \sqrt{2}\left\{\frac{1}{\sqrt{2}} + \frac{\sqrt{(-1)}}{\sqrt{2}}\right\}$$
$$= \sqrt{2}\{\cos\theta + \sqrt{(-1)}\sin\theta\},$$

where for θ we may put $\frac{\pi}{4} + 2n\pi$, where n is any integer

Therefore $$\{1 + \sqrt{(-1)}\}^{\frac{1}{3}} = 2^{\frac{1}{6}}\left\{\cos\frac{\theta}{3} + \sqrt{(-1)}\sin\frac{\theta}{3}\right\},$$

and the three values will be obtained by putting for θ in succession $\frac{\pi}{4}$, $2\pi + \frac{\pi}{4}$, and $4\pi + \frac{\pi}{4}$.

5 Since $\frac{\sin\theta}{\theta}$ is given nearly equal to unity, we may infer that θ is a small angle. Hence we have approximately, by Art 286,

$$\sin\theta = \theta - \frac{\theta^3}{6},$$

thus $$1 - \frac{\theta^2}{6} = \frac{2165}{2166},$$

therefore $$\frac{\theta^2}{6} = \frac{1}{2166},$$

therefore $$\theta^2 = \frac{1}{361},$$

therefore $$\theta = \frac{1}{19}$$

This is the circular measure of the angle, therefore the number of degrees $= \frac{1}{19}$ of $\frac{180}{\pi} = \frac{1}{19}$ of $57^\circ.29 \ldots = 3^\circ$ approximately.

6 $\sin\left(\frac{\pi}{6}+\theta\right)=\cdot 51$

As $\cdot 51$ is very nearly equal to $\sin\frac{\pi}{6}$ we may infer that θ is very small

We have

$$\sin\frac{\pi}{6}\cos\theta+\cos\frac{\pi}{6}\sin\theta=\cdot 51,$$

therefore $$\frac{1}{2}\left(1-\frac{\theta^2}{2}\right)+\frac{\sqrt{3}}{2}\theta=\cdot 51 \text{ approximately.}$$

Hence, neglecting θ^2, we have $\frac{\sqrt{3}}{2}\theta=\frac{1}{100}$, and therefore $\theta=\frac{1}{50\sqrt{3}}$

Then if we retain the term in θ^2 we have

$$\theta=\frac{1}{50\sqrt{3}}+\frac{\theta^2}{2\sqrt{3}};$$

and putting for θ^2 its approximate value, we have for a closer approximation

$$\theta=\frac{1}{50\sqrt{3}}+\frac{1}{2\sqrt{3}}\left(\frac{1}{50\sqrt{3}}\right)^2$$

$$=\frac{1}{50\sqrt{3}}+\frac{1}{15000\sqrt{3}}.$$

The same result will be obtained if we solve the quadratic equation $\theta=\frac{1}{50\sqrt{3}}+\frac{\theta^2}{2\sqrt{3}}$ in the usual way, select the least root, and take its approximate value See *Algebra*, Art 526, Example (3)

7 $$x^{12}=1=\cos 2\kappa\pi\pm\iota\sin 2\kappa\pi,$$

$$x=(\cos 2\kappa\pi\pm\iota\sin 2\kappa\pi)^{\frac{1}{12}}=\cos\frac{\kappa\pi}{6}\pm\iota\sin\frac{\kappa\pi}{6} \quad (1).$$

The roots of $x^4+x^2+1=0$ are the roots of $x^6-1=0$, except $x=\pm 1$

That is they are given by

$$x=\cos\frac{2m\pi}{6}\pm\iota\sin\frac{2m\pi}{6},$$

where m has the values 1 and 2

Hence the roots required are the values of the expression (1) when $\kappa=2$ or 4

8 As in Art 279, $x_1\ x_2\ x_3 \ldots$ ad inf

$$=\left(\cos\frac{\pi}{2}+\iota\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{4}+\iota\sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{8}+\iota\sin\frac{\pi}{8}\right)\ldots$$

$$=\cos\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\right)\pi+\iota\sin\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\right)\pi$$

$$=\cos\frac{\frac{1}{2}\pi}{1-\frac{1}{2}}+\iota\sin\frac{\frac{1}{2}\pi}{1-\frac{1}{2}}$$

$$=\cos\pi+\iota\sin\pi=-1$$

9 Put $y=x^2$ From the given equation

$$(y+1)^2=(2x\cos\theta)^2=4y\cos^2\theta;$$

$$y^2-2y(1-2\cos^2\theta)+1=0,$$

or

$$y^2-2y\cos 2\theta+1=0$$

The values of y given by this equation are the squares of the values of x found from the given equation

10 (i) $a+b=\cos 2\alpha+\cos 2\beta+\iota(\sin 2\alpha+\sin 2\beta)$

$$=2\cos(\alpha+\beta)\cos(\alpha-\beta)+2\iota\sin(\alpha+\beta)\cos(\alpha-\beta)$$

$$=2\cos(\alpha-\beta)\{\cos(\alpha+\beta)+\iota\sin(\alpha+\beta)\}$$

(ii) $a-b=\cos 2\alpha-\cos 2\beta+\iota(\sin 2\alpha-\sin 2\beta)$

$$=-2\sin(\alpha+\beta)\sin(\alpha-\beta)+2\iota\cos(\alpha+\beta)\sin(\alpha-\beta)$$

$$=2\iota\sin(\alpha-\beta)\sin\{\cos(\alpha+\beta)+\iota\sin(\alpha+\beta)\}$$

(iii) $ab+cd=(\cos 2\alpha+\iota\sin 2\alpha)(\cos 2\beta+\iota\sin 2\beta)$

$$+(\cos 2\gamma+\iota\sin 2\gamma)(\cos 2\delta+\iota\sin 2\delta)$$

$$=\cos(2\alpha+2\beta)+\iota\sin(2\alpha+2\beta)+\cos(2\gamma+2\delta)+\iota\sin(2\gamma+2\delta)$$

$$=2\cos(\alpha+\beta-\gamma-\delta)\{\cos(\alpha+\beta+\gamma+\delta)+\iota\sin(\alpha+\beta+\gamma+\delta)\},$$

as in (i)

(iv) from (i),

$$a+b=2\cos(\alpha-\beta)\{\cos(\alpha+\beta)+\iota\sin(\alpha+\beta)\},$$

$$c+d=2\cos(\gamma-\delta)\{\cos(\gamma+\delta)+\iota\sin(\gamma+\delta)\},$$

$(a+b)(c+d)$

$$=4\cos(\alpha-\beta)\cos(\gamma-\delta)\{\cos(\alpha+\beta)+\iota\sin(\alpha+\beta)\}\{\cos(\gamma+\delta)+\iota\sin(\gamma+\delta)\}$$

$$=4\cos(\alpha-\beta)\cos(\gamma-\delta)\{\cos(\alpha+\beta+\gamma+\delta)+\iota\sin(\alpha+\beta+\gamma+\delta)\}$$

11 (i) As in Art 279,

$$abcd = \cos(2\alpha+2\beta+2\gamma+2\delta) + \iota\sin(2\alpha+2\beta+2\gamma+2\delta),$$

$$(abcd)^{\frac{1}{2}} + (abcd)^{-\frac{1}{2}}$$

$$= \cos(\alpha+\beta+\gamma+\delta) + \iota\sin(\alpha+\beta+\gamma+\delta)$$
$$+ \cos(\alpha+\beta+\gamma+\delta) + \iota\sin(-\alpha-\beta-\gamma-\delta)$$
$$= 2\cos(\alpha+\beta+\gamma+\delta)$$

(ii) $(ab)^{\frac{1}{2}}(cd)^{-\frac{1}{2}} + (ab)^{-\frac{1}{2}}(cd)^{\frac{1}{2}}$

$$= \{\cos(\alpha+\beta) + \iota\sin(\alpha+\beta)\}\{\cos(-\gamma-\delta) + \iota\sin(-\gamma-\delta)\}$$
$$+ \{\cos(-\alpha-\beta) + \iota\sin(-\alpha-\beta)\}\{\cos(\gamma+\delta) + \iota\sin(\gamma+\delta)\}$$
$$= \cos(\alpha+\beta-\gamma-\delta) + \iota\sin(\alpha+\beta-\gamma-\delta)$$
$$+ \cos(-\alpha-\beta+\gamma+\delta) + \iota\sin(-\alpha-\beta+\gamma+\delta) \qquad \text{(Art 279)}$$
$$= 2\cos(\alpha+\beta-\gamma-\delta)$$

12 $$a+b = \cos 2\alpha + \cos 2\beta + \sqrt{(-1)}\{\sin 2\alpha + \sin 2\beta\}$$
$$= 2\cos(\alpha+\beta)\cos(\alpha-\beta) + 2\sqrt{(-1)}\sin(\alpha+\beta)\cos(\alpha-\beta)$$
$$= 2\cos(\alpha-\beta)\{\cos(\alpha+\beta) + \sqrt{(-1)}\sin(\alpha+\beta)\}$$

Thus $\dfrac{b}{a+b} = \dfrac{\cos 2\beta + \sqrt{(-1)}\sin 2\beta}{2\cos(\alpha-\beta)\{\cos(\alpha+\beta) + \sqrt{(-1)}\sin(\alpha+\beta)\}}$,

multiply both numerator and denominator by $\cos(\alpha+\beta) - \sqrt{(-1)}\sin(\alpha+\beta)$,

thus we get $$\frac{\cos(\beta-\alpha) + \sqrt{(-1)}\sin(\beta-\alpha)}{2\cos(\alpha-\beta)}$$

Similarly $\dfrac{c}{a+c} = \dfrac{\cos(\gamma-\alpha) + \sqrt{(-1)}\sin(\gamma-\alpha)}{2\cos(\alpha-\gamma)}$

Therefore $\dfrac{bc}{(a+b)(a+c)} = \dfrac{\cos(\beta+\gamma-2\alpha) + \sqrt{(-1)}\sin(\beta+\gamma-2\alpha)}{4\cos(\alpha-\beta)\cos(\alpha-\gamma)}$.

13 $$\{\cos\theta + \cos\phi + \sqrt{(-1)}(\sin\theta + \sin\phi)\}^n$$
$$= \left\{2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} + 2\sqrt{(-1)}\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\right\}^n$$
$$= 2^n\left(\cos\frac{\theta-\phi}{2}\right)^n\left\{\cos\frac{\theta+\phi}{2} + \sqrt{(-1)}\sin\frac{\theta+\phi}{2}\right\}^n$$
$$= 2^n\left(\cos\frac{\theta-\phi}{2}\right)^n\left\{\cos\frac{n}{2}(\theta+\phi) + \sqrt{(-1)}\sin\frac{n}{2}(\theta+\phi)\right\}$$

Similarly $\{\cos\theta+\cos\phi-\sqrt{(-1)}\,(\sin\theta+\sin\phi)\}^n$

$$=2^n\left(\cos\frac{\theta-\phi}{2}\right)^n\left\{\cos\frac{n}{2}(\theta+\phi)-\sqrt{(-1)}\sin\frac{n}{2}(\theta+\phi)\right\}$$

Hence by addition we get $2^{n+1}\left(\cos\frac{\theta-\phi}{2}\right)^n\cos\frac{n}{2}(\theta+\phi)$

14 Since $1-c^2=(nc-1)^2=n^2c^2-2nc+1,$

$$c(1+n^2)=2n$$

$$\frac{c}{2n}(1+nx)\left(1+\frac{n}{x}\right)=\frac{c}{2n}\left\{1+n^2+n\left(x+\frac{1}{x}\right)\right\}$$

$$=1+\frac{c}{2}\left(x+\frac{1}{x}\right)$$

$$=1+c\cos\theta$$

15 $x-b=\cos 2\theta+\sqrt{(-1)}\sin 2\theta-\cos 2\beta-\sqrt{(-1)}\sin 2\beta$

$$=2\sin(\beta-\theta)\{\sin(\beta+\theta)-\sqrt{(-1)}\cos(\beta+\theta)\}$$

$$=\frac{2\sin(\beta-\theta)}{\sqrt{(-1)}}\{\cos(\beta+\theta)+\sqrt{(-1)}\sin(\beta+\theta)\}$$

In like manner

$$a-b=\frac{2\sin(\beta-a)}{\sqrt{(-1)}}\{\cos(\beta+a)+\sqrt{(-1)}\sin(\beta+a)\}$$

Therefore $\dfrac{x-b}{a-b}=\dfrac{\sin(\beta-\theta)}{\sin(\beta-a)}\,\dfrac{\cos(\beta+\theta)+\sqrt{(-1)}\sin(\beta+\theta)}{\cos(\beta+a)+\sqrt{(-1)}\sin(\beta+a)}$, multiply both numerator and denominator by $\cos(\beta+a)-\sqrt{(-1)}\sin(\beta+a)$,

thus we get $\dfrac{\sin(\theta-\beta)}{\sin(a-\beta)}\{\cos(\theta-a)+\sqrt{(-1)}\sin(\theta-a)\}$

Similarly we transform $\dfrac{x-c}{a-c}$, and thus we obtain

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}=\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(a-\beta)\sin(a-\gamma)}\{\cos 2(\theta-a)+\sqrt{(-1)}\sin 2(\theta-a)\}$$

In like manner we transform $\dfrac{(x-c)(x-a)}{(b-c)(b-a)}$ and $\dfrac{(x-a)(x-b)}{(c-a)(c-b)}$ Then by equating to zero the coefficient of the imaginary part we obtain

$$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(a-\beta)\sin(a-\gamma)}\sin 2(\theta-a)+\frac{\sin(\theta-\gamma)\sin(\theta-a)}{\sin(\beta-\gamma)\sin(\beta-a)}\sin 2(\theta-\beta)$$

$$+\frac{\sin(\theta-a)\sin(\theta-\beta)}{\sin(\gamma-a)\sin(\gamma-\beta)}\sin 2(\theta-\gamma)=0$$

And then by equating the real parts we have

$$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}\cos 2(\theta-\alpha)+\frac{\sin(\theta-\gamma)\sin(\theta-\alpha)}{\sin(\beta-\gamma)\sin(\beta-\alpha)}\cos 2(\theta-\beta)$$
$$+\frac{\sin(\theta-\alpha)\sin(\theta-\beta)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}\cos 2(\theta-\gamma)=1$$

16. We have $\dfrac{1}{(x-a)(x-b)}=\dfrac{1}{(a-b)(x-a)}-\dfrac{1}{(a-b)(x-b)}$. (1)

Now $x-a=2\iota\sin(\theta-\alpha)\{\cos(\theta+\alpha)+\iota\sin(\theta+\alpha)\}$,

$x-b=2\iota\sin(\theta-\beta)\{\cos(\theta+\beta)+\iota\sin(\theta+\beta)\}$;

$$\frac{1}{(x-a)(x-b)}$$
$$=\frac{1}{-4\sin(\theta-\alpha)\sin(\theta-\beta)\{\cos(2\theta+\alpha+\beta)+\iota\sin(2\theta+\alpha+\beta)\}}$$
$$=\frac{\cos(2\theta+\alpha+\beta)-\iota\sin(2\theta+\alpha+\beta)}{-4\sin(\theta-\alpha)\sin(\theta-\beta)}$$

In the same way

$$\frac{1}{(a-b)(x-a)}=\frac{\cos(\theta+2\alpha+\beta)-\iota\sin(\theta+2\alpha+\beta)}{-4\sin(\alpha-\beta)\sin(\theta-\alpha)},$$

$$\frac{1}{(a-b)(x-b)}=\frac{\cos(\theta+\alpha+2\beta)-\iota\sin(\theta+\alpha+2\beta)}{-4\sin(\alpha-\beta)\sin(\theta-\beta)},$$

substituting these expressions in equation (1) and equating the real parts we have

$$\frac{\cos(2\theta+\alpha+\beta)}{\sin(\theta-\alpha)\sin(\theta-\beta)}=\frac{\cos(\theta+2\alpha+\beta)}{\sin(\theta-\alpha)\sin(\alpha-\beta)}-\frac{\cos(\theta+\alpha+2\beta)}{\sin(\theta-\beta)\sin(\alpha-\beta)}$$

17 For x write $\cos\alpha+\iota\sin\alpha$, for y write $\cos\beta+\iota\sin\beta$. Then

$$x^r y^s=(\cos\alpha+\iota\sin\alpha)^r(\cos\beta+\iota\sin\beta)^s$$
$$=(\cos r\alpha+\iota\sin r\alpha)(\cos s\beta+\iota\sin s\beta)$$
$$=\cos(r\alpha+s\beta)+\iota\sin(r\alpha+s\beta),$$

$$x^m+x^{m-1}y+\quad+y^m=\Sigma\cos(r\alpha+s\beta)+\iota\Sigma\sin(r\alpha+s\beta),$$

where r and s are positive integers, such that $r+s=m$

$$\Sigma\cos(r\alpha+s\beta)=\text{the real part of }\frac{x^{m+1}-y^{m+1}}{x-y}$$

Now

$$x^{m+1}-y^{m+1}=\cos(m+1)\alpha+\iota\sin(m+1)\alpha-\cos(m+1)\beta-\iota\sin(m+1)\beta$$

$$=2\iota\sin\frac{(m+1)(\alpha-\beta)}{2}\left\{\cos\frac{(m+1)(\alpha+\beta)}{2}+\iota\sin\frac{(m+1)(\alpha+\beta)}{2}\right\}$$

$$x-y=2\iota\sin\frac{\alpha-\beta}{2}\left\{\cos\frac{\alpha+\beta}{2}+\iota\sin\frac{\alpha+\beta}{2}\right\}$$

$$\frac{x^{m+1}-y^{m+1}}{x-y}=\frac{\sin\frac{1}{2}(m+1)(\alpha-\beta)}{\sin\frac{1}{2}(\alpha-\beta)}\left\{\cos\frac{m}{2}(\alpha+\beta)+\iota\sin\frac{m}{2}(\alpha+\beta)\right\}$$

Hence the sum required is

$$\frac{\sin\frac{1}{2}(m+1)(\alpha-\beta)}{\sin\frac{1}{2}(\alpha-\beta)}\cos\frac{m}{2}(\alpha+\beta)$$

18 $\left(a-\frac{1}{a}\right)^2=\left(a+\frac{1}{a}\right)^2-4=4\cos^2\alpha-4=-4\sin^2\alpha$

$$a-\frac{1}{a}=2\iota\sin\alpha$$

$$a=\cos\alpha+\iota\sin\alpha,\qquad \frac{1}{a}=\cos\alpha-\iota\sin\alpha,$$

and $$a^p=\cos p\alpha+\iota\sin p\alpha,\quad \frac{1}{a^p}=\cos p\alpha-\iota\sin p\alpha,$$

$$abc\quad+\frac{1}{abc}=(\cos\alpha+\iota\sin\alpha)(\cos\beta+\iota\sin\beta)$$
$$+(\cos\alpha-\iota\sin\alpha)(\cos\beta-\iota\sin\beta)$$
$$=\cos(\alpha+\beta+\gamma+\quad)+\iota\sin(\alpha+\beta+\gamma+\quad)$$
$$+\cos(\alpha+\beta+\gamma+\quad)-\iota\sin(\alpha+\beta+\gamma+\quad)$$
$$=2\cos(\alpha+\beta+\gamma+\quad)$$

$$a^pb^qc^r\quad+\frac{1}{a^pb^qc^r}=(\cos p\alpha+\iota\sin p\alpha)(\cos q\beta+\iota\sin q\beta)$$
$$+(\cos p\alpha-\iota\sin p\alpha)(\cos q\beta-\iota\sin q\beta)$$
$$=\cos(p\alpha+q\beta+\quad)+\iota\sin(p\alpha+q\beta+\quad)$$
$$+\cos(p\alpha+q\beta+\quad)-\iota\sin(p\alpha+q\beta+\quad)$$
$$=2\cos(p\alpha+q\beta+\quad)$$

19 $\sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$

$$=\frac{1}{2}\left\{\frac{2^2\theta^2}{\lfloor 2}-\frac{2^4\theta^4}{\lfloor 4}+\frac{2^6\theta^6}{\lfloor 6}-\quad\right\},\qquad\text{(Art 286)}$$

$$\cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$$

$$=\frac{1}{2}\left\{2-\frac{2^2\theta^2}{\lfloor 2}+\frac{2^4\theta^4}{\lfloor 4}-\quad\right\}$$

20 $4\sin^2\theta\cos\theta=2\sin\theta\sin 2\theta=\cos\theta-\cos 3\theta$

$$=1-\frac{\theta^2}{\lfloor 2}+\frac{\theta^4}{\lfloor 4}-\qquad+(-1)^n\cdot\frac{\theta^{2n}}{\lfloor 2n}+$$

$$-1+\frac{3^2\theta^2}{\lfloor 2}-\frac{3^4\theta^4}{\lfloor 4}+\quad\cdot-(-1)^n\cdot\frac{3^{2n}\theta^{2n}}{\lfloor 2n}-$$

$$=4\theta^2-\frac{20\theta^4}{6}+\qquad+(-1)^{n+1}\frac{3^{2n}-1}{\lfloor 2n}\cdot\theta^{2n}+$$

$$\sin^2\theta\cos\theta=\theta^2-\frac{5\theta^4}{6}+\qquad+(-1)^{n+1}\frac{3^{2n}-1}{4\lfloor 2n}\cdot\theta^{2n}+\quad.$$

21 $4\sin^3\theta=3\sin\theta-\sin 3\theta$

$$=3\left(\theta-\frac{\theta^3}{\lfloor 3}+\frac{\theta^5}{\lfloor 5}-\quad\right)-\left(3\theta-\frac{3^3\theta^3}{\lfloor 3}+\frac{3^5\theta^5}{\lfloor 5}-\quad\right)$$

$$=\frac{\theta^3}{\lfloor 3}(3^3-3)-\frac{\theta^5}{\lfloor 5}(3^5-3)+\frac{\theta^7}{\lfloor 7}(3^7-3)-$$

$$=3(3^2-1)\left\{\frac{\theta^3}{\lfloor 3}-\frac{\theta^5}{\lfloor 5}(3^2+1)+\frac{\theta^7}{\lfloor 7}(3^4+3^2+1)-\quad\right\};$$

$$\frac{1}{6}\sin^3\theta=\frac{\theta^3}{\lfloor 3}-\frac{\theta^5}{\lfloor 5}(3^2+1)+\frac{\theta^7}{\lfloor 7}(3^4+3^2+1)-$$

22 From Ex. 3, Ch. VIII we obtain

$$\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)$$

$$=-\frac{1}{4}\{\sin 2(\beta-\gamma)+\sin 2(\gamma-\alpha)+\sin 2(\alpha-\beta)\},$$

and from Art 286, $\sin 2(\beta-\gamma)$

$$=2(\beta-\gamma)-\frac{2^3(\beta-\gamma)^3}{\lfloor 3}+\quad\cdot+(-1)^n\frac{2^{2n+1}(\beta-\gamma)^{2n+1}}{\lfloor 2n+1}$$

Hence the term of the $(2n+1)^{\text{th}}$ degree is

$$-\frac{2^{2n-1}(-1)^n}{\lfloor\underline{2n+1}}\{(\beta-\gamma)^{2n+1}+(\gamma-\alpha)^{2n+1}+(\alpha-\beta)^{2n+1}\}$$

23 Solving the equation as a quadratic we have

$$x^5=\frac{\pm 5\sqrt{5}-11}{2}$$

$$\text{Now } \left(\frac{\pm\sqrt{5}-1}{2}\right)^5=\frac{\pm 25\sqrt{5}-5\;\,25\pm 10\;\,5\sqrt{5}-10\;\,5\pm 5\sqrt{5}-1}{2^5}$$

$$=\frac{\pm 80\sqrt{5}-176}{2\times 16}=\frac{\pm 5\sqrt{5}-11}{2};$$

$$x=\frac{\pm 5-1}{2}\,.\,\omega,$$

where ω is any one of the fifth roots of unity, that is

$$\omega=\cos\frac{2r\pi}{5}+\iota\sin\frac{2r\pi}{5},$$

or

$$\omega=\cos\frac{2r\pi}{5}\pm\iota\sin\frac{2r\pi}{5},$$

these expressions including the same set of values

24. $$x^{16}-2x^8+1=45x^8;$$

$$x^8\pm 3\sqrt{5}\;\,x^4=1$$

$$x^8\pm 3\sqrt{5}\;\,x^4+\left(\frac{3\sqrt{5}}{2}\right)^2=\frac{49}{4},$$

$$x^4=\frac{7\pm 3\sqrt{5}}{2}\times(\pm 1)$$

$$\text{Now } \left(\frac{1\pm\sqrt{5}}{2}\right)^4=\frac{1\pm 4\sqrt{5}+30\pm 20\sqrt{5}+25}{8\times 2}=\frac{7\pm 3\sqrt{5}}{2},$$

$$x=\frac{1\pm\sqrt{5}}{2}(\pm 1)^{\frac{1}{4}}$$

$$=\frac{1\pm\sqrt{5}}{2}(\cos r\pi\pm\iota\sin r\pi)^{\frac{1}{4}}$$

$$=\frac{1\pm\sqrt{5}}{2}\left(\cos\frac{r\pi}{4}\pm\iota\sin\frac{r\pi}{4}\right)$$

25 Multiply the second equation by $\sqrt{-1}$ and add; therefore

$$a(x^2-y^2+2\iota xy)+\iota\alpha(x^2-y^2+2\iota xy)+(b+\beta\iota)(x+y\iota)+c+\gamma\iota=0,$$

or
$$(a+\iota\alpha)(x+\iota y)^2+(b+\iota\beta)(x+\iota y)+c+\iota\gamma=0$$

From this quadratic equation we can find $x+\iota y$, and in the same way we can obtain a quadratic equation for $x-\iota y$

26 Suppose $$\tan x=a_1x+\frac{a_3x^3}{\lfloor 3}+\frac{a_5x^5}{\lfloor 5}+\ldots,$$

then $$\sin x=\cos x\left\{a_1x+\frac{a_3x^3}{\lfloor 3}+\frac{a_5x^5}{\lfloor 5}+\ldots\right\}.$$

Substitute for $\sin x$ and $\cos x$ by Art. 286, thus

$$x-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}-\frac{x^7}{\lfloor 7}+\ldots$$
$$=\left\{1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}-\frac{x^6}{\lfloor 6}+\ldots\right\}\left\{a_1x+\frac{a_3x^3}{\lfloor 3}+\frac{a_5x^5}{\lfloor 5}+\ldots\right\}$$

Then, according to the known principles of Algebra, we may equate the coefficient of any power of x on the left-hand side to the coefficient of the same power obtained by working out the product on the right-hand side Take, for instance, the coefficient of x^{2n+1}, thus we obtain

$$\frac{(-1)^n}{\lfloor 2n+1}=\frac{a_{2n+1}}{\lfloor 2n+1}-\frac{a_{2n-1}}{2\lfloor 2n-1}+\frac{a_{2n-3}}{\lfloor 4\lfloor 2n-3}\ldots+(-1)^n\frac{a_1}{\lfloor 2n}$$

Multiply by $\lfloor 2n+1$ and transpose, thus we get

$$a_{2n+1}=\frac{(2n+1)2n}{2}a_{2n-1}-\frac{(2n+1)2n(2n-1)(2n-2)}{\lfloor 4}a_{2n-3}$$
$$+\ldots+(2n+1)(-1)^{n+1}a_1+(-1)^n$$

27 Let $\theta\cot\theta=a_0+a_2\theta^2+a_4\theta^4+\ldots$;

then $$\theta\cos\theta=\sin\theta\{a_0+a_2\theta^2+a_4\theta^4+\ldots\}$$

Substitute for $\cos\theta$ and $\sin\theta$ by Art 286, thus

$$\theta\left(1-\frac{\theta^2}{\lfloor 2}+\frac{\theta^4}{\lfloor 4}-\frac{\theta^6}{\lfloor 6}+\ldots\right)$$
$$=\left\{\theta-\frac{\theta^3}{\lfloor 3}+\frac{\theta^5}{\lfloor 5}-\frac{\theta^7}{\lfloor 7}+\ldots\right\}\{a_0+a_2\theta^2+a_4\theta^4+\ldots\}.$$

Equate the coefficients of θ^{2n+1}, thus

$$\frac{(-1)^n}{\lfloor 2n}=a_{2n}-\frac{a_{2n-2}}{\lfloor 3}+\frac{a_{2n-4}}{\lfloor 5}-\ldots+\frac{(-1)^na_0}{\lfloor 2n+1}$$

Transpose, thus we get

$$a_{2n}=\frac{a_{2n-2}}{\underline{|3}}-\frac{a_{2n-4}}{\underline{|5}}+\quad+\frac{(-1)^{n-1}a_0}{\underline{|2n+1}}+\frac{(-1)^n}{\underline{|2n}}.$$

To find the first four terms of $\theta\cot\theta$ we have the following equations

$$1=a_0,$$

$$-\frac{1}{\underline{|2}}=a_2-\frac{a_0}{\underline{|3}},$$

$$\frac{1}{\underline{|4}}=a_4-\frac{a_2}{\underline{|3}}+\frac{a_0}{\underline{|5}},$$

$$-\frac{1}{\underline{|6}}=a_6-\frac{a_4}{\underline{|3}}+\frac{a_2}{\underline{|5}}-\frac{a_0}{\underline{|7}}.$$

Hence we obtain

$$a_0=1,\qquad a_2=\frac{1}{\underline{|3}}-\frac{1}{2}=\frac{1}{6}-\frac{1}{2}=-\frac{1}{3},\qquad a_4=\frac{1}{\underline{|4}}-\frac{1}{3\,\underline{|3}}-\frac{1}{\underline{|5}}=-\frac{1}{45},$$

$$a_6=-\frac{1}{\underline{|6}}-\frac{1}{45\,\underline{|3}}+\frac{1}{3\,\underline{|5}}+\frac{1}{\underline{|7}}=-\frac{2}{945}.$$

28 Let $\sec\theta=a_0+a_2\theta^2+a_4\theta^4+\quad,$

then $$1=\cos\theta\,(a_0+a_2\theta^2+a_4\theta^4+\quad)$$

$$=\left(1-\frac{\theta^2}{\underline{|2}}+\frac{\theta^4}{\underline{|4}}-\frac{\theta^6}{\underline{|6}}+\quad\right)(a_0+a_2\theta^2+a_4\theta^4+\quad)$$

Then equating to zero the coefficient of θ^{2n} in the expression on the right-hand side we get

$$0=a_{2n}-\frac{a_{2n-2}}{\underline{|2}}+\frac{a_{2n-4}}{\underline{|4}}-\frac{a_{2n-6}}{\underline{|6}}+\quad+\frac{(-1)^n}{\underline{|2n}}a_0$$

Transpose, then we obtain $a_{2n}=\dfrac{a_{2n-2}}{\underline{|2}}-\dfrac{a_{2n-4}}{\underline{|4}}+\quad+\dfrac{(-1)^{n+1}a_0}{\underline{|2n}}$

29 Let r denote the radius, and θ the circular measure of the angle, then the length of the arc is $r\theta$

The chord of the arc is $2r\sin\frac{\theta}{2}$, and the chord of half the arc is $2r\sin\frac{\theta}{4}$

Now let it be required to determine two numbers l and m, such that approximately

$$l\times 2r\sin\frac{\theta}{2}+m\times 2r\sin\frac{\theta}{4}=r\theta$$

Expand $\sin\frac{\theta}{4}$ and $\sin\frac{\theta}{2}$ by Art 286 Thus

$$2l\left\{\frac{\theta}{2}-\frac{1}{\underline{|3}}\left(\frac{\theta}{2}\right)^3+\ldots\right\}+2m\left\{\frac{\theta}{4}-\frac{1}{\underline{|3}}\left(\frac{\theta}{4}\right)^3+\ \right\}=\theta$$

Neglect all powers of θ above θ^3; then to make this formula hold we must put

$$l+\frac{m}{2}=1,\quad \frac{l}{(2)^3}+\frac{m}{(4)^3}=0$$

Therefore $m=-8l$; therefore $-3l=1$.

Thus $$l=-\frac{1}{3}\text{ and }m=\frac{8}{3}.$$

This establishes the rule.

30 Proceed as in Example 29

The chord of one-fourth of the arc is $2r\sin\frac{\theta}{8}$.

Let it be required to determine the numbers l, m, n such that approximately

$$l\times 2r\sin\frac{\theta}{2}+m\times 2r\sin\frac{\theta}{4}+n\times 2r\sin\frac{\theta}{8}=r\theta$$

In this case we can make the approximation closer than in Example 29; for we shall retain θ^5 and neglect only the higher powers Thus

$$2l\left\{\frac{\theta}{2}-\frac{1}{\underline{|3}}\left(\frac{\theta}{2}\right)^3+\frac{1}{\underline{|5}}\left(\frac{\theta}{2}\right)^5\right\}+2m\left\{\frac{\theta}{4}-\frac{1}{\underline{|3}}\left(\frac{\theta}{4}\right)^3+\frac{1}{\underline{|5}}\left(\frac{\theta}{4}\right)^5\right\}$$
$$+2n\left\{\frac{\theta}{8}-\frac{1}{\underline{|3}}\left(\frac{\theta}{8}\right)^3+\frac{1}{\underline{|5}}\left(\frac{\theta}{8}\right)^5\right\}=\theta$$

Hence we must put

$$l+\frac{m}{2}+\frac{n}{4}=1,\quad \frac{l}{(2)^3}+\frac{m}{(4)^3}+\frac{n}{(8)^3}=0,\quad \frac{l}{(2)^5}+\frac{m}{(4)^5}+\frac{n}{(8)^5}=0$$

The values of l, m, n given by these equations are

$$l=\frac{1}{45},\quad m=-\frac{40}{45},\quad n=\frac{256}{45}$$

31 Put $a_1=r_1\cos\theta_1$, $b_1=r_1\sin\theta_1$ and make similar substitutions for a_2, b_2, &c, then

$$r_1^2=a_1^2+b_1^2,\quad \tan\theta_1=\frac{b_1}{a_1}.$$

Now $$a_1+\iota b_1=r_1(\cos\theta_1+\iota\sin\theta_1),$$

$$(a_1+\iota b_1)(a_2+\iota b_2)\ldots$$
$$=r_1 r_2\ldots r_n(\cos\theta_1+\iota\sin\theta_1)(\cos\theta_2+\iota\sin\theta_2)\ldots,$$
$$=r_1 r_2 r_3\ldots r_n\{\cos(\theta_1+\theta_2+\ldots+\theta_n)+\iota\sin(\theta_1+\theta_2+\ldots+\theta_n)\},$$
$$A=r_1 r_2\ldots r_n\cos(\theta_1+\theta_2+\ldots+\theta_n),$$
$$B=r_1 r_2\ldots r_n\sin(\theta_1+\theta_2+\ldots+\theta_n);$$
$$A^2+B^2=r_1^2 r_2^2\ldots r_n^2=(a_1^2+b_1^2)(a_2^2+b_2^2)\ldots(a_n^2+b_n^2),$$

and $$\frac{B}{A}=\tan(\theta_1+\theta_2+\ldots+\theta_n),$$

$$\tan^{-1}\frac{B}{A}=\theta_1+\theta_2+\ldots+\theta_n=\tan^{-1}\frac{b_1}{a_1}+\tan^{-1}\frac{b_2}{a_2}+\ldots.$$

XX.

1 Proceed as in Art 294 Thus we obtain

$$-2^{4n+1}(\sin\theta)^{4n+2}=\cos(4n+2)\theta-(4n+2)\cos 4n\theta$$
$$+\frac{(4n+2)(4n+1)}{\lfloor 2}\cos(4n-2)\theta-\ldots$$
$$-\frac{(4n+2)(4n+1)\ldots(2n+2)}{2\lfloor 2n+1}.$$

2 Proceed as in Art 295 Thus we obtain

$$2^{4n}(\sin\theta)^{4n+1}=\sin(4n+1)\theta-(4n+1)\sin(4n-1)\theta$$
$$+\frac{(4n+1)4n}{\lfloor 2}\sin(4n-3)\theta-\ldots$$
$$+\frac{(4n+1)4n(4n-1)\ldots(2n+2)}{\lfloor 2n}\sin\theta$$

3 Proceed as in Art 292 Thus we obtain

$$2^{2n-1}\cos^{2n}\theta=\cos 2n\theta+2n\cos(2n-2)\theta+\frac{2n(2n-1)}{\lfloor 2}\cos(2n-4)\theta$$
$$+\ldots\frac{2n(2n-1)\ldots(n+1)}{2\lfloor n}.$$

4 By Art 292,

$$2^7\cos^8\theta=\cos 8\theta+8\cos 6\theta+\frac{8\cdot 7}{1\cdot 2}\cos 4\theta+\frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3}\cos 2\theta+\frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{1}{2},$$

$$\therefore\ 128\cos^8\theta=\cos 8\theta+8\cos 6\theta+28\cos 4\theta+56\cos 2\theta+35.$$

5 As in Art 292, we have

$$2^6\cos^7\theta=\cos 7\theta+7\cos 5\theta+\frac{7\cdot 6}{2}\cos 3\theta+\frac{7\cdot 6\cdot 5}{2\cdot 3}\cos\theta$$

$$=\cos 7\theta+7\cos 5\theta+21\cos 3\theta+35\cos\theta$$

6. As in Art 302,

$$2^7 i^5\sin^5\theta\cos^2\theta=\left(x-\frac{1}{x}\right)^5\left(x+\frac{1}{x}\right)^2$$

$$=x^7-3x^5+x^3+5x-\frac{5}{x}-\frac{1}{x^3}+\frac{3}{x^5}-\frac{1}{x^7}$$

$$=2i(\sin 7\theta-3\sin 5\theta+\sin 3\theta+5\sin\theta),$$

$$64\sin^5\theta\cos^2\theta=\sin 7\theta-3\sin 5\theta+\sin 3\theta+5\sin\theta$$

7 $$2^6 i^4\sin^4\theta\cos^2\theta=\left(x-\frac{1}{x}\right)^4\left(x+\frac{1}{x}\right)^2$$

$$=x^6-2x^4-x^2+4-\frac{1}{x^2}-\frac{2}{x^4}+\frac{1}{x^6}$$

$$=2(\cos 6\theta-2\cos 4\theta-\cos 2\theta+2),$$

$$32\sin^4\theta\cos^2\theta=\cos 6\theta-2\cos 4\theta-\cos 2\theta+2$$

8 $$2^9 i^7\sin^7\theta\cos^2\theta=\left(x-\frac{1}{x}\right)^7\left(x+\frac{1}{x}\right)^2$$

$$=x^9-5x^7+8x^5-14x+\frac{14}{x}-\frac{8}{x^5}+\frac{5}{x^7}-\frac{1}{x^9}$$

$$=2i(\sin 9\theta-5\sin 7\theta+8\sin 5\theta-14\sin\theta).$$

$$-256\sin^7\theta\cos^2\theta=\sin 9\theta-5\sin 7\theta+8\sin 5\theta-14\sin\theta$$

9 $1+\cos 10\theta=2(\cos 5\theta)^2$

By Art 298,

$$2\cos 5\theta=(2\cos\theta)^5-5(2\cos\theta)^3+5(2\cos\theta),$$

$$\therefore\ \cos 5\theta=16\cos^5\theta-20\cos^3\theta+5\cos\theta,$$

$$\therefore\ 1+\cos 10\theta=2(16\cos^5\theta-20\cos^3\theta+5\cos\theta)^2.$$

10 $$\frac{1+\cos 9\theta}{1+\cos\theta}=\frac{\cos^2\frac{9\theta}{2}}{\cos^2\frac{\theta}{2}}.$$

$$\frac{\cos\frac{9\theta}{2}}{\cos\frac{\theta}{2}}=\frac{2\cos\frac{9\theta}{2}\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}=\frac{\sin 5\theta-\sin 4\theta}{\sin\theta}$$

$$=\frac{\sin 4\theta\cos\theta+\cos 4\theta\sin\theta-\sin 4\theta}{\sin\theta}$$

$$=(\cos\theta-1)\,.\,4\cos\theta\cos 2\theta+\cos 4\theta$$
$$=4(\cos\theta-1)\quad\cos\theta\quad(2\cos^2\theta-1)+2(2\cos^2\theta-1)^2-1$$
$$=4(2\cos^4\theta-2\cos^3\theta-\cos^2\theta+\cos\theta)+8\cos^4\theta-8\cos^2\theta+1$$
$$=16\cos^4\theta-8\cos^3\theta-12\cos^2\theta+4\cos\theta+1,$$
$$1+\cos 9\theta=(1+\cos\theta)(16\cos^4\theta-8\cos^3\theta-12\cos^2\theta+4\cos\theta+1)^2$$

11 It is required to prove that

$$1-x^2=(1-2x\cos\theta+x^2)(1+2x\cos\theta+2x^2\cos 2\theta+2x^3\cos 3\theta+\quad)$$

The coefficient of x^n on the right-hand side, $(n>2)$

$$=2\cos n\theta-4\cos\theta\cos(n-1)\theta+2\cos(n-2)\theta$$
$$=4\cos\theta\cos(n-1)\theta-4\cos\theta\cos(n-1)\theta=0$$

The coefficient of $x^2=1-4\cos^2\theta+2\cos 2\theta=-1$

The coefficient of $x=-2\cos\theta+2\cos\theta=0$

Hence the result follows.

12 From the preceding example we have

$$2x\cos\theta+2x^2\cos 2\theta+2x^3\cos 3\theta+\quad=\frac{1-x^2}{1-2x\cos\theta+x^2}-1$$
$$=\frac{2x\cos\theta-2x^2}{1-2x\cos\theta+x^2}$$

Divide by $2x$; therefore

$$\frac{\cos\theta-x}{1-2x\cos\theta+x^2}=\cos\theta+x\cos 2\theta+x^2\cos 3\theta+$$

13. $(1-2x\cos\theta+x^2)(x\sin\theta+x^2\sin 2\theta+x^3\sin 3\theta+\quad)$

$$=x\sin\theta+x^2(\sin 2\theta-2\sin\theta\cos\theta)+$$
$$+x^n\{\sin n\theta-2\cos\theta\sin(n-1)\theta+\sin(n-2)\theta\}+$$
$$=x\sin\theta+\Sigma x^n\{2\sin(n-1)\theta\cos\theta-2\cos\theta\sin(n-1)\theta\}$$
$$=x\sin\theta,$$
$$\frac{x\sin\theta}{1-2x\cos\theta+x^2}=x\sin\theta+x^2\sin 2\theta+x^3\sin 3\theta+\quad.$$

14 $$\frac{1-x^2}{1-2x\cos\theta+x^2}=(1-x^2)\{1-x(2\cos\theta-x)\}^{-1}$$
$$=(1-x^2)\{1+x(2\cos\theta-x)+\quad+x^r(2\cos\theta-x)^r+\quad\}$$
$$=(1-x^2)\{1+x(c-x)+\quad+x^r(c-x)^r+\quad\},$$

where $c=2\cos\theta$

The coefficient of x^n in this expression is equal to the coefficient of x^n − coefficient of x^{n-2}, in the expression within the brackets.

The coefficient of x^n in $x^r(c-x)^r$ is the coefficient of x^{n-r} in the expansion of $(c-x)^r$, namely

$$(-1)^{n-r}\frac{\lfloor r}{\lfloor n-r\,\lfloor 2r-n}\cdot c^{2r-n}$$

Hence $$2\cos n\theta=\frac{\lfloor n}{\lfloor n}\cdot c^n-\frac{\lfloor n-1}{\lfloor 1\,\lfloor n-2}\cdot c^{n-2}+\frac{\lfloor n-2}{\lfloor 2\,\lfloor n-4}\cdot c^{n-4}-$$

$$-\left[\frac{\lfloor n-2}{\lfloor n-2}c^{n-2}-\frac{\lfloor n-3}{\lfloor 1\,\lfloor n-4}c^{n-4}+\frac{\lfloor n-4}{\lfloor 2\,\lfloor n-6}c^{n-6}-\quad\right]$$

The coefficient of c^{n-2r}

$$(-1)^r\left[\frac{\lfloor n-r}{\lfloor r\,\lfloor n-2r}+\frac{\lfloor n-r-1}{\lfloor r-1\,\lfloor n-2r}\right]$$

$$=(-1)^r\ \frac{\lfloor n-r-1}{\lfloor r-1\,\lfloor n-2r}\left(\frac{n-r}{r}+1\right)=(-1)^r\cdot n\cdot\frac{\lfloor n-r-1}{\lfloor r\,\lfloor n-2r}$$

Hence $$2\cos n\theta=c^n-nc^{n-2}+\ldots+(-1)^r n\cdot\frac{\lfloor n-r-1}{\lfloor r\,\lfloor n-2r}\cdot c^{n-2r}+\ldots$$

Again from Ex 13,

$$\Sigma x^n\frac{\sin n\theta}{\sin\theta}=\frac{x}{1-2x\cos\theta+x^2}$$

$$=x\{1+x(c-x)+\ldots+x^r(c-x)^r+\ldots\}$$

Hence $\dfrac{\sin n\theta}{\sin\theta}$ = the coefficient of x^{n-1} in the expansion of the expression within the brackets

As before the coefficient of x^{n-1}

$$=\frac{\lfloor n-1}{\lfloor n-1}\ c^{n-1}-\frac{\lfloor n-2}{\lfloor 1\,\lfloor n-3}\cdot c^{n-3}+\ldots+(-1)^r\frac{\lfloor n-r-1}{\lfloor r\,\lfloor n-2r-1}c^{n-2r-1}+$$

$$=c^{n-1}-\frac{(n-2)}{1}\cdot c^{n-3}+\frac{(n-3)(n-4)}{1\ 2}c^{n-5}-$$

15 Put 2θ for θ in Art 292; therefore

$$2^n(\cos 2\theta)^n=\cos 2n\theta+n\cos(2n-4)\theta+\frac{n(n-1)}{1.2}\cos(2n-8)\theta+\ldots$$

16. $x+h=\sin(A+Bh+Ch^2+\ \ )$

$$=\sin A\cos(Bh+Ch^2+\ \ )+\cos A\sin(Bh+Ch^2+\ \ )$$

$$=\sin A\left\{1-\frac{1}{2}(Bh+Ch^2+\ \ )^2+\ \ \right\}+\cos A\{Bh+Ch^2+\ \ \}$$

$$=\sin A+B\cos A\ \ h+\text{higher powers of } h,$$

$x=\sin A$ and $B\cos A=1$,

$$B=\frac{1}{\cos A}=\frac{1}{\sqrt{1-x^2}}$$

17 $\sin^{-1}x=A_1x+A_3x^3+\ \ $,

$$\sin^{-1}(x+h)=A_1(x+h)+A_3(x+h)^3+\ ..$$

$$=A_1x+A_3x^3+\ \ +h\{A_1+3A_3x^2+5A_5x^5+\ \ \}+\ \ .$$

But from Ex 16,

$$\sin^{-1}(x+h)=\sin^{-1}x+\frac{h}{\sqrt{1-x^2}}+$$

$$A_1+3A_3x^2+5A_5x^4+\ \ =\frac{1}{\sqrt{1-x^2}}=(1-x^2)^{-\frac{1}{2}}$$

$$=1+\frac{1}{2}x^2+\frac{1}{2}\frac{3}{4}x^4+\ \ ,$$

$$A_1=1,\ A_3=\frac{1}{2}\ \ \frac{1}{3},\ A_5=\frac{1}{2}\frac{3}{4}\ \ \frac{1}{5},\ \&c,$$

$$\sin^{-1}x=A_1x+A_3x^3+A_5x^5+$$

$$=x+\frac{1}{2}\ \ \frac{x^3}{3}+\frac{1}{2}\frac{3}{4}\ \ \frac{x^5}{5}+$$

18 Put $-n$ for x in Example 11, therefore

$$1-2n\cos\theta+2n^2\cos 2\theta-\ \ =\frac{1-n^2}{1+2n\cos\theta+n^2}=\frac{\frac{1-n^2}{1+n^2}}{1+\frac{2n}{1+n^2}\cos\theta}$$

$$\frac{1}{1+e\cos\theta}=\frac{1+n^2}{1-n^2}\{1-2n\cos\theta+2n^2\cos 2\theta-\ \ \}$$

Since $$e=\frac{2n}{1+n^2},\ 1-e^2=1-\frac{4n^2}{(1+n^2)^2}=\left(\frac{1-n^2}{1+n^2}\right)^2$$

$$\frac{1}{1+e\cos\theta}=\frac{1}{\sqrt{1-e^2}}\{1-2n\cos\theta+2n^2\cos 2\theta-\ \ \}$$

19 If x be $< \frac{\pi}{4}$, we have

$$(1+\tan^2 x)^{-2} = 1 - 2\tan^2 x + 3\tan^4 x - \dots$$

Hence the given series

$$= 2\tan^2 x \times (1+\tan^2 x)^{-2} = 2\tan^2 x \times (\sec^2 x)^{-2}$$

$$= 2\tan^2 x \cos^4 x = \frac{1}{2}\sin^2 2x$$

20 If x be $< \frac{\pi}{4}$,

$$\log(1+\tan^2 x) = \tan^2 x - \frac{1}{2}\tan^4 x + \frac{1}{3}\tan^6 x - \dots;$$

$$\frac{1}{2}\tan^2 x - \frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x - \dots = \frac{1}{2}\log\sec^2 x = \log\sec x.$$

21 $\cos^n\theta\cos n\theta + \iota\cos^n\theta\sin n\theta = (1-\iota\tan\theta)^{-n}$

$$= 1 + n\iota\tan\theta + \frac{n(n+1)}{1.2}\iota^2\tan^2\theta + \frac{n(n+1)(n+2)}{\underline{|3}}\iota^3\tan^3\theta \dots$$

$$= 1 - \frac{n(n+1)}{\underline{|2}}\tan^2\theta + \frac{n(n+1)(n+2)(n+4)}{\underline{|4}}\tan^4\theta - \dots$$

$$+ \iota\left\{n\tan\theta - \frac{n(n+1)(n+2)}{\underline{|3}}\tan^3\theta + \dots\right\}.$$

Equating the real and imaginary parts we obtain the two required results

22 By Art 295 we have, when n is odd,

$$2^{n-1}(-1)^{\frac{n-1}{2}}\sin^n\theta = \sin n\theta - n\sin(n-2)\theta$$

$$+ \frac{n(n-1)}{1\ 2}\sin(n-4)\theta \dots \text{ to } \frac{n+1}{2} \text{ terms} \qquad (1)$$

Expand the sines in powers of θ,

$$\sin^n\theta = \left(\theta - \frac{\theta^3}{\underline{|3}} + \dots\right)^n = \theta^n - \&c \qquad (2)$$

If p be $<n$ the coefficient of θ^p is zero in this series

Since $\quad \sin r\theta = r\theta - \frac{r^3\theta^3}{\underline{|3}} + \dots + (-1)^{\frac{p-1}{2}}\frac{r^p\theta^p}{\underline{|p}} + \dots$

the coefficient of θ^p in the expansion of the right-hand side of equation (1) is

$$\frac{(-1)^{\frac{p-1}{2}}}{\underline{|p}}\left\{n^p - n(n-2)^p + \frac{n(n-1)}{\underline{|2}}(n-4)^p - \dots\right\}.$$

This series is therefore zero

But if $p=n$ the coefficient of θ^p, that is θ^n, in equation (2) is unity Hence

$$2^{n-1}(-1)^{\frac{n-1}{2}}=\frac{(-1)^{\frac{n-1}{2}}}{\underline{|n}}\{n^n-n(n-2)^n+\dots\}$$

Hence the sum required is $2^{n-1}\,\underline{|n}$

23 For $\cos\theta+\iota\sin\theta$ write x, so that

$$2\cos\theta=x+\frac{1}{x},\qquad 2\cos r\theta=x^r+\frac{1}{x^r}$$

Then we have

$$\frac{\cos\theta}{\underline{|n-1}\,\underline{|n+1}}+\frac{\cos 2\theta}{\underline{|n-2}\,\underline{|n+2}}+\frac{\cos 3\theta}{\underline{|n-3}\,\underline{|n+3}}+\dots+\frac{\cos n\theta}{\underline{|2n}}$$

$$=\frac{1}{2}\left\{\frac{x}{\underline{|n-1}\,\underline{|n+1}}+\frac{x^2}{\underline{|n-2}\,\underline{|n+2}}+\dots+\frac{x^{n-1}}{\underline{|1}\,\underline{|2n-1}}+\frac{x^n}{\underline{|2n}}\right.$$

$$\left.+\frac{1}{\underline{|n-1}\,\underline{|n+1}}\cdot\frac{1}{x}+\frac{1}{\underline{|n-2}\,\underline{|n+2}}\cdot\frac{1}{x^2}+\dots+\frac{1}{\underline{|2n}}\cdot\frac{1}{x^n}\right\}$$

$$=\frac{1}{2\,\underline{|2n}}\left\{x^n+2n\cdot x^{n-1}+\frac{2n(2n-1)}{1\cdot 2}\cdot x^{n-2}+\dots\right.$$

$$\left.+\frac{\underline{|2n}}{\underline{|n}\,\underline{|n}}+\dots+\frac{1}{x^n}-\frac{\underline{|2n}}{\underline{|n}\,\underline{|n}}\right\}$$

$$=\frac{1}{2\,\underline{|2n}}(x^{\frac{1}{2}}+x^{-\frac{1}{2}})^{2n}-\frac{1}{2\,\underline{|n}\,\underline{|n}}$$

$$=\frac{1}{2\,\underline{|2n}}(x+x^{-1}+2)^n-\frac{1}{2\,\underline{|n}\,\underline{|n}}$$

$$=\frac{2^{n-1}}{\underline{|2n}}(1+\cos\theta)^n-\frac{1}{2\,\underline{|n}\,\underline{|n}}$$

24 Let $x=\cos\theta+\iota\sin\theta$ Hence the given series

$$=\frac{1}{2}\left\{x^n\sin^n\phi+nx^{n-1}\sin^{n-1}\phi\sin(\theta-\phi)+\dots\right.$$

$$\left.+\frac{1}{x^n}\sin^n\phi+n\cdot\frac{1}{x^{n-1}}\cdot\sin^{n-1}\phi\sin(\theta-\phi)+\dots\right\}$$

$$=\frac{1}{2}\{x\sin\phi+\sin(\theta-\phi)\}^n+\frac{1}{2}\left\{\frac{1}{x}\sin\phi+\sin(\theta-\phi)\right\}^n$$

$$=\frac{1}{2}\{\cos\theta\sin\phi+\iota\sin\theta\sin\phi+\sin\theta\cos\phi-\cos\theta\sin\phi\}^n$$

$$+\frac{1}{2}\{\cos\theta\sin\phi-\iota\sin\theta\sin\phi+\sin\theta\cos\phi-\cos\theta\sin\phi\}^n$$

$$=\frac{1}{2}\sin^n\theta(\cos\phi+\iota\sin\phi)^n+\frac{1}{2}\sin^n\theta(\cos\phi-\iota\sin\phi)^n$$

$$=\sin^n\theta\cos n\phi$$

25 For all values of n we have (Arts 301, 299)

$$\cos n\theta=1-\frac{n^2}{1\cdot 2}\sin^2\theta+\frac{n^2(n^2-2^2)}{\underline{|4}}\sin^4\theta-\frac{n^2(n^2-2^2)(n^2-4^2)}{\underline{|6}}\sin^6\theta+\ldots$$

Now $$\cos n\theta=1-\frac{n^2\theta^2}{2}+\ldots$$

Hence we have by equating the coefficients of n^2,

$$\frac{\theta^2}{2}=\frac{1}{1\cdot 2}\sin^2\theta+\frac{2^2}{\underline{|4}}\sin^4\theta+\frac{2^2\cdot 4^2}{\underline{|6}}\sin^6\theta+\ldots$$

$$=\frac{1}{1\cdot 2}\sin^2\theta+\frac{2}{1\cdot 3}\cdot\frac{\sin^4\theta}{4}+\frac{2\cdot 4}{1\cdot 3\cdot 5}\cdot\frac{\sin^6\theta}{6}+\ldots$$

26 For all values of n we have (Arts 301, 299)

$$\sin n\theta=n\sin\theta-\frac{n(n^2-1)}{\underline{|3}}\sin^3\theta+\frac{n(n^2-1)(n^2-3^2)}{\underline{|5}}\sin^5\theta-\ldots$$

Now $$\sin n\theta=n\theta-\frac{n^3\theta^3}{\underline{|3}}+\ldots$$

Hence we have by equating the coefficients of n^3,

$$\frac{1}{6}\theta^3=\frac{1}{\underline{|3}}\sin^3\theta+\frac{1^2+3^2}{\underline{|5}}\sin^5\theta+\frac{1^2\cdot 3^2+1^2\cdot 5^2+3^2\cdot 5^2}{\underline{|7}}\sin^7\theta+\ldots$$

27 For all values of n we have (Arts 301, 299)

$$\sin n\theta\sec\theta=n\left\{\sin\theta-\frac{n^2-2^2}{\underline{|3}}\sin^3\theta+\frac{(n^2-2^2)(n^2-4^2)}{\underline{|5}}\sin^5\theta-\ldots\right\}$$

Now $$\sin n\theta=n\theta-\frac{n^3\theta^3}{\underline{|3}}+\ldots$$

Hence we have by equating the coefficients of n,

$$\theta\sec\theta=\sin\theta+\frac{2^2}{\underline{|3}}\sin^3\theta+\frac{2^2\cdot 4^2}{\underline{|5}}\sin^5\theta+\ldots$$

$$=\sin\theta+\frac{2}{3}\sin^3\theta+\frac{2\cdot 4}{3\cdot 5}\sin^5\theta+\ldots$$

28 Since for all values of n,

$$\cos n\theta \sec\theta = 1 - \frac{n^2-1}{1 \cdot 2}\sin^2\theta + \frac{(n^2-1)(n^2-3^2)}{\lfloor 4}\sin^4\theta - \dots$$

and
$$\cos n\theta = 1 - \frac{n^2\theta^2}{1 \cdot 2} + \dots$$

we have, by equating the terms without n,

$$\sec\theta = 1 + \frac{1}{2}\sin^2\theta + \frac{1^2 \cdot 3^2}{\lfloor 4}\sin^4\theta + \dots$$

$$= 1 + \frac{1}{2}\sin^2\theta + \frac{1 \cdot 3}{2 \cdot 4}\sin^4\theta + \dots$$

29 In the result of Example 27, write $\sin\theta = x$. Then $\theta = \sin^{-1}x$ and $\cos\theta = \sqrt{1-x^2}$. Hence

$$\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + \frac{2}{3}x^3 + \frac{2 \cdot 4}{3 \cdot 5}x^5 + \dots \qquad (1)$$

Let
$$\tan^{-1}y = \sin^{-1}x = \theta.$$

Then
$$y = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{x}{\sqrt{1-x^2}},$$

$$x^2 = \frac{y^2}{1+y^2}, \text{ and } x\sqrt{1-x^2} = \frac{y}{1+y^2}$$

Hence equation (1) becomes

$$\tan^{-1}y = x\sqrt{1-x^2}\left\{1 + \frac{2}{3}x^2 + \frac{2 \cdot 4}{3 \cdot 5}x^4 + \dots\right\}$$

$$= \frac{y}{1+y^2}\left\{1 + \frac{2}{3} \cdot \frac{y^2}{1+y^2} + \frac{2 \cdot 4}{3 \cdot 5}\left(\frac{y^2}{1+y^2}\right)^2 + \dots\right\}$$

30 From Example 25,

$$\frac{\theta^2}{2} = \Sigma_{n=0}^{n=\infty}\frac{2^2 \cdot 4^2 \dots (2n)^2}{\lfloor 2n+2}\sin^{2n+2}\theta$$

$$= \Sigma_{n=0}^{n=\infty}\frac{2^{2n}\lfloor n\lfloor n}{\lfloor 2n+2} \cdot \sin^{2n+2}\theta$$

Put $\theta = \frac{\pi}{6}$, so that $\sin\theta = \frac{1}{2}$

$$\frac{\pi^2}{72} = \Sigma_{n=0}^{n=\infty}\frac{2^{2n}\lfloor n\lfloor n}{\lfloor 2n+2} \cdot \frac{1}{2^{2n+2}}$$

$$\pi^2 = 18\,\Sigma_{n=0}^{n=\infty}\frac{\lfloor n\lfloor n}{\lfloor 2n+2}$$

31 $$\frac{1+\cos(2m+1)\theta}{1+\cos\theta}=\frac{\cos^2\frac{2m+1}{2}\theta}{\cos^2\frac{\theta}{2}}=\left(\frac{2\sin\frac{\theta}{2}\cos\frac{2m+1}{2}\theta}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)^2$$

$$=\left\{\frac{\sin(m+1)\theta}{\sin\theta}-\frac{\sin m\theta}{\sin\theta}\right\}^2$$

Now by Art 300, if m be even, we obtain by putting $n=m+1$,

$$(-1)^{\frac{m}{2}}\frac{\sin(m+1)\theta}{\sin\theta}=1-\frac{m(m+2)}{1\ 2}\cos^2\theta+\frac{(m-2)m(m+2)(m+4)}{\lfloor 4}\cos^4\theta-$$

$$-(-1)^{\frac{m}{2}}\frac{\sin m\theta}{\sin\theta}=m\cos\theta-\frac{(m-2)m(m+2)}{\lfloor 3}\cos^3\theta+$$

$$(-1)^{\frac{m}{2}}\left\{\frac{\sin(m+1)\theta}{\sin\theta}-\frac{\sin m\theta}{\sin\theta}\right\}$$

$$=1+m\cos\theta-\frac{m(m+2)}{1\ 2}\cos^2\theta-\frac{(m-2)m(m+2)}{\lfloor 3}\cos^3\theta+$$

$$\frac{1+\cos(2m+1)\theta}{1+\cos\theta}=\left(1+m\cos\theta-\frac{m(m+2)}{1\ 2}\cos^2\theta-\qquad\right)^2$$

32 Let $\cos\theta+\iota\sin\theta=x$, then $2\cos 2\theta=x^2+\frac{1}{x^2}$

$$\log\frac{(1+n)^4\cos^2\theta+(1-n)^4\sin^2\theta}{(1+n)^2\cos^2\theta+(1-n)^2\sin^2\theta}$$

$$=\log\frac{(1+6n^2+n^4)(\cos^2\theta+\sin^2\theta)+4n(1+n^2)(\cos^2\theta-\sin^2\theta)}{(1+n^2)(\cos^2\theta+\sin^2\theta)+2n(\cos^2\theta-\sin^2\theta)}$$

$$=\log\frac{(1+n^2)^2+4n^2+4n(1+n^2)\cos 2\theta}{1+n^2+2n\cos 2\theta}$$

$$=\log\frac{(1+n^2)^2+2n(1+n^2)\left(x^2+\frac{1}{x^2}\right)+4n^2}{1+n^2+n\left(x^2+\frac{1}{x^2}\right)}$$

$$=\log\frac{\{(1+n^2)+2nx^2\}\left\{(1+n^2)+\frac{2n}{x^2}\right\}}{(1+nx^2)\left(1+\frac{n}{x^2}\right)}$$

$$=2\log(1+n^2)+\log\frac{(1+cx^2)\left(1+\frac{c}{x^2}\right)}{(1+nx^2)\left(1+\frac{n}{x^2}\right)},\text{ where } c=\frac{2n}{1+n^2}.$$

$$=2\log(1+n^2)+\log(1+cx^2)+\log\left(1+\frac{c}{x^2}\right)-\log(1+nx^2)-\log\left(1+\frac{n}{x^2}\right)$$

$$=2\log(1+n^2)+(c-n)\,x^2-\frac{1}{2}(c^2-n^2)\,x^4+\frac{1}{3}(c^3-n^3)\,x^6-$$

$$+(c-n)\frac{1}{x^2}-\frac{1}{2}(c^2-n^2)\frac{1}{x^4}+\frac{1}{3}(c^3-n^3)\frac{1}{x^6}-$$

$$=2\log(1+n^2)+2\left\{(c-n)\cos 2\theta-\frac{1}{2}(c^2-n^2)\cos 4\theta+\qquad\right\}$$

33 From Art 298 we have

$$\cos n\theta=2^{n-1}\cos^n\theta-2^{n-3}\;n\cos^{n-2}\theta+$$

Hence the given equation is equivalent to

$$\cos n\theta=\cos n\alpha$$

This is satisfied by $\qquad n\theta=2\kappa\pi\pm n\alpha,$

or,

$$\theta=\frac{2\kappa}{n}\pi\pm\alpha$$

The roots of the given equation are therefore the different values of

$$\cos\left(\frac{2\kappa}{n}\pi\pm\alpha\right)$$

Since $\cos\left\{-\alpha+\dfrac{2(n-r)}{n}\pi\right\}=\cos\left(2\pi-\alpha-\dfrac{2r\pi}{n}\right)=\cos\left(\dfrac{2r\pi}{n}+\alpha\right),$

the different values of $\cos\theta$ may be written

$$\cos\alpha,\ \cos(\alpha+\phi),\ \cos(\alpha+2\phi),\qquad \cos\{\alpha+(n-1)\phi\},$$

where

$$n\phi=2\pi$$

When n is odd we have

$$\cos n\theta=2^{n-1}\cos^n\theta-2^{n-3}n\cos^{n-2}\theta+\qquad+(-1)^{\frac{n-1}{2}}n\cos\theta$$

The corresponding equation is therefore

$$2^{n-1}\cos^n\theta-2^{n-3}n\cos^{n-2}\theta+\qquad+(-1)^{\frac{n-1}{2}}n\cos\theta=\cos n\alpha$$

34 From Example 33, $\cos\alpha$, $\cos(\alpha+\phi)$, $\qquad\cos\{\alpha+(n-1)\phi\}$ are the roots of

$$2^{n-1}x^n-2^{n-3}\;nx^{n-2}+\qquad-(-1)^{\frac{n}{2}}\;\frac{n^2}{2}x^2+(-1)^{\frac{n}{2}}-\cos n\alpha=0$$

or of

$$2^{n-1}x^n-2^{n-3}\;nx^{n-2}+\qquad+(-1)^{\frac{n-1}{2}}nx-\cos n\alpha=0,$$

according as n is even or odd

(1) $\cos\alpha \quad \cos(\alpha+\phi) \qquad =$ the product of the roots

$$=2^{1-n}\{(-1)^{\frac{n}{2}}-\cos n\alpha\} \text{ or } 2^{1-n}\cos n\alpha,$$

according as n is even or odd.

(2) $\cos\alpha+\cos(\alpha+\phi)+ \qquad =$ the sum of the roots

$=$ coefficient of $x^{n-1}=0$.

(3) $\sec\alpha+\sec(\alpha+\phi)+ \qquad = \dfrac{\text{product of roots } n-1 \text{ at a time}}{\text{product of all the roots}}$

$$= -\frac{\text{coefficient of } x}{\text{absolute term}}$$

$$=0 \text{ or } \frac{(-1)^{\frac{n-1}{2}} n}{\cos n\alpha}$$

(4) $\sec\alpha\sec(\alpha+\phi)+\sec\alpha\sec(\alpha+2\phi)+\sec(\alpha+\phi)\sec(\alpha+2\phi)+$

$$=\frac{\text{coefficient of } x^2}{\text{absolute term}}=\frac{-(-1)^{\frac{n}{2}}\frac{n^2}{2}}{(-1)^{\frac{n}{2}}-\cos n\alpha}, \text{ or } 0,$$

$\sec^2\alpha+\sec^2(\alpha+\phi)+$

$=\{\sec\alpha+\sec(\alpha+\phi)+ \qquad \}^2-2\sec\alpha\sec(\alpha+\phi)-$

$$=0+\frac{n^2}{1-(-1)^{\frac{n}{2}}\cos n\alpha}, \text{ or } \frac{n^2}{\cos^2 n\alpha}+0, \text{ from (3)}$$

(5) $\tan^2\alpha+\tan^2(\alpha+\phi)+$

$=\sec^2\alpha-1+\sec^2(\alpha+\phi)-1+$

$=\sec^2\alpha+\sec^2(\alpha+\phi)+ \qquad -n$

$$=\frac{n^2}{1-(-1)^{\frac{n}{2}}\cos n\alpha}-n, \text{ or } \frac{n^2}{\cos^2 n\alpha}-n, \text{ from (4)}$$

(6) In (1) put $\frac{\pi}{2}+\alpha$ for α, therefore

$$(-1)^n\sin\alpha\sin(\alpha+\phi) \quad =2^{1-n}\left\{(-1)^{\frac{n}{2}}-\cos\left(\frac{n\pi}{2}+n\alpha\right)\right\} \text{ or } 2^{1-n}\cos\left(\frac{n\pi}{2}+n\alpha\right)$$

according as n is even or odd

If n is even, $\cos\left(\dfrac{n\pi}{2}+n\alpha\right)=(-1)^{\frac{n}{2}}\cos n\alpha$

If n is odd, $\cos\left(\dfrac{n\pi}{2}+n\alpha\right)=(-1)^{\frac{n+1}{2}}\sin n\alpha$

$$\sin\alpha\ldots\sin(\alpha+\phi) \qquad =2^{1-n}\ (-1)^{\frac{n}{2}}(1-\cos n\alpha) \text{ or } 2^{1-n}(-1)^{\frac{n-1}{2}}\sin n\alpha$$

35 Since $z = \cos\theta + \iota\sin\theta,\ z^{-1} = \cos\theta - \iota\sin\theta,$

$$z^n + z^{-n} = 2\cos n\theta,\ z^n - z^{-n} = 2\iota\sin n\theta$$

$$\frac{2\iota\sin\theta\ x}{1 - 2x\cos\theta + x^2} = \frac{1}{1 - zx} - \frac{1}{1 - z^{-1}x}$$

$$= 1 + zx + z^2x^2 + \quad - (1 + z^{-1}x + z^{-2}x^2 + \quad)$$

$$= (z - z^{-1})x + (z^2 - z^{-2})x^2 + (z^3 - z^{-3})x^3 +$$

$$= 2\iota x\sin\theta + 2\iota x^2\sin 2\theta + 2\iota x^3\sin 3\theta + \quad ,$$

$$\frac{2\sin\theta\ x}{1 - 2x\cos\theta + x^2} = 2x\sin\theta + 2x^2\sin 2\theta + 2x^3\sin 3\theta +$$

36 $$\frac{z}{1 - zx} + \frac{z^{-1}}{1 - z^{-1}x} = \frac{z + z^{-1} - 2x}{1 - (z + z^{-1})x + x^2}$$

$$= \frac{2\cos\theta - 2x}{1 - 2x\cos\theta + x^2}$$

$$\frac{2\cos\theta - 2x}{1 - 2x\cos\theta + x^2} = \frac{z}{1 - zx} + \frac{z^{-1}}{1 - z^{-1}x}$$

$$= z(1 + zx + z^2x^2 + \quad) + z^{-1}(1 + z^{-1}x + z^{-2}x^2 + \quad)$$

$$= (z + z^{-1}) + (z^2 + z^{-2})x + (z^3 + z^{-3})x^2 +$$

$$= 2\cos\theta + 2x\cos 2\theta + 2x^2\cos 3\theta +$$

37 Let $z = \cos\theta + \iota\sin\theta$

$$\log(1 - 2x\cos\theta + x^2) = \log\{1 - x(z + z^{-1}) + x^2\}$$

$$= \log(1 - zx)(1 - z^{-1}x)$$

$$= \log(1 - zx) + \log(1 - z^{-1}x)$$

$$= -\left(zx + \frac{1}{2}z^2x^2 + \frac{1}{3}z^3x^3 + \quad \right)$$

$$-\left(z^{-1}x + \frac{1}{2}z^{-2}x^2 + \frac{1}{3}z^{-3}x^3 + \quad \right)$$

$$= -(z + z^{-1})x - \frac{1}{2}(z^2 + z^{-2})x^2 - \frac{1}{3}(z^3 + z^{-3})x^3 -$$

$$= -2x\cos\theta - \frac{1}{2}\ 2x^2\cos 2\theta - \frac{1}{3}\ 2x^3\cos 3\theta -$$

38 Let $z=\cos\theta+\iota\sin\theta$, then $\iota(z-z^{-1})=-2\sin\theta$

$$\frac{2x\cos\theta}{1-2x\sin\theta+x^2}=\frac{x(z+z^{-1})}{1+\iota(z-z^{-1})x-\iota^2x^2}=\frac{x(z+z^{-1})}{(1+\iota zx)(1-\iota z^{-1}x)}$$

$$=\frac{\iota}{1+\iota zx}-\frac{\iota}{1-\iota z^{-1}x}$$

$$=\iota(1-\iota zx+\iota^2z^2x^2-\iota^3z^3x^3+\dots)$$

$$-\iota(1+\iota z^{-1}x+\iota^2z^{-2}x^2+\iota^3z^{-3}x^3+\dots)$$

$$=(z+z^{-1})x-\iota(z^2-z^{-2})x^2-(z^3+z^{-3})x^3+\iota(z^4-z^{-4})x^4+\dots$$

$$=2x\cos\theta+2x^2\sin 2\theta-2x^3\cos 3\theta-2x^4\sin 4\theta+\dots$$

This series may be deduced at once from Example 13

39 Let $z=\cos\theta+\iota\sin\theta$, $a=\cos\phi+\iota\sin\phi$

Then $az=\cos(\theta+\phi)+\iota\sin(\theta+\phi)$, $a^{-1}z^{-1}=\cos(\theta+\phi)-\iota\sin(\theta+\phi)$,

$$az+a^{-1}z^{-1}=2\cos(\theta+\phi)$$

Also $a^{-1}z^r=(\cos r\theta+\iota\sin r\theta)\{\cos(-\phi)+\iota\sin(-\phi)\}$

$$=\cos(r\theta-\phi)+\iota\sin(r\theta-\phi),$$

and $az^{-r}=\cos(r\theta-\phi)-\iota\sin(r\theta-\phi)$,

$$a^{-1}z^r+az^{-r}=2\cos(r\theta-\phi)$$

Now $$\frac{a^{-1}}{1-zx}+\frac{a}{1-z^{-1}x}=\frac{a+a^{-1}-x(az+a^{-1}z^{-1})}{1-(z+z^{-1})x+x^2}$$

$$=\frac{2\cos\phi-2x\cos(\theta+\phi)}{1-2x\cos\theta+x^2}$$

$$\frac{2\cos\phi-2x\cos(\theta+\phi)}{1-2x\cos\theta+x^2}=a^{-1}(1+zx+z^2x^2+\dots)$$

$$+a(1+z^{-1}x+z^{-2}x^2+\dots)$$

$$=a+a^{-1}+x(a^{-1}z+az^{-1})+x^2(a^{-1}z^2+az^{-2})+\dots$$

$$=2\cos\phi+2x\cos(\theta-\phi)+2x^2\cos(2\theta-\phi)+\dots$$

40 The general term is

$$(-1)^r\,\frac{n}{2}\,\frac{(n-r-1)(n-r-2)\dots(n-2r+1)}{r!}(2\cos\theta)^{n-2r}$$

$$=(-1)^r\,\frac{n}{2}\,\frac{(n-r-1)!}{(n-2r)!\,r!}(2\cos\theta)^{n-2r}$$

$$=(-1)^r \frac{n}{2} \cdot \frac{(n-r-1)(n-r-2)\quad(r+1)}{\lfloor n-2r} 2^{n-2r}(\cos\theta)^{n-2r}$$

$$=(-1)^r n \frac{(2n-2r-2)(2n-2r-4)\quad(2r+2)}{\lfloor n-2r}(\cos\theta)^{n-2r}$$

Put $$n-2r=s$$

the general term is

$$(-1)^{\frac{n-s}{2}} n \frac{(n+s-2)(n+s-4)\quad(n-s+4)(n-s+2)}{\lfloor s} \cos^s\theta$$

$$=(-1)^{\frac{n-s}{2}} n \frac{\{n^2-(s-2)^2\}\{n^2-(s-4)^2\}}{\lfloor s} \cos^s\theta$$

The last term in the numerator is n^2-1^2 if n be odd, and n if n be even

If n be even, s must be even, therefore

$$\cos n\theta=(-1)^{\frac{n}{2}}\left\{1-\frac{n^2}{2}\cos^2\theta+\frac{n^2(n^2-2^2)}{\lfloor 4}\cos^4\theta+\qquad\right\}$$

If n be odd, s must be odd, therefore

$$\cos n\theta=(-1)^{\frac{n-1}{2}}\left\{n\cos\theta-\frac{n(n^2-1)}{\lfloor 3}\cos^3\theta+\qquad\right\}$$

These are the two series for $\cos n\theta$ given in Art 300

41 Let $z=\cos\theta+\iota\sin\theta$ Then

$$\frac{1}{(1-zx)^2}-\frac{1}{(1-z^{-1}x)^2}=\frac{2x(z-z^{-1})-x^2(z^2-z^{-2})}{\{1-(z+z^{-1})x+x^2\}^2}$$

$$=\frac{2\iota\{2x\sin\theta-x^2\sin 2\theta\}}{(1-2x\cos\theta+x^2)^2}$$

Hence $\dfrac{2\iota\{2x\sin\theta-x^2\sin 2\theta\}}{(1-2x\cos\theta+x^2)^2}$

$$=(1-zx)^{-2}-(1-z^{-1}x)^{-2}$$

$$=1+2zx+3z^2x^2+4z^3x^3+\qquad-(1+2z^{-1}x+3z^{-2}x^2+4z^{-4}x^4+\qquad)$$

$$=2x(z-z^{-1})+3x^2(z^2-z^{-2})+4x^3(z^3-z^{-3})+$$

$$=2\iota\{2x\sin\theta+3x^2\sin 2\theta+4x^3\sin 3\theta+\qquad\}$$

Therefore

$$\frac{2x\sin\theta(1-x\cos\theta)}{(1-2x\cos\theta+x^2)^2}=2x\sin\theta+3x^2\sin 2\theta+4x^3\sin 3\theta+$$

XXI.

1. $$\frac{\sin 2\theta}{1-\cos 2\theta}=\frac{e^{2\theta\iota}-e^{-2\theta\iota}}{2\iota\left(1-\dfrac{e^{2\theta\iota}+e^{-2\theta\iota}}{2}\right)}=\frac{e^{2\theta\iota}-e^{-2\theta\iota}}{\iota(2-e^{2\theta\iota}-e^{-2\theta\iota})}$$

$$=\frac{\iota(e^{2\theta\iota}-e^{-2\theta\iota})}{e^{2\theta\iota}+e^{-2\theta\iota}-2}=\frac{\iota(e^{\theta\iota}+e^{-\theta\iota})(e^{\theta\iota}-e^{-\theta\iota})}{(e^{\theta\iota}-e^{-\theta\iota})^2}$$

$$=\frac{\iota(e^{\theta\iota}+e^{-\theta\iota})}{e^{\theta\iota}-e^{-\theta\iota}}=\frac{e^{\theta\iota}+e^{-\theta\iota}}{2}\div\frac{e^{\theta\iota}-e^{-\theta\iota}}{2\iota}=\frac{\cos\theta}{\sin\theta}$$

2. Let the angle opposite to the smaller side be $\frac{\pi}{4}-\theta$, and the angle opposite to the larger side $\frac{\pi}{4}+\theta$. Thus

$$\frac{\sin\left(\frac{\pi}{4}-\theta\right)}{\sin\left(\frac{\pi}{4}+\theta\right)}=\frac{49}{51},$$

therefore
$$\frac{\sin\left(\frac{\pi}{4}+\theta\right)-\sin\left(\frac{\pi}{4}-\theta\right)}{\sin\left(\frac{\pi}{4}+\theta\right)+\sin\left(\frac{\pi}{4}-\theta\right)}=\frac{51-49}{51+49}=\frac{1}{50};$$

therefore
$$\frac{2\sin\theta\cos\frac{\pi}{4}}{2\cos\theta\sin\frac{\pi}{4}}=\frac{1}{50},$$

therefore
$$\tan\theta=\frac{1}{50}.$$

But by Art. 293 we have

$$\theta=\tan\theta-\frac{1}{3}\tan^3\theta+\ldots,$$

thus
$$\theta=\cdot 02-\frac{1}{3}(\cdot 02)^3+\frac{1}{5}(\cdot 02)^5-\ldots$$

If we stop at the first term we have $\theta=\cdot 02$.

Then the number of degrees in the angle $=\cdot 02\times 57\cdot 29577951\ldots=1\cdot 14591559$; and this $=1^\circ\,8'\,45''$.

3 We have, as in Art 229,

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$$

Hence by Art 293 the circular measure of $\frac{A-B}{2}$

$$=k-\frac{k^3}{3}+\frac{k^5}{5}-\quad,$$

where k stands for $\frac{a-b}{a+b}\cot\frac{C}{2}$

Therefore the number of degrees in $\frac{A-B}{2}$

$$=\frac{180}{\pi}\left\{k-\frac{k^3}{3}+\frac{k^5}{5}-\quad\right\}$$

Also $\frac{A+B}{2}=90^0-\frac{C}{2}$ Thus A is found by taking the upper sign, and B by taking the lower sign in

$$90^0-\frac{C}{2}\pm\frac{180^0}{\pi}\left\{k-\frac{k^3}{3}+\frac{k^5}{5}-\quad\right\}$$

4 $$\frac{\sin A}{\sin C}=\frac{a}{c},\qquad \frac{\sin B}{\sin C}=\frac{b}{c},$$

therefore $$\frac{\sin A-\sin B}{\sin C}=\frac{a-b}{c},$$

therefore $$\frac{\sin\frac{A-B}{2}\cos\frac{A+B}{2}}{\sin\frac{C}{2}\cos\frac{C}{2}}=\frac{a-b}{c},$$

therefore $$\sin\frac{A-B}{2}=\frac{a-b}{c}\cos\frac{C}{2}$$

$$=\frac{a-b}{c}\sin\frac{A+B}{2}$$

$$=\frac{a-b}{c}\sin\left(\frac{A-B}{2}+B\right)$$

Hence by Art 298 the circular measure of $\frac{A-B}{2}$

$$=n\sin B+\frac{n^2}{2}\sin 2B+\frac{n^3}{3}\sin 3B+\quad,$$

where n stands for $\frac{a-b}{c}$ Therefore the circular measure of $A-B$

$$=2n\sin B+n^2\sin 2B \text{ nearly}$$

5
$$\frac{b}{a}=\frac{\sin B}{\sin A}=\frac{e^{B\iota}-e^{-B\iota}}{e^{A\iota}-e^{-A\iota}}=\frac{e^{B\iota}(1-e^{-2B\iota})}{e^{A\iota}(1-e^{-2A\iota})}$$

Take the logarithms thus

$$\log b-\log a=B\iota-A\iota+\log(1-e^{-2B\iota})-\log(1-e^{-2A\iota})$$
$$=(B-A)\iota-\left\{e^{-2B\iota}+\frac{1}{2}e^{-4B\iota}+\frac{1}{3}e^{-6B\iota}+\ldots\right\}$$
$$+e^{-2A\iota}+\frac{1}{2}e^{-4A\iota}+\frac{1}{3}e^{-6A\iota}+\ldots$$

Now $\qquad e^{-2B\iota}=\cos 2B-\iota\sin 2B,\quad e^{-2A\iota}=\cos 2A-\iota\sin 2A,$

and so on. Then, as the real and imaginary parts of the expression must be separately equal, we have

$$\log b-\log a=\cos 2A-\cos 2B+\frac{1}{2}(\cos 4A-\cos 4B)$$
$$+\frac{1}{3}(\cos 6A-\cos 6B)+\ldots$$

6 By Art 294,

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots$$
$$=\frac{2}{1\cdot 3}+\frac{2}{5\cdot 7}+\frac{2}{9\cdot 11}+\ldots,$$

therefore
$$\frac{\pi}{8}=\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\frac{1}{9\cdot 11}+\ldots$$

7 Let $\qquad A+B\iota=\log(m+n\iota),$

therefore $\qquad e^{A+B\iota}=m+n\iota,$

therefore $\qquad m+n\iota=e^{A}e^{B\iota}=e^{A}(\cos B+\iota\sin B),$

therefore $\qquad m=e^{A}\cos B,$

and $\qquad n=e^{A}\sin B$

By division $\qquad \dfrac{n}{m}=\tan B$

By squaring and adding $\qquad m^2+n^2=e^{2A};$

therefore $\qquad 2A=\log(m^2+n^2)$

8 $$\cos(\theta+\phi\iota)=\cos\theta\cos\phi\iota-\sin\theta\sin\phi\iota$$
$$=\cos\theta\frac{e^{-\phi}+e^{\phi}}{2}-\sin\theta\frac{e^{-\phi}-e^{\phi}}{2\iota}$$
$$=\cos\theta\frac{e^{-\phi}+e^{\phi}}{2}+\iota\sin\theta\frac{e^{-\phi}-e^{\phi}}{2},$$

this is of the form $\alpha+\beta\iota$ where

$$\alpha=\cos\theta\frac{e^{-\phi}+e^{\phi}}{2},\quad\text{and }\beta=\sin\theta\frac{e^{-\phi}-e^{\phi}}{2}$$

9 $$\sin(\theta+\phi\iota)=\sin\theta\cos\phi\iota+\cos\theta\sin\phi\iota$$
$$=\sin\theta\frac{e^{-\phi}+e^{\phi}}{2}+\cos\theta\frac{e^{-\phi}-e^{\phi}}{2\iota}$$
$$=\sin\theta\frac{e^{-\phi}+e^{\phi}}{2}-\iota\cos\theta\frac{e^{-\phi}-e^{\phi}}{2},$$

this is of the form $\alpha+\beta\iota$ where

$$\alpha=\sin\theta\frac{e^{-\phi}+e^{\phi}}{2},\quad\text{and }\beta=-\cos\theta\frac{e^{-\phi}-e^{\phi}}{2}.$$

10 $$\log u=(p+q\iota)\log(a+b\iota)$$
$$=(p+q\iota)\log\sqrt{(a^2+b^2)}\left\{\frac{a}{\sqrt{(a^2+b^2)}}+\frac{b\iota}{\sqrt{(a^2+b^2)}}\right\}$$
$$=(p+q\iota)\log\sqrt{(a^2+b^2)}\{\cos\gamma+\iota\sin\gamma\},$$

where $$\cos\gamma=\frac{a}{\sqrt{(a^2+b^2)}},\quad\text{and }\sin\gamma=\frac{b}{\sqrt{(a^2+b^2)}},$$

$$=(p+q\iota)\log e^{\gamma\iota}\sqrt{(a^2+b^2)}$$
$$=(p+q\iota)\{\log e^{\gamma\iota}+\log\sqrt{(a^2+b^2)}\}$$
$$=(p+q\iota)\{\gamma\iota+\log\sqrt{(a^2+b^2)}\}$$
$$=p\log\sqrt{(a^2+b^2)}-q\gamma+\{p\gamma+q\log\sqrt{(a^2+b^2)}\}\iota$$

This is of the form $\alpha+\beta\iota$, where

$$\alpha=p\log\sqrt{(a^2+b^2)}-q\gamma,\quad\text{and }\beta=p\gamma+q\log\sqrt{(a^2+b^2)}$$

11 By Example 10 we can express $\log(a+b\iota)^{p+q\iota}$ in the form $\alpha+\beta\iota$, therefore

$$(a+b\iota)^{p+q\iota}=e^{\alpha+\beta\iota}=e^{\alpha}e^{\beta\iota}=e^{\alpha}(\cos\beta+\iota\sin\beta),$$

and this is of the form $\lambda+\iota\mu$, where $\lambda=e^{\alpha}\cos\beta$, and $\mu=e^{\alpha}\sin\beta$

12 $$\{\sin(\alpha-\theta)+e^{\alpha\iota}\sin\theta\}^n=\{\sin(\alpha-\theta)+(\cos\alpha+\iota\sin\alpha)\sin\theta\}^n$$
$$=(\sin\alpha\cos\theta+\iota\sin\alpha\sin\theta)^n=\sin^n\alpha(\cos\theta+\iota\sin\theta)^n$$
$$=\sin^n\alpha(\cos n\theta+\iota\sin n\theta)$$

Again $\sin^{n-1}\alpha\,\{\sin(\alpha-n\theta)+e^{\alpha\iota}\sin n\theta\}$

$$=\sin^{n-1}\alpha\,\{\sin(\alpha-n\theta)+(\cos\alpha+\iota\sin\alpha)\sin n\theta\}$$
$$=\sin^{n-1}\alpha\,\{\sin\alpha\cos n\theta+\iota\sin\alpha\sin n\theta\}$$
$$=\sin^{n}\alpha\,(\cos n\theta+\iota\sin n\theta)$$

thus the two expressions agree

In a similar way we may proceed when we take the lower sign in the expressions

13 $\tan(\alpha+\iota\beta)=\dfrac{\sin(\alpha+\iota\beta)}{\cos(\alpha+\iota\beta)}$

$$=\frac{\sin\alpha\,(e^{\beta}+e^{-\beta})+\iota\cos\alpha\,(e^{\beta}-e^{-\beta})}{\cos\alpha\,(e^{\beta}+e^{-\beta})-\iota\sin\alpha\,(e^{\beta}-e^{-\beta})}$$

$$=\frac{\{\sin\alpha\,(e^{\beta}+e^{-\beta})+\iota\cos\alpha\,(e^{\beta}-e^{-\beta})\}\,\{\cos\alpha\,(e^{\beta}+e^{-\beta})+\iota\sin\alpha\,(e^{\beta}-e^{-\beta})\}}{\cos^2\alpha\,(e^{\beta}+e^{-\beta})^2+\sin^2\alpha\,(e^{\beta}-e^{-\beta})^2}$$

$$=\frac{\sin\alpha\cos\alpha\{(e^{\beta}+e^{-\beta})^2-(e^{\beta}-e^{-\beta})^2\}+\iota\,(e^{2\beta}-e^{-2\beta})\,(\sin^2\alpha+\cos^2\alpha)}{e^{2\beta}+e^{-2\beta}+2\,(\cos^2\alpha-\sin^2\alpha)}$$

$$=\frac{2\sin2\alpha+\iota\,(e^{2\beta}-e^{-2\beta})}{e^{2\beta}+2\cos2\alpha+e^{-2\beta}}$$

14 Put $a=r\cos\theta$, $b=r\sin\theta$, so that

$$r=(a^2+b^2)^{\frac{1}{2}}\ \text{and}\ \tan\theta=\frac{b}{a}.$$

Then $(a+b\iota)^{\frac{m}{n}}=r^{\frac{m}{n}}(\cos\theta+\iota\sin\theta)^{\frac{m}{n}}=r^{\frac{m}{n}}\left(\cos\dfrac{m\theta}{n}+\iota\sin\dfrac{m\theta}{n}\right),$

and $(a-b\iota)^{\frac{m}{n}}=r^{\frac{m}{n}}(\cos\theta-\iota\sin\theta)^{\frac{m}{n}}=r^{\frac{m}{n}}\left(\cos\dfrac{m\theta}{n}-\iota\sin\dfrac{m\theta}{n}\right),$

$$\therefore\ (a+b\iota)^{\frac{m}{n}}+(a-b\iota)^{\frac{m}{n}}=2r^{\frac{m}{n}}\cos\frac{m\theta}{n}$$

$$=2\,(a^2+b^2)^{\frac{m}{2n}}\cos\left(\frac{m}{n}\tan^{-1}\frac{b}{a}\right)$$

15 Since $2m=\log_e a\,.\log_a\dfrac{x}{y}=\log_e\dfrac{x}{y},$

$$m=\log_e\sqrt{\frac{x}{y}},\ \text{or}\ e^{m}=\sqrt{\frac{x}{y}}.$$

Now $$\frac{x+y}{2}=\frac{1}{2}\left(\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\right)\sqrt{xy}=\frac{1}{2}(e^m+e^{-m})\sqrt{xy}.$$

$$\log_a\frac{x+y}{2}=\log_a\sqrt{xy}+\log_a\frac{1}{2}(e^m+e^{-m})$$

16 $$\frac{\sin(\alpha+\beta\iota)}{\sin(\alpha-\beta\iota)}+\frac{\sin(\alpha-\beta\iota)}{\sin(\alpha+\beta\iota)}=\frac{\sin^2(\alpha+\beta\iota)+\sin^2(\alpha-\beta\iota)}{\sin(\alpha+\beta\iota)\sin(\alpha-\beta\iota)}$$

$$=\frac{2\sin^2\alpha\cos^2\beta\iota+2\cos^2\alpha\sin^2\beta\iota}{\sin^2\alpha-\sin^2\beta\iota}$$

$$=\frac{2\sin^2\alpha(e^\beta+e^{-\beta})^2-2\cos^2\alpha(e^\beta-e^{-\beta})^2}{4\sin^2\alpha+(e^\beta-e^{-\beta})^2}$$

$$=\frac{4(\sin^2\alpha+\cos^2\alpha)-2(e^{2\beta}+e^{-2\beta})(\cos^2\alpha-\sin^2\alpha)}{4\sin^2\alpha+(e^\beta-e^{-\beta})^2}$$

$$=\frac{4-2(e^{2\beta}+e^{-2\beta})\cos 2\alpha}{4\sin^2\alpha+(e^\beta-e^{-\beta})^2}$$

17 Taking logarithms

$$(\alpha+\beta\iota)\log_e a=(p+q\iota)\log_e(x+y\iota)=(p+q\iota)(f+g\iota),$$

where $$f+g\iota=\log_e(x+y\iota),$$

$$g=\tan^{-1}\frac{y}{x},\qquad f=\frac{1}{2}\log_e(x^2+y^2),\ (\text{Ex } 7)$$

Hence $$\alpha\log_e a=pf-qg=\frac{1}{2}p\log_e(x^2+y^2)-q\tan^{-1}\frac{y}{x}.$$

$$\alpha=\frac{1}{2}p\log_a(x^2+y^2)-q\tan^{-1}\frac{y}{x}\log_a e$$

Also $$f+g\iota=\frac{\alpha+\beta\iota}{p+q\iota}\log_e a$$

$$=\frac{(\alpha+\beta\iota)(p-q\iota)}{p^2+q^2}\log_e a,$$

$$f=\frac{\alpha p+\beta q}{p^2+q^2}\log_e a,$$

$$\log_a(x^2+y^2)=\frac{2(\alpha p+\beta q)}{p^2+q^2}$$

18
$$\sin(\theta+\phi\iota)=e^{\alpha+\beta\iota},$$
$$e^{\iota\theta-\phi}-e^{-\iota\theta+\phi}=2\iota e^{\alpha}\cdot e^{\beta\iota},$$
$$e^{-\phi}(\cos\theta+\iota\sin\theta)-e^{\phi}(\cos\theta-\iota\sin\theta)=2\iota e^{\alpha}(\cos\beta+\iota\sin\beta),$$
$$\therefore (e^{-\phi}-e^{\phi})\cos\theta=-2e^{\alpha}\sin\beta \quad\ldots\quad \text{(i)},$$
and
$$(e^{-\phi}+e^{\phi})\sin\theta=\ 2e^{\alpha}\cos\beta \quad\ldots\quad \text{(ii)}$$

Square and add,
$$e^{-2\phi}+e^{2\phi}-2\cos 2\theta=4e^{2\alpha},$$
or
$$2\cos 2\theta=e^{2\phi}+e^{-2\phi}-4e^{2\alpha}.$$

From (i) and (ii) we have by division
$$\frac{(1-e^{2\phi})\cos\theta}{(1+e^{2\phi})\sin\theta}=\frac{-\sin\beta}{\cos\beta};$$
$$\cos\theta\cos\beta+\sin\theta\sin\beta=e^{2\phi}(\cos\theta\cos\beta-\sin\theta\sin\beta);$$
$$\cos(\theta-\beta)=e^{2\phi}\cos(\theta+\beta)$$

19 Let $\log\log(\alpha+\beta\iota)=A+B\iota$. From Ex. 7,
$$\log(\alpha+\beta\iota)=\log\sqrt{(\alpha^2+\beta^2)}+\iota\tan^{-1}\frac{\beta}{\alpha};$$
$$\therefore e^{A+B\iota}=\log(\alpha+\beta\iota)=\log\sqrt{(\alpha^2+\beta^2)}+\iota\tan^{-1}\frac{\beta}{\alpha},$$
i e
$$e^{A}(\cos B+\iota\sin B)=\log\sqrt{(\alpha^2+\beta^2)}+\iota\tan^{-1}\frac{\beta}{\alpha}$$
$$e^{A}\cos B=\log\sqrt{(\alpha^2+\beta^2)},$$
$$e^{A}\sin B=\tan^{-1}\frac{\beta}{\alpha}$$

Square and add, therefore
$$e^{2A}=\{\log\sqrt{(\alpha^2+\beta^2)}\}^2+\left(\tan^{-1}\frac{\beta}{\alpha}\right)^2,$$
or
$$A=\frac{1}{2}\log\left[\{\log\sqrt{(\alpha^2+\beta^2)}\}^2+\left(\tan^{-1}\frac{\beta}{\alpha}\right)^2\right].$$

Also
$$\tan B=\frac{\tan^{-1}\frac{\beta}{\alpha}}{\log\sqrt{(\alpha^2+\beta^2)}},$$
or
$$B=\tan^{-1}\frac{\tan^{-1}\frac{\beta}{\alpha}}{\log\sqrt{(\alpha^2+\beta^2)}}.$$

20 $$\cos\left(\frac{\pi}{4}+\iota\phi\right)=\frac{1-\tan^2\left(\frac{\pi}{8}+\frac{1}{2}\phi\iota\right)}{1+\tan^2\left(\frac{\pi}{8}+\frac{1}{2}\phi\iota\right)}=\frac{1-\alpha-\beta\iota}{1+\alpha+\beta\iota}$$

$$\cos(\phi\iota)-\sin(\phi\iota)=\frac{\sqrt{2}\,(1-\alpha-\beta\iota)}{1+\alpha+\beta\iota},$$

$$e^{-\phi}+e^{\phi}+\iota(e^{-\phi}-e^{\phi})=\frac{2\sqrt{2}\,(1-\alpha-\beta\iota)}{1+\alpha+\beta\iota}$$

Multiply up and equate real and imaginary parts, therefore

$$\left.\begin{aligned}&(e^{\phi}+e^{-\phi})(1+\alpha)+\beta(e^{\phi}-e^{-\phi})=2\sqrt{2}\,(1-\alpha)\\ \text{and}\quad&(e^{\phi}+e^{-\phi})\beta-(1+\alpha)(e^{\phi}-e^{-\phi})=-2\sqrt{2}\beta\end{aligned}\right\}$$

$$\left.\begin{aligned}&\alpha(e^{\phi}+e^{-\phi}+2\sqrt{2})+\beta(e^{\phi}-e^{-\phi})=2\sqrt{2}-e^{\phi}-e^{-\phi}\\ \text{and}\quad&-\alpha(e^{\phi}-e^{-\phi})+\beta(e^{\phi}+e^{-\phi}+2\sqrt{2})=e^{\phi}-e^{-\phi}\end{aligned}\right\}\quad(1),$$

$$\alpha\{(e^{\phi}+e^{-\phi}+2\sqrt{2})^2+(e^{\phi}-e^{-\phi})^2\}=8-(e^{\phi}+e^{-\phi})^2-(e^{\phi}-e^{-\phi})^2,$$

$$\alpha\{2e^{2\phi}+2e^{-2\phi}+4\sqrt{2}\,(e^{\phi}+e^{-\phi})+8\}=8-2e^{2\phi}-2e^{-2\phi},$$

$$\alpha\{(e^{\phi}+\sqrt{2})^2+(e^{-\phi}+\sqrt{2})^2\}=4-e^{2\phi}-e^{-2\phi},$$

which gives the first result

Again, eliminating α from equations (1) we obtain similarly

$$2\beta\{(e^{\phi}+\sqrt{2})^2+(e^{-\phi}+\sqrt{2})^2\}=4\sqrt{2}\,(e^{\phi}-e^{-\phi}),$$

which gives the second result.

21 $$-1=\cos\pi+\iota\sin\pi$$
$$=e^{\pi\iota},$$
$$\log_e(-1)=2n\pi\iota+\pi\iota$$
$$=(2n+1)\pi\iota$$

22 Let $$\sin^{-1}\sqrt{-1}=\theta$$
$$\sin\theta=\sqrt{-1},$$
and $$\cos\theta=\sqrt{2},$$
$$e^{\theta\iota}=\cos\theta+\iota\sin\theta=\sqrt{2}-1,$$
$$\theta\iota=2n\pi\iota+\log(\sqrt{2}-1),\qquad\text{(Art 310)}$$
$$\theta=2n\pi-\iota\log(\sqrt{2}-1)$$

Different forms of the value of θ may be found in consequence of the double sign which must be given to the square root

23. Let
$$\sin^{-1} a=\theta,$$
$$\therefore \sin\theta=a,$$
$$\cos\theta=\iota\sqrt{(a^2-1)},$$
$$\therefore e^{\iota\theta}=\cos\theta+\iota\sin\theta=\iota\{\sqrt{(a^2-1)}+a\},$$
$$\therefore \iota\theta=2n\pi\iota+\log(\sqrt{a^2-1}+a)+\log\sqrt{-1}$$
$$=\iota\frac{\kappa\pi}{2}+\log(\sqrt{a^2-1}+a). \qquad \text{(Ex. 21.)}$$
$$\therefore \theta=\frac{\kappa\pi}{2}-\iota\log(\sqrt{a^2-1}+a).$$

24 Let
$$\iota^\iota=x,$$
$$\iota\log_e\sqrt{-1}=\log_e x;$$
$$\log_e x=\iota^2(2n+1)\frac{\pi}{2}=-(2n+1)\frac{\pi}{2}. \qquad \text{(Ex. 21)}$$
$$x=e^{-(2n+1)\frac{\pi}{2}}.$$

25. Let
$$\sin^{-1}(\iota a)=\theta,$$
$$\sin\theta=\iota a,$$
$$\cos\theta=\sqrt{(1-\sin^2\theta)}=\sqrt{(1+a^2)},$$
$$e^{\theta\iota}=\sqrt{(1+a^2)}-a;$$
$$\theta\iota=2n\pi\iota+\log(\sqrt{1+a^2}-a),$$
$$\therefore \theta=2n\pi-\iota\log(\sqrt{1+a^2}-a)$$
$$=2n\pi+\iota\log(\sqrt{1+a^2}+a).$$

26 From the given equation we must have
$$\iota^{A+B\iota}=A+B\iota.$$
Now
$$\iota=\cos\frac{\pi}{2}+\iota\sin\frac{\pi}{2}=e^{\frac{\pi\iota}{2}}.$$
Hence
$$A+B\iota=e^{\frac{\pi A}{2}\iota-\frac{\pi B}{2}}=e^{-\frac{\pi B}{2}}\cdot e^{\frac{\pi A}{2}\iota}$$
$$=e^{-\frac{\pi B}{2}}\left(\cos\frac{\pi A}{2}+\iota\sin\frac{\pi A}{2}\right).$$
$$A=e^{-\frac{\pi B}{2}}\cos\frac{\pi A}{2},$$
$$B=e^{-\frac{\pi B}{2}}\sin\frac{\pi A}{2}.$$

Square and add; therefore

$$A^2+B^2=e^{-\pi B}$$

Also, by division, $$\frac{B}{A}=\tan\frac{\pi A}{2}.$$

27 Put $$x-a=r_1\cos\theta_1, \quad y=r_1\sin\theta_1 \qquad \text{(i)},$$

and $$x+a=r_2\cos\theta_2, \quad y=r_2\sin\theta_2 \qquad \text{(ii)}.$$

Then $$\log\frac{x+\iota y-a}{x+\iota y+a}=\log\frac{r_1(\cos\theta_1+\iota\sin\theta_1)}{r_2(\cos\theta_2+\iota\sin\theta_2)}$$

$$=\log\frac{r_1e^{\iota\theta_1}}{r_2e^{\iota\theta_2}}$$

$$=\log\frac{r_1}{r_2}+\iota(\theta_1-\theta_2).$$

From (i), $$r_1^2=(x-a)^2+y^2, \ \tan\theta_1=\frac{y}{x-a}.$$

From (ii), $$r_2^2=(x+a)^2+y^2, \ \tan\theta_2=\frac{y}{x+a},$$

$$\log\frac{x+\iota y-a}{x+\iota y+a}=\frac{1}{2}\log\frac{(x-a)^2+y^2}{(x+a)^2+y^2}$$

$$+\iota\left\{\tan^{-1}\frac{y}{x-a}-\tan^{-1}\frac{y}{x+a}\right\}.$$

28 Making the same substitutions as in the last example, we have

$$\log(x+\iota y-a)(x+\iota y+a)=\log(r_1e^{\iota\theta_1}\ r_2e^{\iota\theta_2})$$

$$=\log(r_1r_2)+\iota(\theta_1+\theta_2)$$

$$=\log[(x-a)^2+y^2]^{\frac{1}{2}}[(x+a)^2+y^2]^{\frac{1}{2}}$$

$$+\iota\left\{\tan^{-1}\frac{y}{x-a}+\tan^{-1}\frac{y}{x+a}\right\}.$$

29. Put $a=r\cos\theta$, $b=r\sin\theta$, so that $\tan\theta=\dfrac{b}{a}$. Then

$$\tan\left\{\iota\log\frac{a-b\iota}{a+b\iota}\right\}=\tan\left\{\iota\log\frac{re^{-\iota\theta}}{re^{\iota\theta}}\right\}=\tan\{\iota(-2\iota\theta)\}$$

$$=\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2ab}{a^2-b^2}$$

30 The expression in brackets

$$=\frac{1}{2}\left(e^{\theta\iota-\phi}+e^{-\theta\iota+\phi}+e^{\theta\iota+\phi}-e^{-\theta\iota-\phi}\right)$$

$$=\frac{1}{2}e^{+\phi}\left(e^{\theta\iota}+e^{-\theta\iota}\right)+\frac{1}{2}e^{-\phi}\left(e^{\theta\iota}-e^{-\theta\iota}\right)$$

$$=e^{\phi}\cos\theta+\iota e^{-\phi}\sin\theta.$$

Put
$$e^{\phi}\cos\theta=e^{\eta}\cos\xi,$$
$$e^{-\phi}\sin\theta=e^{\eta}\sin\xi;$$
$$e^{2\eta}=(e^{2\phi}\cos^2\theta+e^{-2\phi}\sin^2\theta),$$

or
$$\eta=\frac{1}{2}\log(e^{2\phi}\cos^2\theta+e^{-2\phi}\sin^2\theta),$$

and
$$\tan\xi=\frac{e^{-\phi}\sin\theta}{e^{+\phi}\cos\theta}=e^{-2\phi}\tan\theta$$

Therefore $\{\cos(\theta+\phi\iota)+\iota\sin(\theta-\phi\iota)\}^{\alpha+\beta\iota}$

$$=\{e^{\eta}\cos\xi+\iota e^{\eta}\sin\xi\}^{\alpha+\beta\iota}$$

$$=\{e^{\eta}\cdot e^{\iota\xi}\}^{\alpha+\beta\iota}=e^{(\eta+\iota\xi)(\alpha+\beta\iota)}$$

$$=e^{\eta\alpha-\xi\beta}\,e^{\iota(\xi\alpha+\eta\beta)}=e^{\eta\alpha-\xi\beta}\{\cos(\xi\alpha+\eta\beta)+\iota\sin(\xi\alpha+\eta\beta)\}.$$

This is the form required.

31 $\cos^2\theta=\left(1-\frac{\theta^2}{\underline{|2}}+\frac{\theta^4}{\underline{|4}}-\quad\right)\left(1-\frac{\theta^2}{\underline{|2}}+\frac{\theta^4}{\underline{|4}}-\quad\right)$

$$=\Sigma(-1)^n\;\theta^{2n}\left(\frac{1}{\underline{|2n}}+\frac{1}{\underline{|2}\,\underline{|2n-2}}+\frac{1}{\underline{|4}\,\underline{|2n-4}}+\dots\right),$$

$$\sin^2\theta=\left(\theta-\frac{\theta^3}{\underline{|3}}+\quad\right)\left(\theta-\frac{\theta^3}{\underline{|3}}+\cdot\right)$$

$$=\Sigma(-1)^{n-1}\theta^{2n}\left(\frac{1}{\underline{|2n-1}}+\frac{1}{\underline{|3}\,\underline{|2n-3}}+\quad\right)$$

Hence the coefficient of θ^{2n} in $\cos^2\theta+\sin^2\theta$

$$=(-1)^n\left(\frac{1}{\underline{|2n}}-\frac{1}{\underline{|1}\,\underline{|2n-1}}+\frac{1}{\underline{|2}\,\underline{|2n-2}}-\dots\right)$$

$$=(-1)^n \cdot \frac{1}{\lfloor 2n} \left\{1-\frac{2n}{1}+\frac{2n(2n-1)}{1 \cdot 2}- \quad \right\}$$

$$=(-1)^n \cdot \frac{1}{\lfloor 2n}(1-1)^{2n}=0.$$

$$\cos^2\theta+\sin^2\theta=1.$$

32 $$e^u=\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)=\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}=\sec x+\tan x.$$

Put $$\sec x=\cos\theta.$$

Then $$\tan x=\iota\sin\theta \text{ and } \sin x=\iota\tan\theta;$$

$$\therefore e^u=\sec x+\tan x=\cos\theta+\iota\sin\theta=e^{\iota\theta},$$

$$\therefore u=\iota\theta$$

Hence $$e^{\iota x}=\cos x+\iota\sin x=\sec\theta-\tan\theta$$

$$=\sec\iota u+\tan\iota u=\tan\left(\frac{\pi}{4}+\frac{\iota u}{2}\right);$$

$$\iota x=\log\tan\left(\frac{\pi}{4}+\frac{\iota u}{2}\right)$$

$$=\iota u+a_3(\iota u)^3+a_5(\iota u)^5+ \ldots,$$

$$x=u-a_3u^3+a_5u^5- \ldots.$$

33 $$2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{\frac{2}{3}}{1-\frac{1}{9}}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{28}}=\tan^{-1}1=\frac{\pi}{4},$$

$$\frac{\pi}{4}=2\left(\frac{1}{3}-\frac{1}{3}\cdot\frac{1}{3^3}+\frac{1}{5}\cdot\frac{1}{3^5}- \quad\right)+\left(\frac{1}{7}-\frac{1}{3}\cdot\frac{1}{7^3}+ \quad\right) \text{ (Art. 315)}$$

$$=\frac{17}{21}+ \quad +\frac{(-1)^{n+1}}{2n-1}\left(\frac{2}{3}\cdot 3^{2-2n}+7^{1-2n}\right)$$

$$=\frac{17}{21}+ \quad +\frac{(-1)^{n+1}}{2n-1}\left(\frac{2}{3}\cdot 9^{1-n}+7^{1-2n}\right)$$

34. Since $\theta=\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta-$

$$\theta^2=\left(\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta-\quad\right)\left(\tan\theta-\frac{1}{3}\tan^3\theta+\quad\right).$$

Multiplying out, the coefficient of $(-1)^{n+1}\tan^{2n}\theta$

$$=\frac{1}{1}\,\frac{1}{2n-1}+\frac{1}{2n-3}\cdot\frac{1}{3}+\frac{1}{2n-5}\cdot\frac{1}{5}+\quad+\frac{1}{1}\cdot\frac{1}{2n-1}$$

$$=\left(1+\frac{1}{2n-1}+\frac{1}{3}+\frac{1}{2n-3}+\frac{1}{5}+\frac{1}{2n-5}+\cdot\quad+\frac{1}{2n-1}+1\right)\frac{1}{2n}$$

$$=\left(1+\frac{1}{3}+\frac{1}{5}+\quad+\frac{1}{2n-1}\right)\frac{1}{n},$$

$$\theta^2=\tan^2\theta-\left(1+\frac{1}{3}\right)\frac{\tan^4\theta}{2}+\left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{\tan^6\theta}{3}-\quad.$$

35
$$\cos 2a=\frac{\tan\phi}{\tan(\theta+\phi)}.$$

$$\tan^2 a=\frac{1-\cos 2a}{1+\cos 2a}=\frac{\tan(\theta+\phi)-\tan\phi}{\tan(\theta+\phi)+\tan\phi}$$

$$=\frac{\sin(\theta+\phi-\phi)}{\sin(\theta+\phi+\phi)},$$

$$\sin\theta=\tan^2 a\sin(\theta+2\phi)$$

Hence (see Art 321),

$$\theta=\tan^2 a\sin 2\phi+\frac{1}{2}\tan^4 a\sin 4\phi+\frac{1}{3}\tan^6 a\sin 6\phi+\dots.$$

36 Let $a=r\cos\alpha$, $b=r\sin\alpha$, so that

$$\tan\alpha=\frac{b}{a},\quad r=\sqrt{(a^2+b^2)}$$

Then $\log(a\cos\theta+b\sin\theta)=\log\{r\cos(\theta-\alpha)\}$

$$=\log r+\log\frac{1}{2}\{e^{(\theta-\alpha)\iota}+e^{-(\theta-\alpha)\iota}\}$$

$$=\log r+\log\frac{1}{2}+(\theta-\alpha)\iota+\log\{1+e^{-2(\theta-\alpha)\iota}\}$$

$$=\log r+\log\frac{1}{2}+(\theta-\alpha)\iota+\left\{e^{-2(\theta-\alpha)\iota}-\frac{1}{2}e^{-4(\theta-\alpha)\iota}+\ .\right\}.$$

For the exponential terms write $\cos 2(\theta-\alpha)-\iota\sin 2(\theta-\alpha)$ &c. and equate the real parts (assuming that $\log(a\cos\theta+b\sin\theta)$ is real),

$$\log(a\cos\theta+b\sin\theta)=\log\frac{1}{2}\sqrt{(a^2+b^2)}+\cos 2(\theta-\alpha)-\frac{1}{2}\cos 4(\theta-\alpha)+.\ \ .$$

37 Let $\beta=\cos B+\iota\sin B$, so that $\frac{1}{\beta}=\cos B-\iota\sin B$

$$\begin{aligned}
2b^{-n}\cos nA &= (b\cos A-\iota b\sin A)^{-n}+(b\cos A+\iota b\sin A)^{-n}\\
&=(c-a\cos B-\iota a\sin B)^{-n}+(c-a\cos B+\iota a\sin B)^{-n}\\
&=(c-a\beta)^{-n}+\left(c-\frac{a}{\beta}\right)^{-n}\\
&=\frac{1}{c^n}\left\{1+n\,\frac{a}{c}\beta+\frac{n(n+1)}{1\,.\,2}\,\frac{a^2}{c^2}\,.\,\beta^2+\ldots\right.\\
&\qquad\left.+1+n\,.\,\frac{a}{c}\,.\,\frac{1}{\beta}+\frac{n(n+1)}{1\,.\,2}\,\frac{a^2}{c^2}\,\frac{1}{\beta^2}+\ldots\right\}\\
&=\frac{2}{c^n}\left\{1+n\,\frac{a}{c}\cos B+\frac{n(n+1)}{1\,.\,2}\,.\,\frac{a^2}{c^2}\cos 2B+\ldots\right\},
\end{aligned}$$

since
$$\beta^r+\frac{1}{\beta^r}=2\cos rB.$$

$$\begin{aligned}
2b^{-n}\iota\sin nA &= (b\cos A-\iota b\sin A)^{-n}-(b\cos A-\iota b\sin A)^{-n}\\
&=(c-a\beta)^{-n}-\left(c-\frac{a}{\beta}\right)^{-n}, \text{ as before,}\\
&=\frac{1}{c^n}\left\{n\,\frac{a}{c}\left(\beta-\frac{1}{\beta}\right)+\frac{n(n+1)}{1\,.\,2}\,.\,\frac{a^2}{c^2}\left(\beta^2-\frac{1}{\beta^2}\right)+\ldots\right\}\\
&=\frac{2\iota}{c^n}\left\{n\,.\,\frac{a}{c}\sin B+\frac{n(n+1)}{1\,.\,2}\,.\,\frac{a^2}{c^2}\sin 2B+\ldots\right\}.
\end{aligned}$$

38
$$\begin{aligned}
2\cos\left(\theta+\frac{\pi}{4}\right) &= e^{\iota\left(\theta+\frac{\pi}{4}\right)}+e^{-\iota\left(\theta+\frac{\pi}{4}\right)}\\
&=e^{-\iota\left(\theta+\frac{\pi}{4}\right)}\left\{1+e^{\iota\left(2\theta+\frac{\pi}{2}\right)}\right\};
\end{aligned}$$

$$\begin{aligned}
\log 2+\log\cos\left(\theta+\frac{\pi}{4}\right) &= -\iota\left(\theta+\frac{\pi}{4}\right)+\log\left\{1+e^{\iota\left(2\theta+\frac{\pi}{2}\right)}\right\}\\
&=-\iota\left(\theta+\frac{\pi}{4}\right)+e^{\iota\left(2\theta+\frac{\pi}{2}\right)}-\frac{1}{2}e^{2\iota\left(2\theta+\frac{\pi}{2}\right)}+\ldots
\end{aligned}$$

For the exponentials put

$$\cos\left(2\theta+\frac{\pi}{2}\right)+\iota\sin\left(2\theta+\frac{\pi}{2}\right)\ \&c\ ;$$

therefore, assuming that $\cos\left(\theta+\frac{\pi}{4}\right)$ is positive, we have by equating the real parts,

$$\begin{aligned}
&\log\cos\left(\theta+\frac{\pi}{4}\right)\\
&\quad=-\log 2+\cos\left(2\theta+\frac{\pi}{2}\right)-\frac{1}{2}\cos(4\theta+\pi)+\frac{1}{3}\cos\left(6\theta+\frac{3\pi}{2}\right)-\ldots\\
&\quad=-\log 2-\sin 2\theta+\frac{1}{2}\cos 4\theta+\frac{1}{3}\sin 6\theta-\frac{1}{4}\cos 8\theta-\frac{1}{5}\sin 10\theta+\ldots.
\end{aligned}$$

39 Since imaginary roots occur in pairs, if $\kappa e^{\iota\theta}$ be a root, then $\kappa e^{-\iota\theta}$ is also a root, therefore

$$\kappa^3 e^{3\theta\iota} + 3q\kappa e^{\theta\iota} + r = 0 \qquad (\text{i}),$$

and

$$\kappa^3 e^{-3\theta\iota} + 3q\kappa e^{-\theta\iota} + r = 0 \qquad (\text{ii}).$$

Subtract; thus

$$\kappa^3 (e^{3\theta\iota} - e^{-3\theta\iota}) + 3q\kappa (e^{\theta\iota} - e^{-\theta\iota}) = 0;$$

$$\therefore 3q = -\kappa^2 (e^{2\theta\iota} + 1 + e^{-2\theta\iota}) = -\kappa^2 (1 + 2\cos 2\theta)$$

Multiply (i) and (ii) by $e^{-\theta\iota}$, $e^{\theta\iota}$ respectively and subtract; therefore

$$\kappa^3 (e^{2\theta\iota} - e^{-2\theta\iota}) - r (e^{\theta\iota} - e^{-\theta\iota}) = 0;$$

$$r = \kappa^3 (e^{\theta\iota} + e^{-\theta\iota}) = 2\kappa^3 \cos\theta.$$

40 Let $\tan^{-1}(\cos\theta + \iota\sin\theta) = \alpha + \beta\iota$,

$$e^{\iota\theta} = \tan(\alpha + \beta\iota),$$

$$\cos\theta = \frac{1}{2}(e^{\iota\theta} + e^{-\iota\theta}) = \frac{1}{2}\{\tan(\alpha+\beta\iota) + \cot(\alpha+\beta\iota)\}$$

$$= \frac{1}{2\sin(\alpha+\beta\iota)\cos(\alpha+\beta\iota)} = \frac{1}{\sin 2(\alpha+\beta\iota)},$$

$$2\sec\theta = \sin 2\alpha (e^{2\beta} + e^{-2\beta}) + \iota\cos 2\alpha (e^{+2\beta} - e^{-2\beta})$$

Now $\cos\theta$ is real and β cannot be zero, therefore $\cos 2\alpha = 0$,

$$\text{i e } 2\alpha = \pm\frac{\pi}{2} + 2n\pi, \text{ and } \sin 2\alpha = \pm 1.$$

$$e^{2\beta} + e^{-2\beta} = \pm 2\sec\theta,$$

$$e^{4\beta} \pm 2\sec\theta \cdot e^{2\beta} + \sec^2\theta = \sec^2\theta - 1.$$

$$e^{2\beta} = \pm\sec\theta \pm \tan\theta = \pm\tan\left(\frac{\pi}{4} \pm \frac{\theta}{2}\right),$$

$$\beta = \frac{1}{2}\log\left\{\pm\tan\left(\frac{\pi}{4} \pm \frac{\theta}{2}\right)\right\},$$

$$\tan^{-1}(\cos\theta + \iota\sin\theta) = n\pi \pm \frac{\pi}{4} + \frac{1}{2}\iota\log\left\{\pm\tan\left(\frac{\pi}{4} \pm \frac{\theta}{2}\right)\right\} \qquad (\text{i})$$

Now $\tan^{-1}(\cos\theta + \iota\sin\theta) = (\cos\theta + \iota\sin\theta) - \frac{1}{3}(\cos\theta + \iota\sin\theta)^3 +$

$$= \cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta - \ldots$$

$$+ \iota\left(\sin\theta - \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta - \ldots\right).$$

Hence from (i),

$$\cos\theta-\frac{1}{3}\cos 3\theta+\frac{1}{5}\cos 5\theta-\ \ldots=n\pi\pm\frac{\pi}{4} \qquad (1)$$

$$\sin\theta-\frac{1}{3}\sin 3\theta+\frac{1}{5}\sin 5\theta-\ \ldots=\frac{1}{2}\log\left\{\pm\tan\left(\frac{\pi}{4}\pm\frac{\theta}{2}\right)\right\}$$

In (1) if $\theta=0$ the series is equal to $\frac{\pi}{4}$, it is therefore always equal to $\frac{\pi}{4}$ when $\cos\theta$ is positive. If $\cos\theta$ be negative put $\pi-\phi$ for θ and we get

$$-\left(\cos\phi-\frac{1}{3}\cos 3\phi+\ \ldots\right)=-\frac{\pi}{4}$$

41 We have $\dfrac{e^{\theta\iota}-e^{-\theta\iota}}{e^{\theta\iota}+e^{-\theta\iota}}=\iota(x+\tan\alpha)$,

$$e^{2\theta\iota}=\frac{1+\iota x+\iota\tan\alpha}{1-\iota x-\iota\tan\alpha}=\frac{\cos\alpha+\iota\sin\alpha+\iota x\cos\alpha}{\cos\alpha-\iota\sin\alpha-\iota x\cos\alpha}$$

$$=\frac{e^{\alpha\iota}+\iota x\cos\alpha}{e^{-\alpha\iota}-\iota x\cos\alpha}=e^{2\alpha\iota}\ \frac{1+\iota x\cos\alpha\ e^{-\alpha\iota}}{1-\iota x\cos\alpha\,.\,e^{\alpha\iota}},$$

$$2\theta\iota=2\alpha\iota+\log(1+\iota x\cos\alpha e^{-\alpha\iota})-\log(1-\iota x\cos\alpha e^{\alpha\iota})$$

$$=2\alpha\iota+\iota x\cos\alpha e^{-\alpha\iota}-\frac{1}{2}(\iota x\cos\alpha e^{-\alpha\iota})^2+\ \ldots$$

$$+\iota x\cos\alpha\,.\,e^{\alpha\iota}+\frac{1}{2}(\iota x\cos\alpha e^{+\alpha\iota})^2+\ \ldots$$

$$=2\alpha\iota+\left\{\iota x\cos\alpha\,(e^{\alpha\iota}+e^{-\alpha\iota})-\frac{x^2\cos^2\alpha}{2}(e^{2\alpha\iota}-e^{-2\alpha\iota})\right.$$

$$\left.-\frac{x^3\cos^3\alpha}{3}\ \iota(e^{3\alpha\iota}+e^{-3\alpha\iota})+\frac{x^4\cos^4\alpha}{4}(e^{4\alpha\iota}-e^{-4\alpha\iota})+\ \ldots\right\}$$

$$=2\alpha\iota+2\iota\left(x\cos\alpha\ \cos\alpha-\frac{1}{2}x^2\cos^2\alpha\,.\,\sin 2\alpha\right.$$

$$\left.-\frac{1}{3}x^3\cos^3\alpha\cos 3\alpha+\frac{1}{4}x^4\cos^4\alpha\sin 4\alpha+\ \ldots\right),$$

$$\theta=\alpha+x\cos^2\alpha-\frac{1}{2}x^2\cos^2\alpha\sin 2\alpha-\frac{1}{3}x^3\cos^3\alpha\cos 3\alpha+\ \ldots$$

42 Let $\sin^{-1}(\cos\theta+\iota\sin\theta)=\alpha+\beta\iota$

Then $\cos\theta+\iota\sin\theta=\sin(\alpha+\beta\iota)$

$$=\frac{1}{2}\sin\alpha\,(e^{\beta}+e^{-\beta})+\frac{1}{2}\iota\cos\alpha\,(e^{\beta}-e^{-\beta}),$$

$$2\cos\theta=\sin\alpha\,(e^{\beta}+e^{-\beta}),$$

$$2\sin\theta=\cos\alpha\,(e^{\beta}-e^{-\beta}) \qquad (i)$$

$$\frac{\cos^2\theta}{\sin^2\alpha}-\frac{\sin^2\theta}{\cos^2\alpha}=1$$

$$\cos^2\theta\cos^2\alpha-\sin^2\theta\sin^2\alpha=\sin^2\alpha\cos^2\alpha;$$

$$\therefore (1-\sin^2\theta)\cos^2\alpha-\sin^2\theta\sin^2\alpha=\cos^2\alpha-\cos^4\alpha;$$

$$\therefore \sin^2\theta=\cos^4\alpha.$$

Therefore one value of α is $\cos^{-1}\sqrt{(\sin\theta)}$.

Substituting in equation (i),

$$e^{\beta}-e^{-\beta}=2\sqrt{\sin\theta},$$

$$e^{2\beta}-2e^{\beta}\sqrt{\sin\theta}+\sin\theta=1+\sin\theta;$$

$$e^{\beta}=\sqrt{\sin\theta}+\sqrt{1+\sin\theta},$$

or
$$\beta=\log(\sqrt{\sin\theta}+\sqrt{1+\sin\theta})$$

Hence one value of $\sin^{-1}(\cos\theta+\iota\sin\theta)$ is

$$\cos^{-1}\sqrt{(\sin\theta)}+\iota\log(\sqrt{\sin\theta}+\sqrt{1+\sin\theta}).$$

43 If $\log(1+\iota\tan\alpha)=A+B\iota$,

$$1+\iota\tan\alpha=e^{A}\cdot e^{B\iota}=e^{A}(\cos B+\iota\sin B),$$

$$\therefore 1=e^{A}\cos B,$$

$$\tan\alpha=e^{A}\sin B;$$

$$\sec^2\alpha=e^{2A},\text{ or } A=\log\sec\alpha,$$

and
$$\tan\alpha=\tan B,$$

$$\therefore B=n\pi+\alpha.$$

$$\log(1+\iota\tan\alpha)=\log\sec\alpha+\iota(n\pi+\alpha).$$

Hence, expanding $\log(1+\iota\tan\alpha)$, we have

$$\iota\left(\tan\alpha-\frac{1}{3}\tan^3\alpha+\quad\right)+\frac{1}{2}\tan^2\alpha-\frac{1}{4}\tan^4\alpha+\frac{1}{6}\tan^6\alpha-$$

$$=\log\sec\alpha+\iota(n\pi+\alpha).$$

Therefore, equating real and imaginary parts,

$$n\pi+\alpha=\tan\alpha-\frac{1}{3}\tan^3\alpha+$$

$$\log\cos\alpha=-\frac{1}{2}\tan^2\alpha+\frac{1}{4}\tan^4\alpha-\frac{1}{6}\tan^6\alpha+\ldots$$

44 $e^{\theta}(\cos\theta+\iota\sin\theta)=e^{\theta+\iota\theta}=e^{\theta\sqrt{2}\left(\cos\frac{\pi}{4}+\iota\sin\frac{\pi}{4}\right)}$

$$=1+\theta\sqrt{2}\left(\cos\frac{\pi}{4}+\iota\sin\frac{\pi}{4}\right)+\frac{\theta^2 . 2}{\underline{|2}}\left(\cos\frac{2\pi}{4}+\iota\sin\frac{2\pi}{4}\right)$$

$$+\frac{\theta^3\ 2^{\frac{3}{2}}}{\underline{|3}}\left(\cos\frac{3\pi}{4}+\iota\sin\frac{3\pi}{4}\right)+\ ,$$

$$e^{\theta}\cos\theta=1+\theta\sqrt{2}\cos\frac{\pi}{4}+\frac{\theta^2\ 2}{\underline{|2}}\cos\frac{2\pi}{4}+\frac{\theta^3 2^{\frac{3}{2}}}{\underline{|3}}\cos\frac{3\pi}{4}+\ ,$$

$$e^{\theta}\sin\theta=\theta\sqrt{2}\sin\frac{\pi}{4}+\frac{\theta^2\ 2}{\underline{|2}}\sin\frac{2\pi}{2}+\ldots.$$

45 Let $\iota\cos^{-1}(\sin\theta+\cos\theta)=A$,

$$\sin\theta+\cos\theta=\cos(-\iota A)=\frac{1}{2}(e^{A}+e^{-A})$$

$$e^{2A}-2(\sin\theta+\cos\theta)\,e^{A}+(\sin\theta+\cos\theta)^2=-1+(\sin\theta+\cos\theta)^2=\sin 2\theta,$$

$$e^{A}=\sin\theta+\cos\theta\pm\sqrt{\sin 2\theta},$$

$$A=\log(\sin\theta+\cos\theta\pm\sqrt{\sin 2\theta})$$

46 It is clear from geometrical considerations that the value of θ which satisfies the equation is near to $\frac{\pi}{2}$. Let

$$\theta=\frac{\pi}{2}-\phi.$$

$$\cot\phi=2\pi-4\phi,\ \text{or}\ \tan\phi=\frac{1}{2\pi-4\phi}$$

Now $\phi=\tan\phi-\frac{1}{3}\tan^3\phi+\frac{1}{5}\tan^5\phi-$

$$=\frac{1}{2\pi-4\phi}-\frac{1}{3}\cdot\frac{1}{(2\pi-4\phi)^3}+\frac{1}{5}\cdot\frac{1}{(2\pi-4\phi)^5}-$$

$$=\frac{1}{2\pi}\left(1+\frac{2\phi}{\pi}+\frac{4\phi^2}{\pi^2}+\ \right)$$

$$-\frac{1}{3}\ \frac{1}{8\pi^3}\left(1+\frac{6\phi}{\pi}+\frac{24\phi^2}{\pi^2}+.\ \right)$$

$$+\frac{1}{5}\ \frac{1}{32\pi^5}\left(1+\frac{10\phi}{\pi}+\ \right)$$

$$=\frac{1}{2\pi}-\frac{1}{24\pi^3}+\frac{1}{160\pi^5}-\quad+\phi\left(\frac{1}{\pi^2}-\frac{1}{4\pi^4}+\ldots\right)$$

$$+\phi^2\left(\frac{2}{\pi^3}-\frac{1}{\pi^5}+\quad\right)$$

$$=\frac{1}{2\pi}-\frac{1}{24\pi^3}+\frac{1}{160\pi^5}-\quad.$$

$$+\left(\frac{1}{\pi^2}-\frac{1}{4\pi^4}+\quad\right)\left\{\frac{1}{2\pi}-\frac{1}{24\pi^3}+\frac{1}{160\pi^5}-\quad+\phi\left(\frac{1}{\pi^2}-\frac{1}{4\pi^4}+\quad\right)\right\}+$$

$$+\frac{2}{\pi^3}\left\{\frac{1}{2\pi}-\frac{1}{24\pi^3}+\quad\right\}^2.$$

Hence omitting all terms above $\frac{1}{\pi^5}$,

$$\phi=\frac{1}{2\pi}-\frac{1}{24\pi^3}+\frac{1}{160\pi^5}-$$

$$+\left(\frac{1}{\pi^2}-\frac{1}{4\pi^4}+\quad\right)\left(\frac{1}{2\pi}-\frac{1}{24\pi^3}+\frac{1}{2\pi}\,\frac{1}{\pi^2}+\quad\right)+\frac{1}{2\pi^5}+\quad.$$

$$=\frac{1}{2\pi}-\frac{1}{24\pi^3}+\frac{1}{160\pi^5}-\quad+\frac{1}{2\pi^3}-\frac{1}{8\pi^5}+\frac{11}{24\pi^5}+\frac{1}{2\pi^5}$$

$$=\frac{1}{2\pi}+\frac{11}{24\pi^3}+\frac{403}{480}\cdot\frac{1}{\pi^5}+\quad;$$

$$\theta=\frac{\pi}{2}-\frac{1}{2\pi}-\frac{11}{24\pi^3}-\frac{403}{480}\quad\frac{1}{\pi^5}-$$

To find the root which lies between 2π and $\frac{5\pi}{2}$ put

$$\theta=\frac{5\pi}{2}-\phi,\text{ so that }\tan\phi=\frac{1}{10\pi-4\phi}.$$

Then
$$\phi=\tan\phi-\frac{1}{3}\tan^3\phi+\frac{1}{5}\tan^5\phi-$$

$$=\frac{1}{10\pi-4\phi}-\frac{1}{3}\cdot\frac{1}{(10\pi-4\phi)^3}+$$

$$=\frac{1}{10\pi}\left(1+\frac{2\phi}{5\pi}+\frac{4\phi^2}{25\pi^2}+\quad\right)$$

$$-\frac{1}{3000\pi^3}\left(1+\frac{6\phi}{5\pi}+\quad\right)$$

$$=\frac{1}{10\pi}+\frac{\phi}{25\pi^2}-\frac{1}{3000\pi^3}+\text{higher powers}$$

$$=\frac{1}{10\pi}+\frac{1}{25\pi^2}\left\{\frac{1}{10\pi}+\quad\right\}-\frac{1}{3000\pi^3}+$$

$$=\frac{1}{10\pi}+\frac{11}{3000\pi^3}+\quad,$$

$$\theta=\frac{5\pi}{2}-\frac{1}{10\pi}-\frac{11}{3000\pi^3}-\quad.$$

XXII

1
$$\sin^2\alpha=\frac{1}{2}(1-\cos 2\alpha),$$

$$\sin^2(\alpha+\beta)=\frac{1}{2}\{1-\cos 2(\alpha+\beta)\},$$

$$\sin^2(\alpha+2\beta)=\frac{1}{2}\{1-\cos 2(\alpha+2\beta)\},$$

and so on.

Hence the sum of n terms

$$=\frac{n}{2}-\frac{1}{2}\{\cos 2\alpha+\cos 2(\alpha+\beta)+\cos 2(\alpha+2\beta)+\quad\}$$

$$=\frac{n}{2}-\frac{\cos\{2\alpha+(n-1)\beta\}\sin n\beta}{2\sin\beta}.$$

2
$$\sin^3\alpha=\frac{1}{4}(3\sin\alpha-\sin 3\alpha),$$

$$\sin^3(\alpha+\beta)=\frac{1}{4}\{3\sin(\alpha+\beta)-\sin 3(\alpha+\beta)\},$$

$$\sin^3(\alpha+2\beta)=\frac{1}{4}\{3\sin(\alpha+2\beta)-\sin 3(\alpha+2\beta)\},$$

and so on

Hence the sum of n terms

$$=\frac{3}{4}\{\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\quad\}$$

$$-\frac{1}{4}\{\sin 3\alpha+\sin 3(\alpha+\beta)+\sin 3(\alpha+2\beta)+\quad\}$$

$$=\frac{3}{4}\frac{\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{1}{2}\beta}-\frac{1}{4}\frac{\sin\left(3\alpha+\frac{n-1}{2}3\beta\right)\sin\frac{3n\beta}{2}}{\sin\frac{3}{2}\beta}.$$

3 We have $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$,

therefore $\cos^4\theta = \frac{1}{4}(1+\cos 2\theta)^2 = \frac{1}{4}(1+2\cos 2\theta + \cos^2 2\theta)$

$$= \frac{1}{4}\left(1+2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) = \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$$

Apply this transformation to every term of the proposed series, thus the sum of n terms

$$= \frac{3n}{8} + \frac{1}{2}\{\cos 2\alpha + \cos 2(\alpha+\beta) + \cos 2(\alpha+2\beta) + \ldots\}$$

$$+ \frac{1}{8}\{\cos 4\alpha + \cos 4(\alpha+\beta) + \cos 4(\alpha+2\beta) + \ldots\}$$

$$= \frac{3n}{8} + \frac{\cos\{2\alpha+(n-1)\beta\}\sin n\beta}{2\sin\beta} + \frac{\cos\{4\alpha+(n-1)2\beta\}\sin 2n\beta}{8\sin 2\beta}.$$

4 $\sin\theta + \sin 3\theta + \sin 5\theta + \ldots$ to n terms

$$= \frac{\sin\{\theta+(n-1)\theta\}\sin n\theta}{\sin\theta} = \frac{\sin^2 n\theta}{\sin\theta},$$

$\cos\theta + \cos 3\theta + \cos 5\theta + \ldots$ to n terms

$$= \frac{\cos\{\theta+(n-1)\theta\}\sin n\theta}{\sin\theta} = \frac{\sin n\theta\cos n\theta}{\sin\theta}.$$

Divide the former result by the latter, thus we obtain $\tan n\theta$.

5. $\cos A\cos B = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$.

Apply this transformation to every term of the proposed series, thus the sum of n terms

$$= \frac{n}{2}\cos\alpha + \frac{1}{2}\{\cos(2\theta+\alpha) + \cos(2\theta+3\alpha) + \cos(2\theta+5\alpha) + \ldots\}$$

$$= \frac{n}{2}\cos\alpha + \frac{\cos\{2\theta+\alpha+(n-1)\alpha\}\sin n\alpha}{2\sin\alpha}$$

$$= \frac{n}{2}\cos\alpha + \frac{\cos(2\theta+n\alpha)\sin n\alpha}{2\sin\alpha}.$$

6 By Art 330 we have

$\sin\theta - \sin 2\theta + \sin 3\theta - \ldots$ to n terms

$$= \frac{\sin\left\{\theta + \frac{(n-1)(\theta+\pi)}{2}\right\}\sin\frac{n(\theta+\pi)}{2}}{\sin\frac{\theta+\pi}{2}}.$$

And $\cos\theta - \cos 2\theta + \cos 3\theta - \quad$ to n terms

$$=\frac{\cos\left\{\theta+\frac{(n-1)(\theta+\pi)}{2}\right\}\sin\frac{n(\theta+\pi)}{2}}{\sin\frac{\theta+\pi}{2}}$$

Divide the former by the latter the result

$$=\frac{\sin\left\{\theta+\frac{(n-1)(\theta+\pi)}{2}\right\}}{\cos\left\{\theta+\frac{(n-1)(\theta+\pi)}{2}\right\}}=\frac{\sin\left\{\theta+\frac{(n-1)(\theta+\pi)}{2}+\pi\right\}}{\cos\left\{\theta+\frac{(n-1)(\theta+\pi)}{2}+\pi\right\}}$$

$$=\frac{\sin\frac{n+1}{2}(\theta+\pi)}{\cos\frac{n+1}{2}(\theta+\pi)}=\tan\frac{n+1}{2}(\theta+\pi)$$

7 $\sin A\cos B=\frac{1}{2}\sin(A+B)+\frac{1}{2}\sin(A-B)$

Apply this transformation to every term of the proposed series, thus the sum of n terms

$$=\frac{n\sin p\theta}{2}+\frac{1}{2}\{\sin(p+2)\theta+\sin(p+4)\theta+\sin(p+6)\theta+\quad\}$$

$$=\frac{n\sin p\theta}{2}+\frac{\sin(p+1+n)\theta\sin n\theta}{2\sin\theta}.$$

8 $\sin A\sin B=\frac{1}{2}\cos(A-B)-\frac{1}{2}\cos(A+B)$

Apply this transformation to every term of the proposed series, thus the sum of n terms

$$=\frac{n}{2}\cos\alpha-\frac{1}{2}\{\cos 3\alpha+\cos 5\alpha+\cos 7\alpha+\quad\}$$

$$=\frac{n}{2}\cos\alpha-\frac{\cos\{3\alpha+(n-1)\alpha\}\sin n\alpha}{2\sin\alpha}=\frac{n}{2}\cos\alpha-\frac{\cos(n+2)\alpha\sin n\alpha}{2\sin\alpha}$$

9 Suppose that in the preceding result we put for the sines of the angles their values from Art 286, the proposed series becomes an expansion in powers of α, and it is obvious that the coefficient of α^3 is

$$1\ \ 2+2\ \ 3+3\,.\,4\quad+n(n+1)$$

We must therefore find the coefficient of α^3 in the expansion of the expression found for the sum of the Trigonometrical Series, and equate it to the above

17. Put the exponential values for $\cos\theta$, $\cos 2\theta$, $\cos 3\theta$, .. Thus denoting $e^{\iota\theta}$ by z, the proposed series becomes

$$\frac{1}{2}\left\{z-\frac{1}{2}z^2+\frac{1}{3}z^3-\frac{1}{4}z^4+\ \ \right\}+\frac{1}{2}\left\{z^{-1}-\frac{1}{2}z^{-2}+\frac{1}{3}z^{-3}-\frac{1}{4}z^{-4}+\ \ \right\},$$

that is $\quad \frac{1}{2}\log(1+z)+\frac{1}{2}\log(1+z^{-1})$, that is $\frac{1}{2}\log(1+z)(1+z^{-1})$,

that is $\quad \frac{1}{2}\log(2+z+z^{-1})$, that is $\frac{1}{2}\log(2+2\cos\theta)$,

that is $\quad \frac{1}{2}\log\left(4\cos^2\frac{\theta}{2}\right)$, that is $\log\left(2\cos\frac{\theta}{2}\right)$

18 Proceed as in the solution of Example 17. Thus the proposed series becomes

$$\frac{1}{2}\left\{z^2+\frac{1}{3}z^6+\frac{1}{5}z^{10}+\ \ \right\}+\frac{1}{2}\left\{z^{-2}+\frac{1}{3}z^{-6}+\frac{1}{5}z^{-10}+\ \ \right\}$$

$$=\frac{1}{4}\log\frac{1+z^2}{1-z^2}+\frac{1}{4}\log\frac{1+z^{-2}}{1-z^{-2}}=\frac{1}{4}\log\left(\frac{1+z^2}{1-z^2}\times\frac{1+z^{-2}}{1-z^{-2}}\right)$$

$$=\frac{1}{4}\log\frac{2+z^2+z^{-2}}{2-z^2-z^{-2}}=\frac{1}{4}\log\frac{2+2\cos 2\theta}{2-2\cos 2\theta}=\frac{1}{4}\log\frac{1+\cos 2\theta}{1-\cos 2\theta}$$

$$=\frac{1}{4}\log\cot^2\theta=\frac{1}{2}\log\cot\theta$$

19. Put the exponential values for $\sin\theta$, $\sin 2\theta$, $\sin 3\theta$, Thus, denoting $e^{\iota\theta}$ by z, the proposed series becomes

$$\frac{1}{2\iota}\left\{xz-\frac{1}{2}x^2z^2+\frac{1}{3}x^3z^3-\frac{1}{4}x^4z^4+\ \ \right\}$$

$$-\frac{1}{2\iota}\left\{xz^{-1}-\frac{1}{2}x^2z^{-2}+\frac{1}{3}x^3z^{-3}-\frac{1}{4}x^4z^{-4}+\ \ .\right\}.$$

This $\quad =\frac{1}{2\iota}\log(1+xz)-\frac{1}{2\iota}\log(1+xz^{-1})$

$$=\frac{1}{2\iota}\log\frac{1+xz}{1+xz^{-1}}=\frac{1}{2\iota}\log\frac{1+x(\cos\theta+\iota\sin\theta)}{1+x(\cos\theta-\iota\sin\theta)}.$$

Assume $\tan\phi=\dfrac{x\sin\theta}{1+x\cos\theta}$, thus the sum of the proposed series

$$=\frac{1}{2\iota}\log\frac{1+\iota\tan\phi}{1-\iota\tan\phi}=\frac{1}{2\iota}\log\frac{\cos\phi+\iota\sin\phi}{\cos\phi-\iota\sin\phi}$$

$$=\frac{1}{2\iota}\log\frac{e^{\iota\phi}}{e^{-\iota\phi}}=\frac{1}{2\iota}\log e^{2\iota\phi}=\phi=\cot^{-1}\frac{1+x\cos\theta}{x\sin\theta}$$

$$=\cot^{-1}\left(\frac{\operatorname{cosec}\theta}{x}+\cot\theta\right)$$

20 By Art 129 the limit of $\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}$ is $\frac{\sin\theta}{\theta}$, therefore the limit of $\cos\theta\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}$ is $\frac{\cos\theta\sin\theta}{\theta}$, that is $\frac{\sin 2\theta}{2\theta}$. Then take the logarithms of both sides

21. $$\sin\theta\left(\sin\frac{\theta}{2}\right)^2=\frac{1}{2}\sin\theta\,(1-\cos\theta)=\frac{1}{2}\sin\theta-\frac{1}{4}\sin 2\theta,$$

$$2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{4}\right)^2=\sin\frac{\theta}{2}\left(1-\cos\frac{\theta}{2}\right)=\sin\frac{\theta}{2}-\frac{1}{2}\sin\theta,$$

$$4\sin\frac{\theta}{4}\left(\sin\frac{\theta}{8}\right)^2=2\sin\frac{\theta}{4}\left(1-\cos\frac{\theta}{4}\right)=2\sin\frac{\theta}{4}-\sin\frac{\theta}{2},$$

$$8\sin\frac{\theta}{8}\left(\sin\frac{\theta}{16}\right)^2=4\sin\frac{\theta}{8}\left(1-\cos\frac{\theta}{8}\right)=4\sin\frac{\theta}{8}-2\sin\frac{\theta}{4}$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely

$$2^{n-2}\sin\frac{\theta}{2^{n-1}}-\frac{1}{4}\sin 2\theta.$$

22. $$\tan\frac{\theta}{2}\sec\theta=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}\cos\theta}=\frac{\sin\left(\theta-\frac{\theta}{2}\right)}{\cos\frac{\theta}{2}\cos\theta}=\frac{\sin\theta\cos\frac{\theta}{2}-\cos\theta\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}\cos\theta}$$

$$=\tan\theta-\tan\frac{\theta}{2};$$

therefore $$\tan\frac{\theta}{4}\sec\frac{\theta}{2}=\tan\frac{\theta}{2}-\tan\frac{\theta}{4},$$

$$\tan\frac{\theta}{8}\sec\frac{\theta}{4}=\tan\frac{\theta}{4}-\tan\frac{\theta}{8},$$

and so on

Then adding the terms, we see that all cancel on the right-hand side except two, namely

$$\tan\theta-\tan\frac{\theta}{2^n}.$$

23. $\cot\theta\operatorname{cosec}\theta = \dfrac{\cos\theta}{\sin^2\theta} = \dfrac{2\cos^2\frac{\theta}{2}-1}{\sin^2\theta}$

$$= \frac{2\cos^2\frac{\theta}{2}}{4\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}} - \frac{1}{\sin^2\theta} = \frac{1}{2\sin^2\frac{\theta}{2}} - \frac{1}{\sin^2\theta};$$

therefore $$2\cot 2\theta\operatorname{cosec} 2\theta = \frac{1}{\sin^2\theta} - \frac{2}{\sin^2 2\theta},$$

$$4\cot 4\theta\operatorname{cosec} 4\theta = \frac{2}{\sin^2 2\theta} - \frac{4}{\sin^2 4\theta}.$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely

$$\frac{1}{2\sin^2\frac{\theta}{2}} - \frac{2^{n-1}}{\sin^2 2^{n-1}\theta}.$$

24 $$\frac{1}{\sin\theta\sin 2\theta} = \frac{1}{\sin\theta}\cdot\frac{\sin(2\theta-\theta)}{\sin\theta\sin 2\theta} = \frac{1}{\sin\theta}\cdot\frac{\sin 2\theta\cos\theta - \cos 2\theta\sin\theta}{\sin\theta\sin 2\theta}$$

$$= \frac{1}{\sin\theta}(\cot\theta - \cot 2\theta)$$

Similarly $$\frac{1}{\sin 2\theta\sin 3\theta} = \frac{1}{\sin\theta}\,\frac{\sin(3\theta-2\theta)}{\sin 2\theta\sin 3\theta}$$

$$= \frac{1}{\sin\theta}(\cot 2\theta - \cot 3\theta);$$

$$\frac{1}{\sin 3\theta\sin 4\theta} = \frac{1}{\sin\theta}(\cot 3\theta - \cot 4\theta)$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely

$$\frac{1}{\sin\theta}\{\cot\theta - \cot(n+1)\theta\}$$

25 Let $\phi = \theta + \frac{\pi}{2}$, thus the proposed series becomes

$$\frac{1}{\cos\phi\cos 2\phi} + \frac{1}{\cos 2\phi\cos 3\phi} + \frac{1}{\cos 3\phi\cos 4\phi} + \dots$$

Now $$\frac{1}{\cos\phi\cos 2\phi}=\frac{1}{\sin\phi}\frac{\sin(2\phi-\phi)}{\cos\phi\cos 2\phi}=\frac{1}{\sin\phi}(\tan 2\phi-\tan\phi),$$

$$\frac{1}{\cos 2\phi\cos 3\phi}=\frac{1}{\sin\phi}\frac{\sin(3\phi-2\phi)}{\cos 2\phi\cos 3\phi}=\frac{1}{\sin\phi}(\tan 3\phi-\tan 2\phi),$$

$$\frac{1}{\cos 3\phi\cos 4\phi}=\frac{1}{\sin\phi}\frac{\sin(4\phi-3\phi)}{\cos 3\phi\cos 4\phi}=\frac{1}{\sin\phi}(\tan 4\phi-\tan 3\phi)$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely

$$\frac{1}{\sin\phi}\{\tan(n+1)\phi-\tan\phi\}$$

26 $$\operatorname{Tan}^{-1}\frac{1}{1+m+m^2}=\tan^{-1}\frac{1}{m}-\tan^{-1}\frac{1}{1+m};$$

this is obvious, for by taking the tangent of $\tan^{-1}\frac{1}{m}-\tan^{-1}\frac{1}{1+m}$ we obtain $\dfrac{\frac{1}{m}-\frac{1}{m+1}}{1+\frac{1}{m(m+1)}}$, that is $\dfrac{1}{m^2+m+1}$.

Apply this transformation to every term of the proposed series; thus we obtain

$$\tan^{-1}\frac{1}{1}-\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{3}-\tan^{-1}\frac{1}{4}+\ldots,$$

that is $$\tan^{-1}1-\tan^{-1}\frac{1}{n+1},\text{ that is }\frac{\pi}{4}-\tan^{-1}\frac{1}{n+1}.$$

27. $$\operatorname{Tan}^{-1}\frac{x}{1+m(m+1)x^2}=\tan^{-1}(m+1)x-\tan^{-1}mx,$$

this is obvious, for by taking the tangent of $\tan^{-1}(m+1)x-\tan^{-1}mx$, we obtain $\dfrac{(m+1)x-mx}{1+m(m+1)x^2}$, that is $\dfrac{x}{1+m(m+1)x^2}$.

Apply this transformation to every term of the proposed series after the first, thus we obtain

$$\tan^{-1}x+\tan^{-1}2x-\tan^{-1}x+\tan^{-1}3x-\tan^{-1}2x+\ldots,$$

that is $\tan^{-1}nx$.

28 $$\sin A \sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$$

Apply this transformation to every term of the proposed series, thus we obtain

$$\frac{1}{2}(\cos 2a - \cos 4a) + \frac{1}{2}(\cos a - \cos 2a) + \frac{1}{2}\left(\cos\frac{a}{2} - \cos a\right) + \ldots,$$

that is $$\frac{1}{2}\left(\cos\frac{a}{2^{n-2}} - \cos 4a\right).$$

29 $$\frac{1}{\cos\theta + \cos 3\theta} = \frac{1}{2\cos\theta\cos 2\theta} = \frac{1}{2\sin\theta}\cdot\frac{\sin(2\theta-\theta)}{\cos\theta\cos 2\theta}$$

$$= \frac{1}{2\sin\theta}(\tan 2\theta - \tan\theta),$$

$$\frac{1}{\cos\theta + \cos 5\theta} = \frac{1}{2\cos 2\theta\cos 3\theta} = \frac{1}{2\sin\theta}\cdot\frac{\sin(3\theta - 2\theta)}{\cos 2\theta\cos 3\theta}$$

$$= \frac{1}{2\sin\theta}(\tan 3\theta - \tan 2\theta),$$

$$\frac{1}{\cos\theta + \cos 7\theta} = \frac{1}{2\cos 3\theta\cos 4\theta} = \frac{1}{2\sin\theta}\cdot\frac{\sin(4\theta - 3\theta)}{\cos 3\theta\cos 4\theta}$$

$$= \frac{1}{2\sin\theta}(\tan 4\theta - \tan 3\theta).$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely

$$\frac{1}{2\sin\theta}\{\tan(n+1)\theta - \tan\theta\}$$

30 $$\frac{\sin\theta}{\cos 2\theta + \cos\theta} = \frac{\sin\theta}{2\cos\frac{\theta}{2}\cos\frac{3\theta}{2}} = \frac{\sin\theta}{2\cos\frac{\theta}{2}\cos\frac{3\theta}{2}}$$

$$= \frac{1}{4\sin\frac{\theta}{2}}\left\{\frac{1}{\cos\frac{3\theta}{2}} - \frac{1}{\cos\frac{\theta}{2}}\right\},$$

$$\frac{\sin 2\theta}{\cos 4\theta + \cos\theta} = \frac{\sin 2\theta}{2\cos\frac{3\theta}{2}\cos\frac{5\theta}{2}} = \frac{1}{4\sin\frac{\theta}{2}}\left\{\frac{1}{\cos\frac{5\theta}{2}} - \frac{1}{\cos\frac{3\theta}{2}}\right\},$$

$$\frac{\sin 3\theta}{\cos 6\theta+\cos\theta}=\frac{\sin 3\theta}{2\cos\frac{5\theta}{2}\cos\frac{7\theta}{2}}=\frac{1}{4\sin\frac{\theta}{2}}\left\{\frac{1}{\cos\frac{7\theta}{2}}-\frac{1}{\cos\frac{5\theta}{2}}\right\}.$$

Proceeding in this way, and adding the terms, we see that all cancel on the right hand except two, namely

$$\frac{1}{4\sin\frac{\theta}{2}}\left\{\frac{1}{\cos\frac{(2n+1)\theta}{2}}-\frac{1}{\cos\frac{\theta}{2}}\right\}.$$

31
$$\frac{\sin\theta}{1+2\cos\theta}=\frac{\sin\theta}{1+2\left(1-2\sin^2\frac{\theta}{2}\right)}=\frac{\sin\theta}{3-4\sin^2\frac{\theta}{2}}$$

$$=\frac{\sin\theta\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}\left(3-4\sin^2\frac{\theta}{2}\right)}=\frac{\sin\theta\sin\frac{\theta}{2}}{\sin\frac{3\theta}{2}}=\frac{\cos\frac{\theta}{2}-\cos\frac{3\theta}{2}}{2\sin\frac{3\theta}{2}}$$

$$=\frac{2\cos\frac{\theta}{2}-2\cos\frac{3\theta}{2}}{4\sin\frac{3\theta}{2}}=\frac{\cos\frac{\theta}{2}(1+2\cos\theta)+\cos\frac{\theta}{2}-2\cos\theta\cos\frac{\theta}{2}-2\cos\frac{3\theta}{2}}{4\sin\frac{3\theta}{2}}$$

$$=\frac{\cos\frac{\theta}{2}}{4\sin\frac{\theta}{2}}+\frac{\cos\frac{\theta}{2}-2\cos\theta\cos\frac{\theta}{2}-2\cos\frac{3\theta}{2}}{4\sin\frac{3\theta}{2}}$$

$$=\frac{\cos\frac{\theta}{2}}{4\sin\frac{\theta}{2}}-\frac{3\cos\frac{3\theta}{2}}{4\sin\frac{3\theta}{2}}=\frac{1}{4}\cot\frac{\theta}{2}-\frac{3}{4}\cot\frac{3\theta}{2}.$$

Similarly
$$\frac{3\sin 3\theta}{1+2\cos 3\theta}=\frac{3}{4}\cot\frac{3\theta}{2}-\frac{9}{4}\cot\frac{9\theta}{2},$$

$$\frac{3^2\sin 3^2\theta}{1+2\cos 3^2\theta}=\frac{9}{4}\cot\frac{9\theta}{2}-\frac{27}{4}\cot\frac{27\theta}{2}.$$

Proceeding in this way, and adding the terms, we see that all cancel on the right hand except two, namely

$$\frac{1}{4}\cot\frac{\theta}{2}-\frac{3^n}{4}\cot\frac{3^n\theta}{2}.$$

32 $\cot^{-1}\left\{2a^{-1}+\frac{m(m+1)}{2}a\right\}=\cot^{-1}\frac{m}{2}a-\cot^{-1}\frac{m+1}{2}a.$

For if we take the cotangent of $\cot^{-1}\frac{m}{2}a-\cot^{-1}\frac{m+1}{2}a$,

we obtain
$$\frac{\frac{m}{2}a\,.\,\frac{m+1}{2}a+1}{\frac{m+1}{2}a-\frac{m}{2}a},$$

that is
$$\frac{m(m+1)}{2}a+2a^{-1}.$$

Apply this transformation to every term of the proposed series; thus we obtain

$$\cot^{-1}\frac{a}{2}-\cot^{-1}\frac{2a}{2}+\cot^{-1}\frac{2a}{2}-\cot^{-1}\frac{3a}{2}+\cot^{-1}\frac{3a}{2}-\cot^{-1}\frac{4a}{2}+\quad;$$

that is
$$\cot^{-1}\frac{a}{2}-\cot^{-1}\frac{n+1}{2}a.$$

33 $\frac{1}{2}\sec\theta=\frac{1}{2\cos\theta}=\frac{\sin\theta}{2\sin\theta\cos\theta}=\frac{\sin\theta}{\sin 2\theta}=\frac{\sin(2\theta-\theta)}{\sin 2\theta}$

$$=\cos\theta-\cot 2\theta\sin\theta=\sin\theta(\cot\theta-\cot 2\theta);$$

$$\frac{1}{2^2}\sec\theta\sec 2\theta=\frac{1}{2}\sec\theta\sin 2\theta(\cot 2\theta-\cot 4\theta)$$

$$=\sin\theta(\cot 2\theta-\cot 4\theta);$$

$$\frac{1}{2^3}\sec\theta\sec 2\theta\sec 4\theta=\frac{1}{2}\sec\theta\sin 2\theta(\cot 4\theta-\cot 8\theta)$$

$$=\sin\theta(\cot 4\theta-\cot 8\theta).$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely

$$\sin\theta(\cot\theta-\cot 2^n\theta).$$

34. $\tan 2\theta=\frac{\sin 2\theta}{\cos 2\theta}=\frac{\sin^2 2\theta}{\sin 2\theta\cos 2\theta}=\frac{2\sin^2 2\theta}{\sin 4\theta}=\frac{4\sin^2 2\theta}{2\sin 4\theta};$

therefore
$$\frac{1}{2}\log\tan 2\theta=\log 2\sin 2\theta-\frac{1}{2}\log 2\sin 4\theta,$$

$$\frac{1}{2^2}\log\tan 2^2\theta=\frac{1}{2}\log 2\sin 4\theta-\frac{1}{2^2}\log 2\sin 8\theta,$$

$$\frac{1}{2^3}\log\tan 2^3\theta=\frac{1}{2^2}\log 2\sin 8\theta-\frac{1}{2^3}\log 2\sin 16\theta.$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely

$$\log 2 \sin 2\theta - \frac{1}{2^n} \log 2 \sin 2^{n+1} \theta$$

35 $$\cos\frac{\theta}{2} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}} = \frac{\sin\theta}{2\sin\frac{\theta}{2}} = \frac{\sin\theta}{2}\,\frac{2\cos^2\frac{\theta}{4} - \cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= \frac{\sin\theta}{2}\left\{\frac{2\cos^2\frac{\theta}{4}}{2\sin\frac{\theta}{4}\cos\frac{\theta}{4}} - \cot\frac{\theta}{2}\right\} = \frac{\sin\theta}{2}\left\{\cot\frac{\theta}{4} - \cot\frac{\theta}{2}\right\},$$

$$2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2} = 2\cos\frac{\theta}{2}\,\frac{\sin\frac{\theta}{2}}{2}\left\{\cot\frac{\theta}{8} - \cot\frac{\theta}{4}\right\}$$

$$= \frac{\sin\theta}{2}\left\{\cot\frac{\theta}{8} - \cot\frac{\theta}{4}\right\},$$

$$2^2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3} = 2\cos\frac{\theta}{2}\cdot\frac{\sin\frac{\theta}{2}}{2}\left\{\cot\frac{\theta}{16} - \cot\frac{\theta}{8}\right\}$$

$$= \frac{\sin\theta}{2}\left\{\cot\frac{\theta}{16} - \cot\frac{\theta}{8}\right\}.$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely

$$\frac{\sin\theta}{2}\left\{\cot\frac{\theta}{2^{n+1}} - \cot\frac{\theta}{2}\right\}$$

36 Let R denote the radius of the circle, n the number of sides of the polygon Put β for $\frac{\pi}{n}$ Let 2α denote the angular distance of a fixed point in the circumference from one of the angular points, then the angular distances from the other angular points in succession will be

$$2\alpha+2\beta,\ 2\alpha+4\beta,\ 2\alpha+6\beta,\quad 2\alpha+2(n-1)\beta$$

The lengths of the successive chords will be

$$2R\sin\alpha,\ 2R\sin(\alpha+\beta),\ 2R\sin(\alpha+2\beta),\ \ldots\ 2R\sin\{\alpha+(n-1)\beta\}$$

To find the sum of the squares of the chords, we have

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta),$$

and applying this transformation to every term of the proposed series, we obtain

$$2nR^2-2R^2\{\cos 2\alpha+\cos(2\alpha+2\beta)+\cos(2\alpha+4\beta)+\ \ \}.$$

The sum of the series of cosines is zero, as in Art. 327, and thus the result is $2nR^2$.

Next, to find the sum of the fourth powers of the chords We have

$$\sin^4\theta=\frac{3}{8}-\frac{1}{2}\cos 2\theta+\frac{1}{8}\cos 4\theta,$$

and applying this transformation to every term of the proposed series, we obtain

$$6nR^4-8R^4\{\cos 2\alpha+\cos(2\alpha+2\beta)+\cos(2\alpha+4\beta)+\ldots\}$$
$$+2R^4\{\cos 4\alpha+\cos(4\alpha+4\beta)+\cos(4\alpha+8\beta)+\ \ \},$$

that is $6nR^4$

37 Let A be the common vertex, let B, C, be the successive angular points Put β for $\frac{\pi}{n}$

Let PQ be one of the sides of the polygon, such that the arc ABP contains m of the sides, then the angle $AQP=m\beta$, the angle $PAQ=\beta$, and the angle $APQ=\pi-(m+1)\beta$

Let $PQ=c$, and let r_m denote the radius of the circle inscribed in APQ Then

$$r_m\left\{\cot\frac{1}{2}APQ+\cot\frac{1}{2}AQP\right\}=c,$$

therefore $$r_m\left\{\cot\frac{\pi-(m+1)\beta}{2}+\cot\frac{m\beta}{2}\right\}=c,$$

therefore $$r_m\left\{\tan\frac{m+1}{2}\beta+\cot\frac{m\beta}{2}\right\}=c,$$

therefore $$r_m\cos\frac{\beta}{2}=c\cos\frac{m+1}{2}\beta\sin\frac{m\beta}{2}$$
$$=\frac{c}{2}\left\{\sin\frac{2m+1}{2}\beta-\sin\frac{\beta}{2}\right\}.$$

Now there are $n-2$ circles in all which can be drawn, so that we have to sum up the values of

$$\frac{c}{2}\sec\frac{\beta}{2}\left\{\sin\frac{2m+1}{2}\beta-\sin\frac{\beta}{2}\right\}$$

for values of m from 1 to $n-2$ inclusive The sum then is

$$\frac{c}{2}\sec\frac{\beta}{2}\left\{\frac{\sin\left\{\frac{3\beta}{2}+(n-3)\frac{\beta}{2}\right\}\sin\frac{n-2}{2}\beta}{\sin\frac{\beta}{2}}-(n-2)\sin\frac{\beta}{2}\right\},$$

that is $$\frac{c}{2}\sec\frac{\beta}{2}\left\{\frac{\cos\beta}{\sin\frac{\beta}{2}}-(n-2)\sin\frac{\beta}{2}\right\},$$

that is $$\frac{c}{2}\sec\frac{\beta}{2}\left\{\frac{1}{\sin\frac{\beta}{2}}-n\sin\frac{\beta}{2}\right\}.$$

But $c=2r\sin\beta$, thus we get

$$\frac{r\sin\beta}{\cos\frac{\beta}{2}\sin\frac{\beta}{2}}-\frac{rn\sin\beta\sin\frac{\beta}{2}}{\cos\frac{\beta}{2}},\quad\text{that is } 2r-2rn\sin^2\frac{\beta}{2},$$

that is $$2r\left(1-n\sin^2\frac{\pi}{2n}\right).$$

38 Use the notation of the preceding solution. The area of the m^{th} circle

$$=\frac{\pi c^2}{4}\sec^2\frac{\beta}{2}\left\{\sin\frac{2m+1}{2}\beta-\sin\frac{\beta}{2}\right\}^2$$

$$=\frac{\pi c^2}{4}\sec^2\frac{\beta}{2}\left\{\sin^2\frac{2m+1}{2}\beta-2\sin\frac{2m+1}{2}\beta\sin\frac{\beta}{2}+\sin^2\frac{\beta}{2}\right\}$$

$$=\frac{\pi c^2}{8}\sec^2\frac{\beta}{2}\left\{1-\cos(2m+1)\beta-4\sin\frac{2m+1}{2}\beta\sin\frac{\beta}{2}+2\sin^2\frac{\beta}{2}\right\}$$

Then as before we have to sum this expression for the values of m from 1 to $n-2$ inclusive Thus we obtain

$$\frac{\pi c^2}{8}\sec^2\frac{\beta}{2}\left\{(n-2)\left(1+2\sin^2\frac{\beta}{2}\right)-\frac{\cos\{3\beta+(n-3)\beta\}\sin(n-2)\beta}{\sin\beta}\right.$$
$$\left.-4\sin\frac{\beta}{2}\frac{\sin\left\{\frac{3\beta}{2}+\frac{n-3}{2}\beta\right\}\sin\frac{n-2}{2}\beta}{\sin\frac{\beta}{2}}\right\},$$

and this $$=\frac{\pi c^2}{8}\sec^2\frac{\beta}{2}\left\{(n-2)\left(1+2\sin^2\frac{\beta}{2}\right)-2\cos\beta\right\}$$

$$=\frac{\pi c^2}{8}\sec^2\frac{\beta}{2}\left\{n-4+2n\sin^2\frac{\beta}{2}\right\}$$

But $c=2r\sin\beta$, therefore $c^2\sec^2\frac{\beta}{2}=\frac{4r^2\sin^2\beta}{\cos^2\frac{\beta}{2}}=16r^2\sin^2\frac{\beta}{2}$;

so that the result $$=16\pi r^2\sin^2\frac{\pi}{2n}\left\{\frac{n}{4}\sin^2\frac{\pi}{2n}+\frac{n-4}{8}\right\}.$$

39 Let S_n denote the sum of the series; so that

$$S_n = n\sin\theta + (n-1)\sin 2\theta + (n-2)\sin 3\theta + \ldots + \sin n\theta.$$

In like manner let S_{n-1} denote the sum of the series formed by changing n into $n-1$, so that

$$S_{n-1} = (n-1)\sin\theta + (n-2)\sin 2\theta + \ldots + \sin(n-1)\theta;$$

therefore $S_n - S_{n-1} = \sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta$

$$= \frac{\sin\frac{n+1}{2}\theta\,\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} = \frac{1}{2\sin\frac{\theta}{2}}\left\{\cos\frac{\theta}{2} - \cos\frac{2n+1}{2}\theta\right\}.$$

Similarly we have

$$S_{n-1} - S_{n-2} = \frac{1}{2\sin\frac{\theta}{2}}\left\{\cos\frac{\theta}{2} - \cos\frac{2n-1}{2}\theta\right\},$$

$$S_{n-2} - S_{n-3} = \frac{1}{2\sin\frac{\theta}{2}}\left\{\cos\frac{\theta}{2} - \cos\frac{2n-3}{2}\theta\right\};$$

$$\cdot \qquad \cdot$$

$$S_2 - S_1 = \frac{1}{2\sin\frac{\theta}{2}}\left\{\cos\frac{\theta}{2} - \cos\frac{5\theta}{2}\right\},$$

$$S_1 = \frac{1}{2\sin\frac{\theta}{2}}\left\{\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right\}.$$

Hence by addition from this series of equations we obtain

$$S_n = \frac{1}{2\sin\frac{\theta}{2}}\left\{n\cos\frac{\theta}{2} - \cos\frac{3\theta}{2} - \cos\frac{5\theta}{2} \ldots - \cos\frac{2n+1}{2}\theta\right\}$$

$$= \frac{1}{2\sin\frac{\theta}{2}}\left\{n\cos\frac{\theta}{2} - \frac{\cos\left\{\frac{3\theta}{2} + (n-1)\frac{\theta}{2}\right\}\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\right\}$$

$$= \frac{n}{2}\cot\frac{\theta}{2} - \frac{\cos\frac{(n+2)}{2}\theta\,\sin\frac{n\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{n}{2}\cot\frac{\theta}{2} - \frac{\sin(n+1)\theta - \sin\theta}{4\sin^2\frac{\theta}{2}}$$

$$= \frac{n+1}{2}\cot\frac{\theta}{2} - \frac{\sin(n+1)\theta}{4\sin^2\frac{\theta}{2}}.$$

40 Let S_n denote the required sum, and S_{n-1} the sum of the series when n is changed to $n-1$ Thus

$$S_n=(n+1)\,n\sin\theta+n\,(n-1)\sin 2\theta+\quad+2\ 1\sin n\theta,$$

$$S_{n-1}=n\,(n-1)\sin\theta+(n-1)\,(n-2)\sin 2\theta+\quad+2\,.\,1\sin(n-1)\,\theta;$$

therefore $\quad S_n-S_{n-1}=2\,\{n\sin\theta+(n-1)\sin 2\theta+\quad+\sin n\theta\}$,

that is, by Example 39,

$$S_n-S_{n-1}=(n+1)\cot\frac{\theta}{2}-\frac{\sin(n+1)\,\theta}{2\sin^2\frac{\theta}{2}}.$$

Similarly $\quad S_{n-1}-S_{n-2}=n\cot\frac{\theta}{2}-\frac{\sin n\theta}{2\sin^2\frac{\theta}{2}},$

$$S_2-S_1=3\cot\frac{\theta}{2}-\frac{\sin 3\theta}{2\sin^2\frac{\theta}{2}};$$

$$S_1=2\cot\frac{\theta}{2}-\frac{\sin 2\theta}{2\sin^2\frac{\theta}{2}}$$

Hence by addition from this series of equations we obtain

$$S_n=\frac{n\,(n+3)}{2}\cot\frac{\theta}{2}-\frac{1}{2\sin^2\frac{\theta}{2}}\{\sin 2\theta+\sin 3\theta+\quad+\sin(n+1)\,\theta\}$$

$$=\frac{n\,(n+3)}{2}\cot\frac{\theta}{2}-\frac{\sin\left\{2\theta+(n-1)\frac{\theta}{2}\right\}\sin\frac{n\theta}{2}}{2\sin^2\frac{\theta}{2}\sin\frac{\theta}{2}}$$

$$=\frac{n\,(n+3)}{2}\cot\frac{\theta}{2}-\frac{\cos\frac{3\theta}{2}-\cos\frac{2n+3}{2}\theta}{4\sin^3\frac{\theta}{2}}.$$

XXIII.

1. By Art. 286 we have

$$\frac{\sin\theta}{\theta}=1-\frac{\theta^2}{\underline{|3}}+\frac{\theta^4}{\underline{|5}}-\frac{\theta^6}{\underline{|7}}+\ldots;$$

and by Art. 341 we have

$$\frac{\sin\theta}{\theta}=\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots$$

Take the logarithms of the equivalent expressions, thus

$$\log\left\{1-\frac{\theta^2}{\underline{|3}}+\frac{\theta^4}{\underline{|5}}-\frac{\theta^6}{\underline{|7}}+\ldots\right\}$$

$$=\log\left(1-\frac{\theta^2}{\pi^2}\right)+\log\left(1-\frac{\theta^2}{2^2\pi^2}\right)+\log\left(1-\frac{\theta^2}{3^2\pi^2}\right)+\ldots$$

Expand the logarithms, then both sides become series arranged according to powers of θ, and by equating the coefficients of θ^2 we obtain

$$-\frac{\theta^2}{\underline{|3}}=-\theta^2\left(\frac{1}{\pi^2}+\frac{1}{2^2\pi^2}+\frac{1}{3^2\pi^2}+\ldots\right),$$

therefore $$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}.$$

2. Equate the coefficients of θ^4 in the two equivalent series of the preceding solution, thus since

$$\log\left\{1-\frac{\theta^2}{\underline{|3}}+\frac{\theta^4}{\underline{|5}}\ldots\right\}=-\left\{\frac{\theta^2}{\underline{|3}}-\frac{\theta^4}{\underline{|5}}+\ldots\right\}-\frac{1}{2}\left\{\frac{\theta^2}{\underline{|3}}-\frac{\theta^4}{\underline{|5}}+\ldots\right\}^2-\ldots$$

we have $$\frac{1}{\underline{|5}}-\frac{1}{2}\left(\frac{1}{\underline{|3}}\right)^2=-\frac{1}{2\pi^4}\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\right),$$

therefore $$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots=2\pi^4\left(\frac{1}{72}-\frac{1}{120}\right)$$

$$=\frac{\pi^4}{12}\left(\frac{1}{3}-\frac{1}{5}\right)=\frac{\pi^4}{90}.$$

3. Let $$S=\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\ldots;$$

and let $$\Sigma=\frac{1}{1^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\ldots$$

Then
$$S=\frac{1}{1^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\dots$$
$$+\frac{1}{2^n}+\frac{1}{4^n}+\frac{1}{6^n}+\frac{1}{8^n}+\dots$$
$$=\frac{1}{1^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\dots$$
$$+\frac{1}{2^n}\left\{\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\dots\right\}$$
$$=\Sigma+\frac{1}{2^n}S$$

Therefore
$$\Sigma=\frac{2^n-1}{2^n}S.$$

Hence Σ can be found when S is known

If $n=2$ we have $S=\frac{\pi^2}{6}$ by Example 1, and then $\Sigma=\frac{3}{4}\,\frac{\pi^2}{6}=\frac{\pi^2}{8}$.

4 In the preceding solution suppose $n=4$, then we have $S=\frac{\pi^4}{90}$ by Example 2, and therefore $\Sigma=\frac{15}{16}\cdot\frac{\pi^4}{90}=\frac{\pi^4}{96}$.

5 $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots$
$$=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots-\left(\frac{1}{2^2}+\frac{1}{4^2}+\dots\right)$$
$$=\frac{\pi^2}{8}-\frac{1}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right),\text{ by Ex 3,}$$
$$=\frac{\pi^2}{8}-\frac{1}{4}\cdot\frac{\pi^2}{6}=\frac{\pi^2}{12}.$$

6 Let S denote the series of which we require the sum, then
$$\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots\right)^2=2S+\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots$$

Hence by Examples 1 and 2 we have
$$\left(\frac{\pi^2}{6}\right)^2=2S+\frac{\pi^4}{90};$$

therefore
$$S=\frac{\pi^4}{2}\left(\frac{1}{36}-\frac{1}{90}\right)=\frac{\pi^4}{120}.$$

7. The general term of the series is $\dfrac{\frac{1}{2}n(n+1)}{(2n+1)^4}$.

$$\frac{n(n+1)}{2(2n+1)^4}=\frac{1}{8}\cdot\frac{4n^2+4n+1-1}{(2n+1)^4}=\frac{1}{8}\cdot\frac{(2n+1)^2-1}{(2n+1)^4}$$

$$=\frac{1}{8}\left\{\frac{1}{(2n+1)^2}-\frac{1}{(2n+1)^4}\right\}.$$

Hence the series $=\dfrac{1}{8}\left\{\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\ldots-\dfrac{1}{1^4}-\dfrac{1}{3^4}-\dfrac{1}{5^4}-\ldots\right\}$

$$=\frac{1}{8}\left(\frac{\pi^2}{8}-\frac{\pi^4}{96}\right), \text{ Ex 3 and 4,}$$

$$=\frac{\pi^2}{64}\left(1-\frac{\pi^2}{12}\right).$$

8 $\left(\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\ldots\right)^2=\dfrac{1}{1^4}+\dfrac{1}{3^4}+\ldots+2\left\{\dfrac{1}{1^2.3^2}+\dfrac{1}{1^2.5^2}+\dfrac{1}{3^2.5^2}+\ldots\right\}$

$$=\frac{\pi^4}{96}+2\Sigma\Sigma\frac{1}{(2m+1)^2(2n+1)^2}, \text{ Ex. 4},$$

therefore by Example 3,

$$\Sigma\Sigma\frac{1}{(2m+1)^2(2n+1)^2}=\frac{1}{2}\left\{\left(\frac{\pi^2}{8}\right)^2-\frac{\pi^4}{96}\right\}$$

$$=\frac{\pi^4}{2}\left(\frac{1}{64}-\frac{1}{96}\right)=\frac{\pi^4}{384}.$$

9 (1) $\dfrac{1}{n^2(n+1)^2}=\left(\dfrac{1}{n}-\dfrac{1}{n+1}\right)^2=\dfrac{1}{n^2}+\dfrac{1}{(n+1)^2}-\dfrac{2}{n(n+1)}$

$$=\frac{1}{n^2}+\frac{1}{(n+1)^2}-2\left(\frac{1}{n}-\frac{1}{n+1}\right).$$

$$\Sigma_1^\infty\frac{1}{n^2(n+1)^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$$

$$+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$$

$$-2\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\ldots\right)$$

$$=2\cdot\frac{\pi^2}{6}-1-2=\frac{\pi^2}{3}-3.$$

(ii) $$\frac{1}{n^3(n+1)^3} = \left(\frac{1}{n} - \frac{1}{n+1}\right)^3 = \frac{1}{n^3} - \frac{1}{(n+1)^3} - \frac{3}{n(n+1)}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \frac{1}{n^3} - \frac{1}{(n+1)^3} - \frac{3}{n^2(n+1)^2}.$$

$$\Sigma\frac{1}{n^3(n+1)^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \dots$$

$$- \frac{1}{2^3} - \frac{1}{3^3} - \dots \quad - 3\Sigma\frac{1}{n^2(n+1)^2}$$

$$= 1 - 3\left(\frac{\pi^2}{3} - 3\right), \text{ by (i)},$$

$$= 10 - \pi^2$$

(iii) $$\frac{1}{n^4(n+1)^4} = \left(\frac{1}{n} - \frac{1}{n+1}\right)^4$$

$$= \frac{1}{n^4} + \frac{1}{(n+1)^4} - \frac{4}{n^3(n+1)} - \frac{4}{n(n+1)^3} + \frac{6}{n^2(n+1)^2}$$

$$- \frac{4}{n^3(n+1)} - \frac{4}{n(n+1)^3} = - \frac{4}{n(n+1)}\left\{\frac{1}{n^2} + \frac{1}{(n+1)^2}\right\}$$

$$= - \frac{4}{n(n+1)} \cdot \frac{2n(n+1)+1}{n^2(n+1)^2}$$

$$= - \frac{8}{n^2(n+1)^2} - \frac{4}{n^3(n+1)^3}$$

$$\frac{1}{n^4(n+1)^4} = \frac{1}{n^4} + \frac{1}{(n+1)^4} - \frac{2}{n^2(n+1)^2} - \frac{4}{n^3(n+1)^3}$$

$$\Sigma\frac{1}{n^4(n+1)^4} = 2\Sigma\frac{1}{n^4} - 1 - 2\Sigma\frac{1}{n^2(n+1)^2} - 4\Sigma\frac{1}{n^3(n+1)^3}$$

$$= 2\,\frac{\pi^4}{90} - 1 - 2\left(\frac{\pi^2}{3} - 3\right) - 4(10 - \pi^2)$$

$$= \frac{\pi^4}{45} + \frac{10\pi^2}{3} - 35$$

10 By resolving into partial fractions we have

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\,\frac{1}{n} - \frac{1}{n+1} + \frac{1}{2}\,\frac{1}{n+2}.$$

$$\left\{\frac{1}{n(n+1)(n+2)}\right\}^2 = \frac{1}{4}\,\frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{4}\,\frac{1}{(n+2)^2}$$

$$- \frac{1}{n(n+1)} + \frac{1}{2}\,\frac{1}{n(n+2)} - \frac{1}{(n+1)(n+2)}$$

$$= \frac{1}{4}\,\frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{4}\,\frac{1}{(n+2)^2} + \frac{-2n-4+n+1-2n}{2n(n+1)(n+2)}$$

$$= \frac{1}{4}\,\frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{4}\,\frac{1}{(n+2)^2} - \frac{3}{4}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

Hence the given series

$$=\frac{1}{4}\Sigma\frac{1}{n^2}+\Sigma\frac{1}{(n+1)^2}+\frac{1}{4}\Sigma\frac{1}{(n+2)^2}$$

$$-\frac{3}{4}\left\{\frac{1}{1}-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\frac{1}{4}-\frac{1}{6}+\ldots\right\}$$

$$=\frac{1}{4}\cdot\frac{\pi^2}{6}+\left(\frac{\pi^2}{6}-1\right)+\frac{1}{4}\left(\frac{\pi^2}{6}-1-\frac{1}{2^2}\right)-\frac{3}{4}\cdot\frac{3}{2}=\frac{\pi^2}{4}-\frac{37}{16}.$$

11. By Art 342 we have

$$\sin n\phi=2^{n-1}\sin\phi\sin(2\beta+\phi)\sin(4\beta+\phi)\ldots\sin(2n\beta-2\beta+\phi),$$

where $$\beta=\frac{\pi}{2n}.$$

Let $\alpha=\frac{1}{2}\beta$, and let $\phi=\alpha$, then $\sin n\phi=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$; thus

$$\frac{1}{\sqrt{2}}=2^{n-1}\sin\alpha\sin5\alpha\sin9\alpha\ldots\sin(4n-3)\alpha,$$

therefore $$\sin\alpha\sin5\alpha\sin9\alpha\ldots\sin(4n-3)\alpha=2^{-n+\frac{1}{2}}$$

12 By Art 342 we have

$$\frac{\sin\theta}{\theta}=\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots.$$

Put $\frac{\pi}{2}$ for θ, thus

$$\frac{2}{\pi}=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)\ldots,$$

therefore $$\frac{\pi}{2}=\frac{2.2.4.4.6.6.8.8\ldots}{1.3.3.5.5.7.7.9\ldots}.$$

13 By Art 342 we have

$$\cos5A=16\sin(A+18^\circ)\sin(A+54^\circ)\sin(A+90^\circ)\sin(A+126^\circ)\sin(A+162^\circ)$$

$$=16\cos(72^\circ-A)\cos(36^\circ-A)\cos A\cos(A+36^\circ)\cos(A+72^\circ);$$

and $\cos(36^\circ-A)=-\cos(144^\circ+A)$, $\cos(A+36^\circ)=-\cos(144^\circ-A)$;
therefore

$$\cos5A=16\cos(72^\circ-A)\cos(72^\circ+A)\cos A\cos(144^\circ-A)\cos(144^\circ+A).$$

14 Put $\frac{\pi}{6}$ for θ in the expression for $\sin\theta$ in Art 341, thus

$$\frac{1}{2}=\frac{\pi}{6}\left(1-\frac{1}{6^2}\right)\left(1-\frac{1}{2^26^2}\right)\left(1-\frac{1}{3^26^2}\right)\ldots,$$

therefore $$3=\pi\cdot\frac{35}{36}\cdot\frac{143}{144}\cdot\frac{323}{324}\cdot\frac{575}{576}\ldots,$$

therefore $$\pi=3\cdot\frac{36}{35}\cdot\frac{144}{143}\cdot\frac{324}{323}\cdot\frac{576}{575}\ldots$$

15. In the formula for $\cos\theta$ in Art. 344 put $\frac{\pi}{4}$ for θ, thus

$$\frac{1}{\sqrt{2}}=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{2^2 . 3^2}\right)\left(1-\frac{1}{2^2 . 5^2}\right)\left(1-\frac{1}{2^2 . 7^2}\right)\ldots$$

$$=\frac{3 . 35 . 99 . 195 \ldots}{4 . 36 . 100 . 196 \ldots},$$

therefore $$\sqrt{2}=\frac{4 . 36 . 100 . 196 \ldots}{3 . 35 . 99 . 195 \ldots}$$

16. In the formula for $\cos\theta$ in Art. 344 put $\frac{\pi}{6}$ for θ, thus

$$\frac{\sqrt{3}}{2}=\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{3^2 . 3^2}\right)\left(1-\frac{1}{3^2 . 5^2}\right)\left(1-\frac{1}{3^2 . 7^2}\right)\ldots$$

$$=\frac{8 . 80 . 224 . 440 \ldots}{9 . 81 . 225 . 441 \ldots}$$

17. $\sin 5A-\cos 5A=\sqrt{2}\sin(5A-45^0)=\sqrt{2}\sin 5(A-9^0)$.

And by Art. 342 we have $\sin 5(A-9^0)$

$=2^4 \sin B\sin(B+36^0)\sin(B+72^0)\sin(B+108^0)\sin(B+144^0)$,

where $B=A-9^0$,

$=2^4\sin(A-9^0)\sin(A+27^0)\cos(27^0-A)\cos(A+9^0)\sin(A+135^0)$

$=2^4\sin(A-9^0)\sin(A+27^0)\cos(27^0-A)\cos(A+9^0)(\cos A-\sin A)\frac{1}{\sqrt{2}}$.

Therefore $\sin 5A-\cos 5A$

$=2^4\sin(A-9^0)\sin(A+27^0)\cos(A-27^0)\cos(A+9^0)(\cos A-\sin A)$.

18. $$\cos x+\tan y\sin x=\frac{\cos x\cos y+\sin x\sin y}{\cos y}=\frac{\cos(x-y)}{\cos y}.$$

Now by Art. 344

$$\cos(x-y)=\left\{1-\frac{4(x-y)^2}{\pi^2}\right\}\left\{1-\frac{4(x-y)^2}{3^2\pi^2}\right\}\left\{1-\frac{4(x-y)^2}{5^2\pi^2}\right\}\ldots,$$

$$\cos y=\left(1-\frac{4y^2}{\pi^2}\right)\left(1-\frac{4y^2}{3^2\pi^2}\right)\left(1-\frac{4y^2}{5^2\pi^2}\right)\ldots$$

Divide the former by the latter. Then

$$\frac{1-\frac{4(x-y)^2}{\pi^2}}{1-\frac{4y^2}{\pi^2}}=\frac{\pi^2-4(x-y)^2}{\pi^2-4y^2}=\frac{\pi^2-4y^2-4x^2+8xy}{\pi^2-4y^2}=1-\frac{4x^2}{\pi^2-4y^2}+\frac{8xy}{\pi^2-4y^2}$$

$$=\left(1+\frac{2x}{\pi-2y}\right)\left(1-\frac{2x}{\pi+2y}\right).$$

Similarly $$\frac{1-\dfrac{4(x-y)^2}{3^2\pi^2}}{1-\dfrac{y^2}{3^2\pi^2}}=\left(1+\frac{2x}{3\pi-y}\right)\left(1-\frac{2x}{3\pi+y}\right)$$

And so on. Thus the required result is obtained

19 In the last example put $y=\dfrac{\pi}{4}$, therefore

$$\cos x+\sin x=\left(1+\frac{2x}{\frac{\pi}{2}}\right)\left(1-\frac{2x}{\frac{3\pi}{2}}\right)\left(1+\frac{2x}{\frac{5\pi}{2}}\right)\left(1-\frac{2x}{\frac{7\pi}{2}}\right)$$

$$=\left(1+\frac{4x}{\pi}\right)\left(1-\frac{4x}{3\pi}\right)\left(1+\frac{4x}{5\pi}\right)\left(1-\frac{4x}{7\pi}\right)$$

20 $$\cos x-\cot y\sin x=\frac{\cos x\sin y-\sin x\cos y}{\sin y}=\frac{\sin(y-x)}{\sin y}.$$

Now by Art 344

$$\sin(y-x)=(y-x)\left\{1-\frac{(y-x)^2}{\pi^2}\right\}\left\{1-\frac{(y-x)^2}{2^2\pi^2}\right\}\left\{1-\frac{(y-x)^2}{3^2\pi^2}\right\},$$

$$\sin y=y\left(1-\frac{y^2}{\pi^2}\right)\left(1-\frac{y^2}{2^2-\pi^2}\right)\left(1-\frac{y^2}{3^2\pi^2}\right)$$

Divide the former by the latter Then

$$\frac{1-\dfrac{(y-x)^2}{\pi^2}}{1-\dfrac{y^2}{\pi^2}}=\frac{\pi^2-(y-x)^2}{\pi^2-y^2}=\frac{\pi^2-y^2-x^2+2xy}{\pi^2-y^2}$$

$$=1-\frac{x^2}{\pi^2-y^2}+\frac{2xy}{\pi^2-y^2}=\left(1+\frac{x}{\pi-y}\right)\left(1-\frac{x}{\pi+y}\right)$$

Similarly $$\frac{1-\dfrac{(y-x)^2}{2^2\pi^2}}{1-\dfrac{y^2}{2^2\pi^2}}=\left(1+\frac{x}{2\pi-y}\right)\left(1-\frac{x}{2\pi+y}\right)$$

And so on Thus the required result is obtained

21 $$\frac{\cos x-\cos y}{1-\cos y}=\frac{2\sin\frac{1}{2}(y-x)\sin\frac{1}{2}(y+x)}{2\sin^2\frac{y}{2}}$$

Now by Art 320

$$\sin\frac{1}{2}(y-x)=\frac{1}{2}(y-x)\left\{1-\frac{(y-x)^2}{4\pi^2}\right\}\left\{1-\frac{(y-x)^2}{4\cdot 2^2\pi^2}\right\}$$

$$\sin\frac{1}{2}(y+x)=\frac{1}{2}(y+x)\left\{1-\frac{(y+x)^2}{4\pi^2}\right\}\left\{1-\frac{(y+x)^2}{4\cdot 2^2\pi^2}\right\}$$

$$\sin\frac{1}{2}y=\frac{1}{y}y\left(1-\frac{y^2}{4\pi^2}\right)\left(1-\frac{y^2}{4\cdot 2^2\pi^2}\right)$$

Divide the first by the third, and divide the second by the third, and multiply the results together

Then $$\frac{\frac{1}{2}(y-x)}{\frac{1}{2}y}=1-\frac{x}{y},\quad \frac{\frac{1}{2}(x+y)}{\frac{1}{2}y}=1+\frac{x}{y},\quad \left(1-\frac{x}{y}\right)\left(1+\frac{x}{y}\right)=1-\frac{x^2}{y^2}$$

And as in the solution of Example 20,

$$\frac{1-\frac{(y-x)^2}{4\pi^2}}{1-\frac{y^2}{4\pi^2}}=\left(1+\frac{x}{2\pi-y}\right)\left(1-\frac{x}{2\pi+y}\right),$$

$$\frac{1-\frac{(y+x)^2}{4\pi^2}}{1-\frac{y^2}{4\pi^2}}=\left(1-\frac{x}{2\pi-y}\right)\left(1+\frac{x}{2\pi+y}\right),$$

$$\left(1+\frac{x}{2\pi-y}\right)\left(1-\frac{x}{2\pi+y}\right)\left(1-\frac{x}{2\pi-y}\right)\left(1+\frac{x}{2\pi+y}\right)$$

$$=\left\{1-\frac{x^2}{(2\pi-y)^2}\right\}\left\{1-\frac{x^2}{(2\pi+y)^2}\right\}$$

Similarly $$\frac{1-\frac{(y-x)^2}{4\cdot 2^2\pi^2}}{1-\frac{y^2}{4\cdot 2^2\pi^2}}\;\frac{1-\frac{(y+x)^2}{4\cdot 2^2\pi^2}}{1-\frac{y^2}{4\cdot 2^2\pi^2}}=\left\{1-\frac{x^2}{(4\pi-y)^2}\right\}\left\{1-\frac{x^2}{(4\pi+y)^2}\right\}.$$

And so on Thus the required result is obtained

22 $$\frac{\cos x+\cos y}{1+\cos y}=\frac{2\cos\frac{1}{2}(y-x)\cos\frac{1}{2}(y+x)}{2\cos^2\frac{y}{2}}$$

Now by Art 344

$$\cos\frac{1}{2}(y-x)=\left\{1-\frac{(y-x)^2}{\pi^2}\right\}\left\{1-\frac{(y-x)^2}{3^2\pi^2}\right\}\left\{1-\frac{(y-x)^2}{5^2\pi^2}\right\}$$

$$\cos\frac{1}{2}(y+x)=\left\{1-\frac{(y+x)^2}{\pi^2}\right\}\left\{1-\frac{(y+x)^2}{3^2\pi^2}\right\}\left\{1-\frac{(y+x)^2}{5^2\pi^2}\right\}$$

$$\cos\frac{y}{2}=\left(1-\frac{y^2}{\pi^2}\right)\left(1-\frac{y^2}{3^2\pi^2}\right)\left(1-\frac{y^2}{5^2\pi^2}\right)$$

Divide the first by the third, and divide the second by the third, and multiply the two results together; the reductions will be similar to those in the preceding solution

Thus $$\frac{1-\frac{(y-x)^2}{\pi^2}}{1-\frac{y^2}{\pi^2}}\;\frac{1-\frac{(y+x)^2}{\pi^2}}{1-\frac{y^2}{\pi^2}}=\left\{1-\frac{x^2}{(\pi-y)^2}\right\}\left\{1-\frac{x^2}{(\pi+y)^2}\right\},$$

$$\frac{1-\frac{(y-x)^2}{3^2\pi^2}}{1-\frac{y^2}{3^2\pi^2}}\;\frac{1-\frac{(y+x)^2}{3^2\pi^2}}{1-\frac{y^2}{3^2\pi^2}}=\left\{1-\frac{x^2}{(3\pi-y)^2}\right\}\left\{1-\frac{x^2}{(3\pi+y)^2}\right\}$$

And so on Thus the required result is obtained.

Or we may obtain the result in Example 22 by changing y into $\pi-y$ in the result of Example 21

23 $$\frac{\sin x+\sin y}{\sin y}=\frac{2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)}{2\sin\frac{y}{2}\cos\frac{y}{2}}$$

Now in the course of the solution of Example 21 we see that

$$\frac{\sin\frac{1}{2}(x+y)}{\sin\frac{1}{2}y}=\left(1+\frac{x}{y}\right)\left(1-\frac{x}{2\pi-y}\right)\left(1+\frac{x}{2\pi+y}\right)\left(1-\frac{x}{4\pi-y}\right)\left(1+\frac{x}{4\pi+y}\right)$$

And by changing y into $\pi-y$ we see that

$$\frac{\cos\frac{1}{2}(x-y)}{\cos\frac{1}{2}y}=\left(1+\frac{x}{\pi-y}\right)\left(1-\frac{x}{\pi+y}\right)\left(1+\frac{x}{3\pi-y}\right)\left(1-\frac{x}{3\pi+y}\right)$$

Hence by multiplication the required result is obtained

24 We have $\cos x + \tan y \sin x$

$$=1-\frac{x^2}{|\underline{2}}+\frac{x^4}{|\underline{4}}- \quad +\tan y\left(x-\frac{x^3}{|\underline{3}}+ \quad \right),$$

thus the coefficient of x is $\tan y$

Now conceive the factors on the right-hand side of the formula of Example 18 multiplied together, and the product arranged according to powers of x The first term will be unity, the second term will involve x and the coefficient will be

$$\frac{2}{\pi-2y}-\frac{2}{\pi+2y}+\frac{2}{3\pi-y}-\frac{2}{3\pi+y}+\frac{2}{5\pi-y}-\frac{2}{5\pi+y}+$$

Hence by equating the coefficients we obtain the required result

25 Proceed as in Example 24 Then on the left-hand side the coefficient of x will be $-\cot y$, and on the right-hand side

$$-\frac{1}{y}+\frac{1}{\pi-y}-\frac{1}{\pi+y}+\frac{1}{2\pi-y}-\frac{1}{2\pi+y}+ \quad -$$

Equate the coefficients, and then change the signs of both sides, thus we obtain the required result

26 From Example 23

$$1+\frac{\sin x}{\sin y}=\left(1+\frac{x}{y}\right)\left(1+\frac{x}{\pi-y}\right)\left(1-\frac{x}{\pi+y}\right)\left(1+\frac{x}{2\pi+y}\right)\left(1-\frac{x}{2\pi-y}\right)$$

Now $$1+\frac{\sin x}{\sin y}=1+\frac{1}{\sin y}\left(x-\frac{x^3}{6}+ \quad \right)$$

Hence $\dfrac{1}{\sin y}$ is equal to the coefficient of x in the expansion in powers of x of the above expression in factors, namely

$$\frac{1}{y}+\frac{1}{\pi-y}-\frac{1}{\pi+y}+\frac{1}{2\pi+y}-\frac{1}{2\pi-y}+$$

27 In the formula of the preceding example put $\frac{1}{2}\pi-x$ for y, therefore

$$\frac{1}{\cos x}=\frac{2}{\pi-2x}+\frac{2}{\pi+2x}-\frac{2}{3\pi-2x}+\frac{2}{5\pi-2x}-\frac{2}{3\pi+2x}+\frac{2}{5\pi+2x}-$$

$$=\frac{2}{\pi-2x}+\frac{2}{\pi+2x}-\frac{2}{3\pi-2x}-\frac{2}{3\pi+2x}+\frac{2}{5\pi-2x}+\frac{2}{5\pi-2x}-$$

$$=4\pi\left\{\frac{1}{\pi^2-4x^2}-\frac{3}{3^2\pi^2-4x^2}+\frac{5}{5^2\pi^2-4x^2}- \quad \right\},$$

$$\frac{1}{4\pi}\sec x=\frac{1}{\pi^2-4x^2}-\frac{3}{3^2\pi^2-4x^2}+\frac{5}{5^2\pi^2-4x^2}- \quad .$$

28 In the formula of Example 24 put $\frac{\pi}{6}$ for y; then

$$\tan y = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}};$$

thus
$$\frac{1}{\sqrt{3}} = \frac{1}{\pi}\left\{\frac{6}{2} - \frac{6}{4} + \frac{6}{8} - \frac{6}{10} + \frac{6}{14} - \quad\right\}$$
$$= \frac{6}{\pi}\left\{\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{10} + \frac{1}{14} - \quad\right\},$$

therefore
$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} +$$

29 In the formula of Example 25 put $\frac{\pi}{6}$ for y, then

$$\cot y = \cot\frac{\pi}{6} = \sqrt{3};$$

thus
$$\sqrt{3} = \frac{6}{\pi}\left\{1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - . \right\};$$

therefore
$$\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} -$$

30 By Art 342 we have

$$\sin n\phi = 2^{n-1}\sin\phi\sin\left(\phi + \frac{\pi}{n}\right)\sin\left(\phi + \frac{2\pi}{n}\right) \quad \sin\left(\phi + \frac{n-1}{n}\pi\right).$$

Change ϕ into $\phi + \frac{\pi}{2}$, then since n is even we have

$$\sin n\left(\phi + \frac{\pi}{2}\right) = \sin\left(n\phi + \frac{n\pi}{2}\right) = \sin n\phi\cos\frac{n\pi}{2};$$

thus

$$\sin n\phi\cos\frac{n\pi}{2} = 2^{n-1}\cos\phi\cos\left(\phi + \frac{\pi}{n}\right)\cos\left(\phi + \frac{2\pi}{n}\right) \quad \cos\left(\phi + \frac{n-1}{n}\pi\right)$$

Divide the former result by this, then we obtain

$$\sec\frac{n\pi}{2} = \tan\phi\tan\left(\phi + \frac{\pi}{n}\right)\tan\left(\phi + \frac{2\pi}{n}\right) \quad \tan\left(\phi + \frac{n-1}{n}\pi\right)$$

And
$$\sec\frac{n\pi}{2} = \frac{1}{\cos\frac{n\pi}{2}} = \frac{1}{(-1)^{\frac{n}{2}}} = (-1)^{\frac{n}{2}}$$

31 By Art 342 we have

$$2^{m-1}\sin\phi\sin\left(\phi + \frac{\pi}{m}\right)\sin\left(\phi + \frac{2\pi}{m}\right) \quad \sin\left(\phi + \frac{m-1}{m}\pi\right) = \sin m\phi$$

For m put $\frac{n}{2}$, therefore

$$2^{\frac{1}{2}n-1}\sin\phi\sin\left(\phi+\frac{2\pi}{n}\right)\sin\left(\phi+\frac{4\pi}{n}\right)\quad\sin\left(\phi+\frac{n-2}{n}\pi\right)=\sin\frac{n}{2}\phi$$

In this result put $\pi+\phi$ for ϕ Therefore

$$2^{\frac{1}{2}n-1}\sin\left(\phi+\frac{n}{n}\pi\right)\sin\left(\phi+\frac{n+2}{n}\pi\right)\sin\left(\phi+\frac{n+4}{n}\pi\right)\quad\sin\left(\phi+\frac{2n-2}{n}\pi\right)$$
$$=\sin\frac{n}{2}(\phi+\pi)$$

Multiply these two results together, therefore

$$2^{n-1}\sin\phi\sin\left(\phi+\frac{2\pi}{n}\right)\sin\left(\phi+\frac{4\pi}{n}\right)\quad\sin\left(\phi+\frac{2n-2}{n}\phi\right)$$
$$=2\sin\frac{n\phi}{2}\sin\frac{n(\phi+\pi)}{2}=\cos\frac{1}{2}n\pi-\cos n\left(\phi+\frac{\pi}{2}\right)$$

32 See Example 70, Ch XVIII.

33 We have

$$\sin 2\theta=2\theta\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{2^2\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{4^2\pi^2}\right)\quad,$$

$$\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\left(1-\frac{\theta^2}{4^2\pi^2}\right)\quad.$$

Divide the first by the second, thus

$$\frac{\sin 2\theta}{\sin\theta}=2\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\quad,$$

$$\cos\theta=\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)$$

34 From Art 129 we have

$$\sin\theta=\theta\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}$$

But

$$\cos\frac{\theta}{2}=\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\left(1-\frac{\theta^2}{5^2\pi^2}\right)\quad,$$

$$\cos\frac{\theta}{2^2}=\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{6^2\pi^2}\right)\left(1-\frac{\theta^2}{10^2\pi^2}\right)\quad,$$

$$\cos\frac{\theta}{2^3}=\left(1-\frac{\theta^2}{4^2\pi^2}\right)\left(1-\frac{\theta^2}{12^2\pi^2}\right)\left(1-\frac{\theta^2}{20^2\pi^2}\right)\quad,$$

and so on

Multiplying these results together we obtain the expression for $\sin\theta$ in factors

35. It is obvious that

$$x_1x_2x_3\ldots x_n=x_1+x_1(x_2-1)+x_1x_2(x_3-1)+\ldots+x_1x_2\ldots x_{n-1}(x_n-1)$$

Put $x_1=1-\frac{\theta^2}{\pi^2}$, $x_2=1-\frac{\theta^2}{2^2\pi^2}$, $x_3=1-\frac{\theta^2}{3^2\pi^2}$, &c., and make n infinite.

Therefore

$$\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots$$

$$=1-\frac{\theta^2}{\pi^2}-\frac{\pi^2-\theta^2}{\pi^2}\cdot\frac{\theta^2}{2^2\pi^2}-\frac{(\pi^2-\theta^2)(2^2\pi^2-\theta^2)}{1^2\cdot 2^2\cdot\pi^4}\cdot\frac{\theta^2}{3^2\pi^2}-\ldots$$

$$=1-\frac{\theta^2}{\pi^2}+\frac{1}{\pi^4}\cdot\frac{\theta^2(\theta^2-\pi^2)}{1^2\cdot 2^2}-\frac{1}{\pi^6}\cdot\frac{\theta^2(\theta^2-\pi^2)(\theta^2-2^2\pi^2)}{1^2\cdot 2^2\cdot 3^2}+\ldots.$$

Since $$\frac{\sin\theta}{\theta}=\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\ldots,$$

the required result follows.

Again $$\frac{\sin\theta}{\theta}=1-\frac{\theta^2}{\lfloor 3}+\frac{\theta^4}{\lfloor 5}-\ldots,$$

$\frac{1}{\lfloor 5}$ = coefficient of θ^4 in the expansion of

$$1-\frac{\theta^2}{\pi^2}+\frac{1}{\pi^4}\cdot\frac{\theta^2(\theta^2-\pi^2)}{1^2\cdot 2^2}-\frac{1}{\pi^6}\cdot\frac{\theta^2(\theta^2-\pi^2)(\theta^2-2^2\pi^2)}{1^2\cdot 2^2\cdot 3^2}+\ldots;$$

$$\frac{1}{\lfloor 5}=\frac{1}{\pi^4}\left\{\frac{1}{1^2\cdot 2^2}+\frac{1^2+2^2}{1^2\cdot 2^2}\cdot\frac{1}{3^2}+\frac{1^2\cdot 2^2+1^2\cdot 3^2+2^2\cdot 3^2}{1^2\cdot 2^2\cdot 3^2}\cdot\frac{1}{4^2}+\ldots\right\},$$

$$\frac{\pi^4}{120}=\frac{1}{1^2}\cdot\frac{1}{2^2}+\left(\frac{1}{1^2}+\frac{1}{2^2}\right)\frac{1}{3^2}+\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\right)\frac{1}{4^2}+\ldots$$

36. In the identity in the solution of Ex. 35, put

$$x_1=1-\theta^2,\ x_2=1-\frac{\theta^2}{3^2},\ x_3=1-\frac{\theta^2}{5^2},\ \ldots,$$

$$(1-\theta^2)\left(1-\frac{\theta^2}{3^2}\right)\left(1-\frac{\theta^2}{5^2}\right)\ldots$$

$$=1-\theta^2-(1-\theta^2)\cdot\frac{\theta^2}{9}-(1-\theta^2)\left(1-\frac{\theta^2}{9}\right)\frac{\theta^2}{25}-\ldots$$

But $$\cos\frac{1}{2}\pi\theta=(1-\theta^2)\left(1-\frac{\theta^2}{3^2}\right)\ldots,$$ by Art. 341.

$$\cos\frac{1}{2}\pi\theta=1-\theta^2-(1-\theta^2)\frac{\theta^2}{9}-(1-\theta^2)\left(1-\frac{\theta^2}{9}\right)\frac{\theta^2}{25}-\ldots.$$

37 From Example 36 we have

$$\cos\frac{1}{2}\pi\theta=1-\theta^2-(1-\theta^2)\frac{\theta^2}{3^2}-(1-\theta^2)\left(1-\frac{\theta^2}{3^2}\right)\frac{\theta^2}{5^2}-$$

Expanding the right-hand side in powers of θ, the coefficient of θ^4 is

$$\frac{1}{3^2}+\left(1+\frac{1}{3^2}\right)\frac{1}{5^2}+\left(1+\frac{1}{3^2}+\frac{1}{5^2}\right)\frac{1}{7^2}+$$

Also $$\cos\frac{1}{2}\pi\theta=1-\frac{1}{\lfloor 2}\left(\frac{\pi\theta}{2}\right)^2+\frac{1}{\lfloor 4}\left(\frac{\pi\theta}{2}\right)^4-$$

Equating coefficients of θ^4 we have

$$\frac{1}{3^2}+\left(1+\frac{1}{3^2}\right)\frac{1}{5^2}+\left(1+\frac{1}{3^2}+\frac{1}{5^2}\right)\frac{1}{7^2}+\ldots=\frac{1}{\lfloor 4}\left(\frac{\pi}{2}\right)^4=\frac{\pi^4}{384}.$$

38 In the formula

$$\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)$$

put $$\theta=\pi x\sqrt{-1}$$

Then $$\sin\theta=\frac{1}{2\sqrt{-1}}\left(e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}}\right)=\frac{\sqrt{-1}}{2}\left(e^{\pi x}-e^{-\pi x}\right),$$

$$\frac{1}{2}\left(e^{\pi x}-e^{-\pi x}\right)=\pi x\left(1+\frac{x^2}{1^2}\right)\left(1+\frac{x^2}{2^2}\right) \qquad (1)$$

By taking logarithms and differentiating we obtain the required result Otherwise, without using the Differential Calculus, as follows In equation (1) put $x+h$ for x and divide the result by (1),

$$\frac{e^{\pi(x+h)}-e^{-\pi(x+h)}}{e^{\pi x}-e^{-\pi x}}=\frac{x+h}{x}\cdot\frac{1+\frac{(x+h)^2}{1^2}}{1+\frac{x^2}{1^2}}\cdot\frac{1+\frac{(x+h)^2}{2^2}}{1+\frac{x^2}{2^2}}\cdot$$

$$=\left(1+\frac{h}{x}\right)\left(1+\frac{2hx}{1^2+x^2}+\frac{h^2}{1^2+x^2}\right)\left(1+\frac{2hx}{2^2+x^2}+\frac{h^2}{2^2+x^2}\right)$$

Multiplying out this expression the coefficient of h is

$$\frac{1}{x}+\frac{2x}{1^2+x^2}+\frac{2x}{2^2+x^2}+\frac{2x}{3^2+x^2}+$$

This is therefore equal to the coefficient of h in the expansion in powers of h of $\dfrac{e^{\pi(x+h)}-e^{-\pi(x+h)}}{e^{\pi x}-e^{-\pi x}}$.

This expression $= \dfrac{e^{\pi x} \cdot e^{\pi h} - e^{-\pi x} \cdot e^{-\pi h}}{e^{\pi x} - e^{-\pi x}}$

$$= \frac{e^{\pi x}(1+\pi h+\ldots) - e^{-\pi x}(1-\pi h+\ldots)}{e^{\pi x} - e^{-\pi x}}$$

$$= 1 + \pi h \cdot \frac{e^{\pi x} + e^{-\pi x}}{e^{\pi x} - e^{-\pi x}} + \ldots$$

Hence equating coefficients of h and dividing by $2x$, we obtain

$$\frac{1}{2x^2} + \frac{1}{1^2+x^2} + \frac{1}{2^2+x^2} + \frac{1}{3^2+x^2} + \ldots = \frac{\pi}{2x} \cdot \frac{e^{\pi x} + e^{-\pi x}}{e^{\pi x} - e^{-\pi x}}.$$

Hence the required result follows

39 This may be done as in Example 38, by means of the expression for $\cos\theta$ in factors, or it may be deduced from Ex 38 In Example 38 write $2x$ for x Therefore

$$\frac{1}{1^2+4x^2} + \frac{1}{2^2+2^2x^2} + \frac{1}{3^2+4x^2} + \frac{1}{4^2+2^2x^2} + \ldots = \frac{4x}{\pi} \cdot \frac{e^{2\pi x} - e^{-2\pi x}}{e^{2\pi x} - e^{-2\pi x}} - \frac{1}{8x^2}.$$

The left-hand side is

$$\frac{1}{1^2+4x^2} + \frac{1}{3^2+4x^2} + \ldots + \frac{1}{4}\left\{\frac{1}{1^2+x^2} + \frac{1}{2^2+x^2} + \ldots\right\}$$

$$= \frac{1}{1^2+4x^2} + \frac{1}{3^2+4x^2} + \ldots + \frac{\pi}{8x} \cdot \frac{e^{\pi x} + e^{-\pi x}}{e^{\pi x} - e^{-\pi x}} - \frac{1}{8x^2}. \qquad \text{(Ex. 38)}$$

Hence

$$\frac{1}{1^2+4x^2} + \frac{1}{3^2+4x^2} + \ldots = \frac{\pi}{8x}\left[\frac{2(e^{2\pi x} + e^{-2\pi x})}{e^{2\pi x} - e^{-2\pi x}} - \frac{e^{\pi x} + e^{-\pi x}}{e^{\pi x} - e^{-\pi x}}\right]$$

$$= \frac{\pi}{8x}\left[\frac{2(e^{2\pi x} + e^{-2\pi x}) - (e^{\pi x} + e^{-\pi x})^2}{e^{2\pi x} - e^{-2\pi x}}\right]$$

$$= \frac{\pi}{8x} \cdot \frac{(e^{\pi x} - e^{-\pi x})^2}{e^{2\pi x} - e^{-2\pi x}} = \frac{\pi}{8x} \cdot \frac{e^{\pi x} - e^{-\pi x}}{e^{\pi x} + e^{-\pi x}}.$$

40 From Examples 38 and 39 we have, by putting $x=1$,

$$1 + \frac{2}{1^2+1} + \frac{2}{2^2+1} + \frac{2}{3^2+1} + \ldots = \frac{\pi(e^{\pi} + e^{-\pi})}{e^{\pi} - e^{-\pi}}$$

$$\frac{1}{1^2+4} + \frac{1}{3^2+4} + \frac{1}{5^2+4} + \ldots = \frac{\pi}{8} \cdot \frac{e^{\pi} - e^{-\pi}}{e^{\pi} + e^{-\pi}}$$

Multiplying these we obtain the required result

41 $$\lambda(x)=x\left(1-\frac{x}{a}\right)\left(1+\frac{x}{a}\right)\left(1-\frac{x}{2a}\right)\left(1+\frac{x}{2a}\right)\cdots\left(1-\frac{x}{ma}\right)\left(1+\frac{x}{ma}\right),$$

where m is made infinitely great

$$\lambda\left(x+\frac{a}{2}\right)=a\left(\frac{1}{2}+\frac{x}{a}\right)\left(\frac{1}{2}-\frac{x}{a}\right)\left(\frac{3}{2}+\frac{x}{a}\right)\left(\frac{3}{4}-\frac{x}{2a}\right)\left(\frac{5}{4}+\frac{x}{2a}\right)\cdots\left(\frac{2m-1}{2m}-\frac{x}{ma}\right)\left(\frac{2m+1}{2m}+\frac{x}{ma}\right)$$

$$=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdots\frac{(2m-1)}{2m}\,(2m+1)\,\frac{a}{2}\left\{1-\left(\frac{2x}{a}\right)^2\right\}\left\{1-\left(\frac{2x}{3a}\right)^2\right\}\cdots$$

$$=\frac{1}{2}P\,a\mu(x) \qquad (1),$$

where $$P=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdots\frac{(2m-1)}{2m}\,(2m+1)$$

Again $$\mu(x)=\left(1-\frac{2x}{a}\right)\left(1+\frac{2x}{a}\right)\left(1-\frac{2x}{2a}\right)\left(1+\frac{2x}{3a}\right)\cdots\left(1-\frac{2}{2m+1}\frac{x}{a}\right)\left(1+\frac{2}{2m+1}\frac{x}{a}\right),$$

$$\mu\left(x+\frac{a}{2}\right)=-\frac{2x}{a}\left(2+\frac{2x}{a}\right)\left(\frac{2}{3}-\frac{2x}{3a}\right)\left(\frac{4}{3}+\frac{2x}{3a}\right)\left(\frac{4}{5}-\frac{2x}{5a}\right)\cdots\left(\frac{2m}{2m+1}-\frac{2}{2m+1}\frac{x}{a}\right)\left(\frac{2m+2}{2m+1}+\frac{2}{2m+1}\frac{x}{a}\right)$$

$$=-2\,\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots\frac{2m}{(2m-1)}\cdot\frac{2m+2}{2m+1}\cdot\frac{1}{2m+1}\cdot\frac{x}{a}\left(1+\frac{x}{a}\right)\left(1-\frac{x}{a}\right)$$

$$=-\frac{2}{P}\cdot\frac{2m+2}{2m+1}\cdot\frac{x}{a}\left(1+\frac{x}{a}\right)\left(1-\frac{x}{a}\right)$$

$$=-\frac{2}{Pa}\cdot\lambda(x)$$

Now $$\frac{a}{\pi}\sin\frac{\pi x}{a}=x\left(1-\frac{x^2}{1^2a^2}\right)\left(1-\frac{x^2}{2^2a^2}\right)\cdots=\lambda(x),$$

$$\cos\frac{\pi x}{a}=\left(1-\frac{4x^2}{1^2a^2}\right)\left(1-\frac{4x^2}{3^2a^2}\right)\cdots=\mu(x),$$

$$\lambda\left(x+\frac{a}{2}\right)=\frac{a}{\pi}\sin\left(\frac{\pi x}{a}+\frac{\pi}{2}\right)=\frac{a}{\pi}\cos\frac{\pi x}{a}=\frac{a}{\pi}\mu(x).$$

Hence from (1), $$P=\frac{2}{\pi}$$

42 From Art 341

$$\sin\theta = n\sin\frac{\theta}{n}\left(1-\frac{\sin^2\frac{\theta}{2n}}{\sin^2\frac{\pi}{2n}}\right)\left(1-\frac{\sin^2\frac{\theta}{2n}}{\sin^2\frac{2\pi}{2n}}\right)\ldots$$

For θ put $nz\sqrt{-1}$ Then $\sin\theta = \frac{1}{2\sqrt{-1}}(e^{-nz}-e^{nz})$ and

$$\sin\frac{\theta}{2n} = \frac{1}{2\sqrt{-1}}(e^{-\frac{1}{2}z}-e^{\frac{1}{2}z});\ \text{therefore} -\sin^2\frac{\theta}{2n} = u^2$$

Hence

$$e^{nz}-e^{-nz} = n(e^z-e^{-z})\left(1+\frac{u^2}{\sin^2\frac{\pi}{2n}}\right)\left(1+\frac{u^2}{\sin^2\frac{2\pi}{2n}}\right)$$

43 From the expression for $\cos\theta$ in factors (Art. 344) we have

$$\cos\left(\frac{\pi}{2}\sin\theta\right) = \left(1-\frac{\sin^2\theta}{1^2}\right)\left(1-\frac{\sin^2\theta}{3^2}\right)\left(1-\frac{\sin^2\theta}{5^2}\right)$$

$$=\cos^2\theta\left(1-\frac{1}{3^2}+\frac{\cos^2\theta}{3^2}\right)\left(1-\frac{1}{5^2}+\frac{\cos^2\theta}{5^2}\right)$$

$$=\cos^2\theta\left(\frac{2\ 4}{3^2}+\frac{\cos^2\theta}{3^2}\right)\left(\frac{4\ 6}{5^2}+\frac{\cos^2\theta}{5^2}\right)$$

$$=\frac{2\ 4\ 4\ 6}{3.3\ 5\ 5}\cos^2\theta\left(1+\frac{\cos^2\theta}{2\ 4}\right)\left(1+\frac{\cos^2\theta}{4\ 6}\right)$$

$$=\frac{\pi}{4}\cos^2\theta\left(1+\frac{\cos^2\theta}{2\ 4}\right)\left(1+\frac{\cos^2\theta}{4\ 6}\right) \qquad \text{(Ex 12)}$$

44 $$\left(1-\frac{1}{a^2}\right)^{-1}\left(1-\frac{1}{b^2}\right)^{-1}\left(1-\frac{1}{c^2}\right)^{-1}$$

$$=\left(1+\frac{1}{a^2}+\frac{1}{a^4}+\ \right)\left(1+\frac{1}{b^2}+\frac{1}{b^4}+\ .\right)\left(1+\frac{1}{c^2}+\frac{1}{c^4}+\ .\right)$$

$$=1+\frac{1}{a^2}+\frac{1}{b^2}+\quad+\frac{1}{(a^r b^s\ )^2}+$$

$$=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+$$

$$=\frac{\pi^2}{6} \qquad \text{(Ex 1)}.$$

$$\left(1-\frac{1}{a^4}\right)^{-1}\left(1-\frac{1}{b^4}\right)^{-1}\left(1-\frac{1}{c^4}\right)^{-1}$$

$$=\left(1+\frac{1}{a^4}+\frac{1}{a^8}+\quad\right)\left(1+\frac{1}{b^4}+\frac{1}{b^8}+\quad\right)$$

$$=1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+$$

$$=\frac{\pi^4}{90} \quad . \quad . \quad . \quad \text{(Ex 2).}$$

Hence, by division, we obtain

$$\left(1+\frac{1}{a^2}\right)\left(1+\frac{1}{b^2}\right)\left(1+\frac{1}{c^2}\right) \quad =\frac{15}{\pi^2}$$

45 We shall require the value of $\dfrac{\sin\theta}{1-\dfrac{\theta^2}{\pi^2}}$ when $\theta=\pi$

Put $\theta=\pi+h$, then

$$\frac{\sin\theta}{1-\dfrac{\theta^2}{\pi^2}}=\frac{-\sin h}{1-\dfrac{\pi^2+2\pi h+h^2}{\pi^2}}=\frac{\sin h}{\dfrac{2}{\pi}\quad h+\dfrac{h^2}{\pi^2}}=\frac{\sin h}{h}\quad\frac{1}{\dfrac{2}{\pi}+\dfrac{h}{\pi^2}}$$

$$=\frac{\pi}{2}, \text{ when } h=0$$

$$\frac{\sin\theta}{\theta}=\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)$$

Divide both sides by all factors such as $\left(1-\dfrac{\theta^2}{a^2\pi^2}\right)$ where a is a prime Therefore

$$\frac{\sin\theta}{\theta\left(1-\dfrac{\theta^2}{\pi^2}\right)\left(1-\dfrac{\theta^2}{2^2\pi^2}\right)}=\left(1-\frac{\theta^2}{4^2\pi^2}\right)\left(1-\frac{\theta^2}{6^2\pi^2}\right)$$

In this identity put $\theta=\pi$, the left-hand side

$$=\frac{\dfrac{1}{2}}{\left(1-\dfrac{1}{2^2}\right)\left(1-\dfrac{1}{3^2}\right)}$$

$$=\frac{1}{2}\left(1+\frac{1}{2^2}+\frac{1}{2^4}+\quad\right)\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\ .\ \right)\left(1+\frac{1}{5^2}+\frac{1}{5^4}+\quad\right)$$

$$=\frac{1}{2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\quad\right)=\frac{\pi^2}{12}$$

Hence

$$\frac{\pi^2}{12}=\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right) \quad .$$

$$\log\frac{12}{\pi^2}=-\log\left(1-\frac{1}{4^2}\right)-\log\left(1-\frac{1}{6^2}\right)-$$

$$=\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+$$

$$+\frac{1}{2}\left(\frac{1}{4^4}+\frac{1}{6^4}+\ \right)+\frac{1}{3}\left(\frac{1}{4^6}+\frac{1}{6^6}+\ .\right)$$

$$=S_2+\frac{1}{2}S_4+\frac{1}{3}S_6+ \quad .$$

46 Let $a, b, c,$ be all the prime numbers except unity The numbers which are not divisible by the square of a prime other than unity are

$$a, b, c \quad ab, ac, bc, \quad abc$$

Hence the sum required is

$$1+\frac{1}{a^2}+\frac{1}{b^2}+ \quad +\frac{1}{a^2b^2}+\frac{1}{a^2c^2}+\frac{1}{a^2b^2c^2}+$$

$$=\left(1+\frac{1}{a^2}\right)\left(1+\frac{1}{b^2}\right)\left(1+\frac{1}{c^2}\right)$$

$$=\frac{15}{\pi^2} \qquad . \ . \qquad \text{(Ex 44)}$$

47. From Example 44 we have

$$\left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{a^2}\right)\left(1+\frac{1}{b^2}\right)\left(1+\frac{1}{c^2}\right) \quad =\frac{30}{\pi^2},$$

or, since $\tan A$, $\tan B$, represent successive prime numbers,

$$(1+\cot^2 A)(1+\cot^2 B)(1+\cot^2 C) \quad =\frac{30}{\pi^2}.$$

$$\operatorname{cosec}^2 A \,.\, \operatorname{cosec}^2 B \,.\, \operatorname{cosec}^2 C \quad =\frac{30}{\pi^2}.$$

$$\therefore \sin A \,.\, \sin B \,.\, \sin C \quad =\frac{\pi}{\sqrt{30}}$$

48 $$\frac{\sin\theta}{\theta}=\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\quad ,$$

By taking logarithms and differentiating (or from Example 25) we obtain

$$\frac{\cos\theta}{\sin\theta}-\frac{1}{\theta}=-\frac{2\theta}{\pi^2}\left\{\frac{\frac{1}{1^2}}{1-\frac{\theta^2}{\pi^2}}+\frac{\frac{1}{2^2}}{1-\frac{\theta^2}{2^2\pi^2}}+\quad\right\}$$

$$\begin{aligned}\frac{\sin\theta-\theta\cos\theta}{\sin\theta}=\frac{2\theta^2}{\pi^2}&\left\{\frac{1}{1^2}\left(1+\frac{\theta^2}{1^2\pi^2}+\frac{\theta^4}{1^4\pi^4}+\quad\right)\right.\\&+\frac{1}{2^2}\left(1+\frac{\theta^2}{2^2\pi^2}+\frac{\theta^4}{2^4\pi^4}+\quad\right)\\&\left.+\frac{1}{3^2}\left(1+\frac{\theta^2}{3^2\pi^2}+\frac{\theta^4}{3^4\pi^4}+\quad\right)\quad\right\}\\&=2\left\{\frac{S_2}{\pi^2}\theta^2+\frac{S_4}{\pi^4}\theta^4+\frac{S_6}{\pi^6}\theta^6+\quad\right\}\end{aligned}$$

$$\begin{aligned}\frac{1}{2}(\sin\theta-\theta\cos\theta)&=\left\{\frac{S_2}{\pi^2}\theta^2+\frac{S_4}{\pi^4}\theta^4+\quad\right\}\sin\theta\\&=\left\{\frac{S_2}{\pi^2}\theta^2+\frac{S_4}{\pi^4}\theta^4+\quad\right\}\left(\theta-\frac{\theta^3}{\lfloor\underline{3}}+\frac{\theta^5}{\lfloor\underline{5}}-\quad\right)\end{aligned}$$

The coefficient of θ^{2n+1} on the right is

$$-(-1)^n\left\{\frac{S_2}{\pi^2}\quad\frac{1}{\lfloor\underline{2n-1}}-\frac{S_4}{\pi^4}\frac{1}{\lfloor\underline{2n-3}}+\quad-(-1)^n\frac{S_{2n}}{\pi^{2n}}\right\}$$

The coefficient of θ^{2n+1} in the expansion of $\frac{1}{2}(\sin\theta-\theta\cos\theta)$ is

$$+(-1)^n\ \frac{1}{2}\left\{\frac{1}{\lfloor\underline{2n+1}}-\frac{1}{\lfloor\underline{2n}}\right\},$$

or $$\frac{-n}{\lfloor\underline{2n+1}}(-1)^n$$

Hence $$S_2\ \frac{\pi^{2n-2}}{\lfloor\underline{2n-1}}-S_4\ \frac{\pi^{2n-4}}{\lfloor\underline{2n-3}}+\quad-(-1)^n S_{2n}=\frac{n\pi^{2n}}{\lfloor\underline{2n+1}}.$$

If $n=1$ we get $S_2=\dfrac{\pi^2}{\lfloor\underline{3}}$

If $n=2$ we get

$$S_2\cdot\frac{\pi^2}{\lfloor\underline{3}}-S_4\ =\frac{2\pi^4}{\lfloor\underline{5}}$$

$$S_4=\frac{\pi^4}{36}-\frac{2\pi^4}{5\ 4\ 3\ 2}=\frac{\pi^4}{90}$$

49 From Art. 347 we obtain by putting $\frac{\pi}{2n}$ for θ and writing θ instead of ϕ,

$$\cos n\theta = 2^{n-1}\left(\cos\theta - \cos\frac{\pi}{2n}\right)\left(\cos\theta - \cos\frac{3\pi}{2n}\right)\ldots\left(\cos\theta - \cos\frac{2n-1}{2n}\pi\right)$$

Take logarithms, put $\theta+h$ for θ, and subtract Therefore

$$\log\frac{\cos(n\theta+nh)}{\cos n\theta} = \log\frac{\cos(\theta+h)-\cos\frac{\pi}{2n}}{\cos\theta-\cos\frac{\pi}{2n}} + \log\frac{\cos(\theta+h)-\cos\frac{3\pi}{2n}}{\cos\theta-\cos\frac{3\pi}{2n}} + \ldots$$

Expand both sides in powers of h and equate the coefficients of h

$$\log\frac{\cos(n\theta+nh)}{\cos n\theta} = \log(\cos nh - \tan n\theta \sin nh)$$
$$= \log(1 - nh\tan n\theta + \text{higher powers of } h)$$
$$= -nh\tan n\theta + \ldots$$

$$\log\frac{\cos(\theta+h)-\cos\frac{\pi}{2n}}{\cos\theta-\cos\frac{\pi}{2n}} = \log\frac{\cos\theta-\cos\frac{\pi}{2n} - h\sin\theta + \ldots}{\cos\theta-\cos\frac{\pi}{2n}}$$
$$= \log\left(1 - \frac{\sin\theta}{\cos\theta-\cos\frac{\pi}{2n}}\cdot h + \ldots\right)$$
$$= -\frac{\sin\theta}{\cos\theta-\cos\frac{\pi}{2n}}h + \ldots$$

$$\log\frac{\cos(\theta+h)-\cos\frac{3\pi}{2n}}{\cos\theta-\cos\frac{3\pi}{2n}} = -\frac{\sin\theta}{\cos\theta-\cos\frac{3\pi}{2n}}\cdot h + \ldots,$$

and so on Hence

$$n\tan n\theta = \frac{\sin\theta}{\cos\theta-\cos\frac{\pi}{2n}} + \frac{\sin\theta}{\cos\theta-\cos\frac{3\pi}{2n}} + \ldots$$

50 From the last example

$$\frac{n\tan n\theta\cos\theta}{\sin\theta} = \frac{\cos\theta}{\cos\theta-\cos\frac{\pi}{2n}} + \frac{\cos\theta}{\cos\theta-\cos\frac{3\pi}{2n}} + \ldots$$

$$= 1 + \frac{\cos\frac{\pi}{2n}}{\cos\theta-\cos\frac{\pi}{2n}} + 1 + \frac{\cos\frac{3\pi}{2n}}{\cos\theta-\cos\frac{3\pi}{2n}} + \ldots$$

Hence the given series $=n\left\{\frac{\tan n\theta\cos\theta}{\sin\theta}-1\right\}$

$$=n\left\{\frac{\sin n\theta\cos\theta}{\cos n\theta\sin\theta}-1\right\}=\frac{n\sin(n-1)\theta}{\cos n\theta\sin\theta}$$

51. From Art 346, we have

$$x^{2n}-2x^na^n\cos n\theta+a^{2n}=(x^2-2ax\cos\theta+a^2)\left\{x^2-2ax\cos\left(\theta+\frac{2\pi}{n}\right)+a^2\right\}\quad.$$

Take logarithms, write $x+h$ for x, subtract the two equations, expand in powers of h and equate the coefficients of h

$$\log\frac{(x+h)^{2n}-2(x+h)^na^n\cos n\theta+a^{2n}}{x^{2n}-2x^na^n\cos n\theta+a^{2n}}$$

$$=\log\frac{x^{2n}+2nhx^{2n-1}-2x^na^n\cos n\theta-2nx^{n-1}\;\;ha^n\cos n\theta+a^{2n}+}{x^{2n}-2x^na^n\cos n\theta+a^{2n}}$$

$$=\log\left(1+2nhx^{n-1}\;\;\frac{x^n-a^n\cos n\theta}{x^{2n}-2x^na^n\cos n\theta+a^{2n}}+\;\;\right)$$

$$=2nhx^{n-1}\;\;\frac{x^n-a^n\cos n\theta}{x^{2n}-2x^na^n\cos n\theta+a^{2n}}+$$

$$\log\frac{(x+h)^2-2a(x+h)\cos\theta+a^2}{x^2-2ax\cos\theta+a^2}=\log\left(1+2h\;\;\frac{x-a\cos\theta}{x^2-2ax\cos\theta+a^2}+\;\;\right)$$

$$=2h\cdot\frac{x-a\cos\theta}{x^2-2ax\cos\theta+a^2}+\quad,$$

and similarly for the other terms Hence

$$\frac{x^n-a^n\cos n\theta}{x^{2n}-2x^na^n\cos n\theta+a^{2n}}=\frac{1}{nx^{n-1}}\Sigma\frac{x-a\cos\left(\theta-\frac{2r\pi}{n}\right)}{x^2-2ax\cos\left(\theta+\frac{2r\pi}{n}\right)+a^2}$$

52 $\cos n\theta=2^{n-1}\left(\cos\theta-\cos\frac{\pi}{2n}\right)\left(\cos\theta-\cos\frac{3\pi}{2n}\right)\quad.$

If p be less than n we can resolve $\frac{\cos^p\theta}{\cos n\theta}$ into partial fractions Thus

$$\frac{\cos^p\theta}{\cos n\theta}=\frac{A_1}{\cos\theta-\cos\frac{\pi}{2n}}+\frac{A_2}{\cos\theta-\cos\frac{3\pi}{2n}}+\quad+\frac{A_{r+1}}{\cos\theta-\cos\frac{2r+1}{2n}\pi}+$$

To find A_{r+1} multiply up by $\cos\theta - \cos\frac{2r+1}{2n}\pi$ and put $\theta = \frac{2r-1}{2n}\pi$.

Therefore

$$A_{r+1} = \text{limit of } \frac{\cos^p\theta\left(\cos\theta - \cos\frac{2r+1}{2n}\pi\right)}{\cos n\theta} \text{ when } \theta = \frac{2r+1}{2n}\pi$$

$$= \cos^p\frac{2r+1}{2n}\pi \times \text{limit of } \frac{\cos\theta - \cos\frac{2r+1}{2n}\pi}{\cos n\theta}$$

To find this limit put $\theta = \frac{2r+1}{2n}\pi + h$ and make h indefinitely small. Keeping only the first power of h, we have

$$\frac{\cos\theta - \cos\frac{2r+1}{2n}\pi}{\cos n\theta} = \frac{\cos\frac{2r+1}{2n}\pi - h\sin\frac{2r+1}{2n}\pi - \cos\frac{2r+1}{2n}\pi}{\cos\frac{2r+1}{2}\pi - nh\sin\frac{2r+1}{2}\pi}$$

$$= \frac{1}{n}\cdot\frac{\sin\frac{2r+1}{2n}\pi}{\sin(2r+1)\frac{\pi}{2}} = (-1)^r\cdot\frac{1}{n}\sin\frac{2r+1}{2n}\pi$$

Hence $$\frac{\cos^p\theta}{\cos n\theta} = \frac{1}{n}\Sigma(-1)^r\frac{\sin\frac{2r+1}{2n}\pi\cos^p\frac{2r+1}{2n}\pi}{\cos\theta - \cos\frac{2r+1}{2n}\pi}$$

53 From Example 49,

$$\frac{n\tan n\theta}{\sin\theta} = \frac{1}{\cos\theta - \cos\frac{\pi}{2n}} + \frac{1}{\cos\theta - \cos\frac{3\pi}{2n}} + \dots$$

Put $\theta = 0$, then $\cos\theta = 1$, therefore

$$\frac{1}{1-\cos\frac{\pi}{2n}} + \frac{1}{1-\cos\frac{3\pi}{2n}} + \dots = \text{the limit of } n\frac{\tan n\theta}{\sin\theta} \text{ when } \theta = 0$$

$$= \text{the limit of } n^2\cdot\frac{\sin n\theta}{n\theta}\cdot\frac{\theta}{\sin\theta}\cdot\frac{1}{\cos n\theta}$$

$$= n^2;$$

$$\left(\text{vers}\frac{\pi}{2n}\right)^{-1} + \left(\text{vers}\frac{3\pi}{2n}\right)^{-1} + \dots = n^2.$$

54 $\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)$.

For θ put $a-a\sqrt{-1}$, then $\theta^2=-2a^2\sqrt{-1}$,

and $$\sin\theta=\sin a\cos(a\sqrt{-1})-\cos a\sin(a\sqrt{-1})$$

$$=\frac{1}{2}\sin a\,(e^a+e^{-a})-\frac{\sqrt{-1}}{2}\cos a\,(e^a-e^{-a})$$

Hence

$$\frac{1}{2}\sin a\,(e^a+e^{-a})-\frac{1}{2}\cos a\,(e^a-e^{-a})\sqrt{-1}$$

$$=(a-a\sqrt{-1})\left(1+\frac{2a^2}{\pi^2}\sqrt{-1}\right)\left(1+\frac{2a^2}{2^2\pi^2}\sqrt{-1}\right)\left(1+\frac{2a^2}{3^2\pi^2}\sqrt{-1}\right)$$

Therefore, by Ex 31, Ch XIX,

$$\tan^{-1}(-1)+\tan^{-1}\frac{2a^2}{\pi^2}+\tan^{-1}\frac{2a^2}{2^2\pi^2}+\tan^{-1}\frac{2a^2}{3^2\pi^2}+$$

$$=\tan^{-1}\left\{-\frac{\cos a\,(e^a-e^{-a})}{\sin a\,(e^a+e^{-a})}\right\}.$$

$$\tan^{-1}\frac{2a^2}{\pi^2}+\tan^{-1}\frac{2a^2}{2^2\pi^2}+ \quad =\frac{\pi}{4}-\tan^{-1}\left(\frac{e^a-e^{-a}}{e^a+e^{-a}}\cot a\right)$$

55 In Example 19 for x write $\frac{\pi x\sqrt{-1}}{4}$ Then

$$\cos\frac{\pi x\sqrt{-1}}{4}+\sin\frac{\pi x\sqrt{-1}}{4}=\left(1+\frac{x\sqrt{-1}}{1}\right)\left(1-\frac{x\sqrt{-1}}{3}\right)\left(1+\frac{x\sqrt{-1}}{5}\right)$$

$$\cos\frac{\pi x\sqrt{-1}}{4}+\sin\frac{\pi x\sqrt{-1}}{4}=\frac{1}{2}(e^{\frac{\pi x}{4}}+e^{-\frac{\pi x}{4}})+\frac{1}{2}(e^{\frac{\pi x}{4}}-e^{-\frac{\pi x}{4}})\sqrt{-1}$$

Therefore by Ex 31, Ch XIX,

$$\tan^{-1}x-\tan^{-1}\frac{x}{3}+\tan^{-1}\frac{x}{5}- \quad =\tan^{-1}\frac{e^{\frac{\pi x}{4}}-e^{-\frac{\pi x}{4}}}{e^{\frac{\pi x}{4}}+e^{-\frac{\pi x}{4}}}$$

$$=-\tan^{-1}\frac{1-e^{\frac{\pi x}{2}}}{1+e^{\frac{\pi x}{2}}}=\tan^{-1}e^{\frac{1}{2}\pi x}-\frac{\pi}{4}$$

56. We have

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\dots\equiv(x-\alpha)(x-\beta)(x-\gamma)\dots$$

$$\therefore x^n+p_1x^{n-1}y+p_2x^{n-2}y^2+\dots\equiv(x-\alpha y)(x-\beta y)(x-\gamma y)\dots$$

Put $y=\cos\theta+\sqrt{-1}\sin\theta$.

$$\therefore x^n+p_1x^{n-1}\cos\theta+p_2x^{n-2}\cos 2\theta+\dots$$
$$+\sqrt{-1}\,(p_1x^{n-1}\sin\theta+p_2x^{n-2}\sin 2\theta+\dots)$$
$$\equiv(x-\alpha\cos\theta-\alpha\sin\theta\,.\,\sqrt{-1})(x-\beta\cos\theta-\beta\sin\theta\,.\,\sqrt{-1})\dots$$

Therefore by Example 31, Ch. XIX.,

$$\tan^{-1}\frac{\alpha\sin\theta}{\alpha\cos\theta-x}+\tan^{-1}\frac{\beta\sin\theta}{\beta\cos\theta-x}+\dots$$
$$=\tan^{-1}\frac{p_1x^{n-1}\sin\theta+p_2x^{n-2}\sin 2\theta+\dots}{x^n+p_1x^{n-1}\cos\theta+p_2x^{n-2}\cos 2\theta+\dots}.$$

57. $\sin\theta=\theta\left(1-\dfrac{\theta^2}{\pi^2}\right)\left(1-\dfrac{\theta^2}{2^2\pi^2}\right)\dots$

For θ put $\cos\phi+\sqrt{-1}\sin\phi$. Then

$$\sin(\cos\phi+\sqrt{-1}\sin\phi)$$
$$=\sin(\cos\phi)\cos(\sqrt{-1}\sin\phi)+\cos(\cos\phi)\sin(\sqrt{-1}\sin\phi)$$
$$=\frac{1}{2}\sin(\cos\phi)\left(e^{\sin\phi}+e^{-\sin\phi}\right)+\frac{\sqrt{-1}}{2}\cos(\cos\phi)\left(e^{\sin\phi}-e^{-\sin\phi}\right)$$

This expression is therefore

$$=(\cos\phi+\sqrt{-1}\sin\phi)\left(1-\frac{\cos 2\phi+\sqrt{-1}\sin 2\phi}{\pi^2}\right)$$
$$\left(1-\frac{\cos 2\phi+\sqrt{-1}\sin 2\phi}{2^2\pi^2}\right)\dots$$

Hence by Example 31, Ch. XIX.,

$$\tan^{-1}(\tan\phi)-\tan^{-1}\frac{\sin 2\phi}{\pi^2-\cos 2\phi}-\tan^{-1}\frac{\sin 2\phi}{2^2\pi^2-\cos 2\phi}-\dots$$
$$=\tan^{-1}\left\{\frac{\cos(\cos\phi)}{\sin(\cos\phi)}\,\frac{e^{\sin\phi}-e^{-\sin\phi}}{e^{\sin\phi}+e^{-\sin\phi}}\right\}.$$

$$\therefore\tan^{-1}\frac{\sin 2\phi}{\pi^2-\cos 2\phi}+\tan^{-1}\frac{\sin 2\phi}{2^2\pi^2-\cos 2\phi}+\dots$$
$$=\phi-\tan^{-1}\left\{\cot(\cos\phi)\,\frac{e^{\sin\phi}-e^{-\sin\phi}}{e^{\sin\phi}+e^{-\sin\phi}}\right\}$$

58 Let r be the radius of the circle The polygon can be resolved into n triangles, and thus the area of the polygon

$$=\frac{r^2}{2}\{\sin\alpha+\sin 2\alpha+\sin 3\alpha+ \quad +\sin n\alpha\}$$

$$=\frac{r^2}{2}\cdot\frac{\sin\frac{n+1}{2}\alpha\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}.$$

But $$\alpha+2\alpha+3\alpha+ \quad +n\alpha=2\pi,$$

that is $$\frac{n(n+1)\alpha}{2}=2\pi,$$

so that $$\frac{n+1}{2}\alpha=\frac{2\pi}{n}.$$

Now the area of the regular polygon of n sides

$$=\frac{nr^2}{2}\sin\frac{2\pi}{n}=\frac{nr^2}{2}\sin\frac{n+1}{2}\alpha.$$

Hence the ratio of the former area to the latter $=\dfrac{\sin\frac{n\alpha}{2}}{n\sin\frac{\alpha}{2}}$.

59 Let $A, B, C,$ be the angles of the polygon From A draw straight lines to the other angles Let AP be the m^{th} straight line, so that AP subtends at the centre of the circle the angle $m\frac{2\pi}{n}$. Then $AP=2a\sin m\beta$ where $\beta=\frac{\pi}{n}$.

Thus the product of all the straight lines

$$=(2a)^{n-1}\sin\beta\sin 2\beta\sin 3\beta. \quad \sin(n-1)\beta$$
$$=na^{n-1},$$

for by Art 343 we have

$$n=2^{n-1}\sin\beta\sin 2\beta\sin 3\beta \quad (n-1)\beta$$

60 Let $A, B, C,$ be the points of contact of the circle with the circumscribed polygon taken in order. Let O be the fixed point, and suppose the arc OA to subtend an angle 2ϕ at the centre of the circle Then the angle between OA and the tangent at A is ϕ, and the length of the perpendicular from O on this tangent is $OA\sin\phi$, that is $2a\sin^2\phi$. Thus we have

$$p_1=2a\sin^2\phi.$$

Let $\beta=\frac{\pi}{2n}$, then we obtain in a similar way

$$p_2=2a\sin^2(\phi+\beta),$$
$$p_3=2a\sin^2(\phi+2\beta),$$
$$p_4=2a\sin^2(\phi+3\beta),$$

and so on

Thus $\quad p_1p_3p_5 \ldots p_{2n-1}=(2a)^n\sin^2\phi\sin^2(\phi+2\beta)\ldots\sin^2\{\phi+(2n-2)\beta\}$

$$=(2a)^n\left\{\frac{\sin n\phi}{2^{n-1}}\right\}^2, \text{ by Art. 342,}$$

$$=\frac{a^n}{2^{n-2}}\sin^2 n\phi$$

In the same way we have

$$p_2p_4\ldots p_{2n}=(2a)^n\sin^2(\phi+\beta)\sin^2(\phi+3\beta)\ldots\sin^2\{\phi+(2n-1)\beta\}$$
$$=\frac{a^n}{2^{n-2}}\cos^2 n\phi$$

Hence by addition we obtain

$$\frac{a^n}{2^{n-2}}(\sin^2 n\phi+\cos^2 n\phi), \quad \text{that is } \frac{a^n}{2^{n-2}}$$

61 Let $A, B, C, D, \ldots$ be the angular points of the inscribed polygon Let O be the fixed point from which the perpendiculars are drawn Let the arc OA subtend an angle 2α at the centre of the circle, let the arc OB subtend an angle 2β, the arc OC an angle 2γ, and so on.

Let $p_1, p_2, p_3, \ldots$ denote the perpendiculars from O on the sides of the circumscribed polygon which touch the circle at $A, B, C, \ldots$ respectively Then

$$p_1=OA\sin\alpha, \quad p_2=OB\sin\beta, \quad p_3=OC\sin\gamma, \ldots$$

Again, let $q_1, q_2, q_3, \ldots$ denote the perpendiculars from O on the sides of the inscribed polygon $AB, BC, CD, \ldots$ respectively Then

$$q_1=OA\sin OAB=OA\sin(\pi-\beta)=OA\sin\beta,$$

similarly $\qquad q_2=OB\sin\gamma, \quad q_3=OC\sin\delta, \ldots$

Thus $p_1p_2p_3\ldots$ and $q_1q_2q_3\ldots$ are equal, for each is equal to the product of the same series of lengths into the same series of sines

62
$$\sin^2\theta_r=\frac{c^2-\frac{a^2}{r^2}}{c^2}=1-\frac{a^2}{c^2r^2}.$$

$$\sin^2\theta_1\sin^2\theta_2\sin^2\theta_3\ldots=\left(1-\frac{a^2}{c^2}\right)\left(1-\frac{a^2}{2^2c^2}\right)\left(1-\frac{a^2}{3^2c^2}\right)\ldots$$

$$=\frac{\sin\frac{\pi a}{c}}{\frac{\pi a}{c}}.$$

$$\therefore \sin\theta_1\sin\theta_2\sin\theta_3\ldots=\sqrt{\frac{c}{\pi a}\sin\frac{\pi a}{c}}$$

63 $\sin^2\theta_1=\frac{2^2}{1^2+2^2}$, $\sin^2\theta_2=\frac{3^2}{1^2+3^2}$, $\sin^2\theta_3=\frac{4^2}{1+4^2}$,

Now by Art 345

$$\frac{e^x-e^{-x}}{2x}=\left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{2^2\pi^2}\right)\left(1+\frac{x^2}{3^2\pi^2}\right)$$

Put $x=\pi$ Therefore

$$\frac{e^\pi-e^{-\pi}}{2\pi}=2\cdot\frac{2^2+1}{2^2}\cdot\frac{3^2+1}{3^2}\cdot\frac{4^2+1}{4^2}$$

$$=2\cdot\frac{1}{\sin^2\theta_1}\;\frac{1}{\sin^2\theta_2}\;\frac{1}{\sin^2\theta_3}$$

$$\sin\theta_1\sin\theta_2\sin\theta_3=2\sqrt{\frac{\pi}{e^\pi-e^{-\pi}}}$$

64 Let AB be the chord, N its middle point, P_1P_2 the angular points of the polygon, $BON=a$

Let $P_1ON=\theta$, then $P_2ON=\theta+\frac{2\pi}{n}$, $P_3ON=\theta+\frac{4\pi}{n}$ &c

The perpendiculars from P_1, P_2, on AB are

$$a(\cos\alpha-\cos\theta),\ a\left\{\cos\alpha-\cos\left(\theta+\frac{2\pi}{n}\right)\right\},\ a\left\{\cos\alpha-\cos\left(\theta+\frac{4\pi}{n}\right)\right\}$$

Therefore the product of the perpendiculars is

$$a^n(\cos\alpha-\cos\theta)\left\{\cos\alpha-\cos\left(\theta+\frac{2\pi}{n}\right)\right\}\quad\left\{\cos\alpha-\cos\left(\theta+\frac{2n-2}{n}\pi\right)\right\}$$

$$=\frac{a^n}{2^{n-1}}(\cos n\alpha-\cos n\theta)\quad\text{(Art 347).}$$

Now $\theta-\frac{\pi}{n}$ is the angle between ON and the perpendicular on the side P_nP_1 and may therefore be taken to be the angle β.

Then $$\cos n\theta=\cos(n\beta+\pi)=-\cos n\beta$$

Therefore the product of the perpendiculars is

$$\frac{a^n}{2^{n-1}}(\cos n\alpha+\cos n\beta).$$

65 Let a be the radius of the circumscribing circle of the polygon, r the distance of P from the centre, θ the angle between OP and a radius to any angular point

Then $\rho_1^2=r^2-2ra\cos\theta+a^2$, $\rho_2^2=r^2-2ra\cos\left(\theta+\frac{2\pi}{n}\right)+a^2$,

and so on

Now $$r^{2n}-2r^na^n\cos n\theta+a^{2n}$$
$$=(r^2-2ra\cos\theta+a^2)\left\{r^2-2ra\cos\left(\theta+\frac{2\pi}{n}\right)+a^2\right\}\ \ldots$$

Take logarithms and differentiate with respect to r, (or from Ex 51),

$$n\cdot\frac{2r^{2n-1}-2r^{n-1}a^n\cos n\theta}{r^{2n}-2r^na^n\cos n\theta+a^{2n}}=\frac{2r-2a\cos\theta}{r^2-2ra\cos\theta+a^2}$$
$$+\frac{2r-2a\cos\left(\theta+\frac{2\pi}{n}\right)}{r^2-2ra\cos\left(\theta+\frac{2\pi}{n}\right)+a^2}+\ldots$$

Multiply by r, subtract n from the left-hand side, and unity from each fraction on the right;

$$n\cdot\frac{r^{2n}-a^{2n}}{r^{2n}-2r^na^n\cos n\theta+a^{2n}}=\frac{r^2-a^2}{r^2-2ra\cos\theta+a^2}$$
$$+\frac{r^2-a^2}{r^2-2ra\cos\left(\theta+\frac{2\pi}{n}\right)+a^2}+\ldots$$
$$=(r^2-a^2)\left(\frac{1}{\rho_1^2}+\frac{1}{\rho_2^2}+\ldots+\frac{1}{\rho_n^2}\right)$$

which gives the required result.

XXIV.

1 $\sinh 2\theta=2\sinh\theta\cosh\theta=\dfrac{2\sinh\theta\cosh\theta}{\cosh^2\theta-\sinh^2\theta}$

$$=\frac{\dfrac{2\sinh\theta}{\cosh\theta}}{1-\dfrac{\sinh^2\theta}{\cosh^2\theta}}=\frac{2\tanh\theta}{1-\tanh^2\theta}.$$

2. $\cosh 2\theta=\cosh^2\theta+\sinh^2\theta=\dfrac{\cosh^2\theta+\sinh^2\theta}{\cosh^2\theta-\sinh^2\theta}$

$$=\frac{1+\dfrac{\sinh^2\theta}{\cosh^2\theta}}{1-\dfrac{\sinh^2\theta}{\cosh^2\theta}}=\frac{1+\tanh^2\theta}{1-\tanh^2\theta}.$$

3 $\sinh 3\theta=\sinh(2\theta+\theta)=\sinh 2\theta\cosh\theta+\cosh 2\theta\sinh\theta$
$$=2\sinh\theta\cosh^2\theta+(1+2\sinh^2\theta)\sinh\theta$$
$$=2\sinh\theta\,(1+\sinh^2\theta)+\sinh\theta+2\sinh^3\theta$$
$$=3\sinh\theta+4\sinh^3\theta.$$

4 $$\begin{aligned}\cosh 3\theta &= \cosh(2\theta+\theta) = \cosh 2\theta \cosh\theta + \sinh 2\theta \sinh\theta \\ &= (2\cosh^2\theta - 1)\cosh\theta + 2\sinh^2\theta\cosh\theta \\ &= 2\cosh^3\theta - \cosh\theta + 2(\cosh^2\theta - 1)\cosh\theta \\ &= 4\cosh^3\theta - 3\cosh\theta\end{aligned}$$

5 $$\begin{aligned}\sinh(\alpha+\beta+\gamma) &= \sinh(\alpha+\beta)\cosh\gamma + \cosh(\alpha+\beta)\sinh\gamma \\ &= (\sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta)\cosh\gamma \\ &\qquad + (\cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta)\sinh\gamma\end{aligned}$$

6 $$\begin{aligned}\cosh(\alpha+\beta+\gamma) &= \cosh(\alpha+\beta)\cosh\gamma + \sinh(\alpha+\beta)\sinh\gamma \\ &= (\cosh\alpha\cos\beta + \sinh\alpha\sinh\beta)\cosh\gamma \\ &\qquad + (\sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta)\sinh\gamma\end{aligned}$$

7 $$\begin{aligned}&\sinh(\alpha+\beta+\gamma) - \sinh\alpha - \sinh\beta - \sinh\gamma \\ &= 2\cosh\frac{2\alpha+\beta+\gamma}{2}\sinh\frac{\beta+\gamma}{2} - 2\sinh\frac{\beta+\gamma}{2}\cosh\frac{\beta-\gamma}{2} \\ &= 2\sinh\frac{\beta+\gamma}{2}\left(\cosh\frac{2\alpha+\beta+\gamma}{2} - \cosh\frac{\beta-\gamma}{2}\right) \\ &= 4\sinh\frac{\beta+\gamma}{2}\sinh\frac{\gamma+\alpha}{2}\sinh\frac{\alpha+\beta}{2}.\end{aligned}$$

8 $$\begin{aligned}&\cosh(\alpha+\beta+\gamma) + \cosh\alpha + \cosh\beta + \cosh\gamma \\ &= 2\cosh\tfrac{1}{2}(2\alpha+\beta+\gamma)\cosh\tfrac{1}{2}(\beta+\gamma) + 2\cosh\tfrac{1}{2}(\beta+\gamma)\cosh\tfrac{1}{2}(\beta-\gamma) \\ &= 2\cosh\tfrac{1}{2}(\beta+\gamma)\left\{\cosh\tfrac{1}{2}(2\alpha+\beta+\gamma) + \cosh\tfrac{1}{2}(\beta-\gamma)\right\} \\ &= 4\cosh\tfrac{1}{2}(\beta+\gamma)\cosh\tfrac{1}{2}(\gamma+\alpha)\cosh\tfrac{1}{2}(\alpha+\beta)\end{aligned}$$

9 $$\begin{aligned}\cos^2\alpha\cosh^2\beta + \sin^2\alpha\sinh^2\beta &= (1-\sin^2\alpha)\cosh^2\beta + \sin^2\alpha(\cosh^2\beta - 1) \\ &= \cosh^2\beta - \sin^2\alpha\end{aligned}$$

10 $$\begin{aligned}\cosh\theta + \sinh\theta &= \tfrac{1}{2}(e^\theta + e^{-\theta}) + \tfrac{1}{2}(e^\theta - e^{-\theta}) \\ &= e^\theta,\end{aligned}$$

$$\cosh\theta - \sinh\theta = e^{-\theta},$$

$$\begin{aligned}\therefore\ (\cosh\theta \pm \sinh\theta)^n &= e^{\pm n\theta} \\ &= \cosh n\theta \pm \sinh n\theta.\end{aligned}$$

11 $\cosh\alpha+\sinh\alpha=e^{\alpha}$,

$\cosh\beta+\sinh\beta=e^{\beta}$, &c ,

$$(\cosh\alpha+\sinh\alpha)(\cosh\beta+\sinh\beta)(\cosh\gamma+\sinh\gamma)$$
$$=e^{\alpha+\beta+\gamma+}$$
$$=\cosh(\alpha+\beta+\gamma+\quad)+\sinh(\alpha+\beta+\gamma+\quad)$$

12 $\cos(x+\iota y)=\cos x\cos\iota y-\sin x\sin\iota y$
$$=\cos x\cosh y-\iota\sin x\sinh y$$

13 $\sin(x+\iota y)=\sin x\cos\iota y+\cos x\sin\iota y$
$$=\sin x\cosh y+\iota\cos x\sinh y$$

14 $\tan(x+\iota y)=\dfrac{\sin(x+\iota y)}{\cos(x+\iota y)}=\dfrac{\sin x\cosh y+\iota\cos x\sinh y}{\cos x\cosh y-\iota\sin x\sinh y}$

Multiply numerator and denominator by $\cos x\cosh y+\iota\sin x\sinh y$ The denominator becomes

$$\cos^2 x\cosh^2 y+\sin^2 x\sinh^2 y$$
$$=\cosh^2 y-\sin^2 x \text{ by Ex 9}$$
$$=\frac{1}{2}\{2\cosh^2 y-1+1-2\sin^2 x\}=\frac{1}{2}(\cosh 2y+\cos 2x).$$

The numerator becomes

$$\sin x\cos x(\cosh^2 y-\sinh^2 y)+\iota\sinh y\cosh y(\sin^2 x+\cos^2 x)$$
$$=\frac{1}{2}(\sin 2x+\iota\sinh 2y),$$
$$\tan(x+\iota y)=\frac{\sin 2x+\iota\sinh 2y}{\cos 2x+\cosh 2y}.$$

15 From Ex 14,

$$\cot(x+\iota y)=\frac{\cos 2x+\cosh 2y}{\sin 2x+\iota\sinh 2y}$$
$$=\frac{\cos^2 2x-\cosh^2 2y}{\sin^2 2x+\sinh^2 2y}\times\frac{\sin 2x-\iota\sinh 2y}{\cos 2x-\cosh 2y}$$

But $\cos^2 2x-\cosh^2 2y=1-\sin^2 2x-1-\sinh^2 2y=-(\sin^2 2x+\sinh^2 2y)$

$$\cdot\ \cot(x+\iota y)=-\frac{\sin 2x-\iota\sinh 2y}{\cos 2x-\cosh 2y}.$$

16 $$\sec(x+\iota y)=\frac{1}{\cos(x+\iota y)}=\frac{1}{\cos x\cosh y-\iota\sin x\sinh y}$$

Multiply numerator and denominator by $\cos x\cosh y+\iota\sin x\sinh y$.

The denominator becomes (Ex. 14) $\frac{1}{2}\cos 2x+\frac{1}{2}\cosh 2y$.

$$\therefore \sec(x+\iota y)=2\,.\,\frac{\cos x\cosh y+\iota\sin x\sinh y}{\cos 2x+\cosh 2y}.$$

17 $$\operatorname{cosec}(x+\iota y)=\frac{1}{\sin(x+\iota y)}=\frac{1}{\sin x\cosh y+\iota\cos x\sinh y}$$
$$=\frac{\sin x\cosh y-\iota\cos x\sinh y}{\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y}.$$

The denominator $=(1-\cos^2 x)\cosh^2 y+\cos^2 x(\cosh^2 y-1)$
$$=\cosh^2 y-\cos^2 x$$
$$=\frac{1}{2}\{2\cosh^2 y-1+1-2\cos^2 x\}$$
$$=\frac{1}{2}(\cosh 2y-\cos 2x),$$

$$\operatorname{cosec}(x+\iota y)=-2\,.\,\frac{\sin x\cosh y-\iota\cos x\sinh y}{\cos 2x-\cosh 2y}.$$

18. $$\cosh u=\sec\theta,$$
$$\therefore \cosh^2 u-1=\sec^2\theta-1.$$
$$\sinh^2 u=\tan^2\theta,$$
$$\therefore \sinh u=\tan\theta.$$

Also $$\frac{\sinh u}{\cosh u}=\frac{\tan\theta}{\sec\theta}=\sin\theta.$$

Again, $$\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}=\frac{\tanh u}{1+\operatorname{sech} u}=\frac{\sinh u}{1+\cosh u}$$
$$=\frac{2\sinh\frac{1}{2}u\cosh\frac{1}{2}u}{2\cosh^2\frac{1}{2}u}=\tanh\frac{u}{2}.$$

Also, $$\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}=\sec\theta+\tan\theta$$
$$=\cosh u+\sinh u=e^u,$$
$$u=\log_e\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

19 See Ch xxi. Ex 32, and Ch. xxiv Ex. 18

20 and 21. Let $$y=\cosh\theta=\frac{1}{2}(e^{\theta}+e^{-\theta}),$$

and $$z=\sinh\theta=\frac{1}{2}(e^{\theta}-e^{-\theta}),$$

$$y+z=e^{\theta},\text{ and } y-z=e^{-\theta}.$$

$$\begin{aligned}\cosh n\theta&=\frac{1}{2}(e^{n\theta}+e^{-n\theta})\\&=\frac{1}{2}\{(y+z)^n+(y-z)^n\}\\&=y^n+\frac{n(n-1)}{2}y^{n-2}z^2+.\\&=\cosh^n\theta+\frac{n(n-1)}{2}\cosh^{n-2}\theta\sinh^2\theta+\quad;\end{aligned}$$

$$\begin{aligned}\sinh n\theta&=\frac{1}{2}(e^{n\theta}-e^{-n\theta})\\&=\frac{1}{2}\{(y+z)^n-(y-z)^n\}\\&=ny^{n-1}z+\frac{n(n-1)(n-2)}{1\ 2\ 3}.y^{n-3}z^3+\\&=n\cosh^{n-1}\theta\sinh\theta+\frac{n(n-1)(n-2)}{1.2\ 3}\cosh^{n-3}\theta\sinh^3\theta\end{aligned}$$

22 $$2\cosh\theta=e^{\theta}+e^{-\theta},$$

$$\begin{aligned}.\ 2^n\cosh^n\theta&=(e^{\theta}+e^{-\theta})^n\\&=e^{n\theta}+ne^{(n-2)\theta}+\frac{n(n-1)}{1\ 2}e^{(n-4)\theta}+\quad+ne^{-(n-2)\theta}+e^{-n\theta}\\&=(e^{n\theta}+e^{-n\theta})+n(e^{(n-2)\theta}+e^{-(n-2)\theta})+\\&=2\cosh n\theta+2.n\cosh(n-2)\theta+2\ \frac{n(n-1)}{1\ 2}\cosh(n-4)\theta+\end{aligned}$$

23 $\cosh\theta+\cosh 2\theta+\cosh 3\theta+$

$$\begin{aligned}&=\frac{1}{2}\{e^{\theta}+e^{2\theta}+e^{3\theta}+\quad+e^{n\theta}+e^{-\theta}+e^{-2\theta}+\quad+e^{-n\theta}\}\\&=\frac{1}{2}\left\{\frac{e^{\theta}-e^{(n+1)\theta}}{1-e^{\theta}}+\frac{e^{-\theta}-e^{(n+1)\theta}}{1-e^{-\theta}}\right\}\end{aligned}$$

$$=\frac{1}{2}\left\{\frac{e^{\frac{\theta}{2}}-e^{\left(n+\frac{1}{2}\right)\theta}}{e^{-\frac{\theta}{2}}-e^{\frac{\theta}{2}}}+\frac{e^{-\frac{\theta}{2}}-e^{-\left(n+\frac{1}{2}\right)\theta}}{e^{\frac{\theta}{2}}-e^{-\frac{\theta}{2}}}\right\}$$

$$=\frac{1}{2}\frac{e^{\left(n+\frac{1}{2}\right)\theta}-e^{-\left(n+\frac{1}{2}\right)\theta}-(e^{\frac{\theta}{2}}-e^{\frac{\theta}{2}})}{e^{\frac{\theta}{2}}-e^{-\frac{\theta}{2}}}$$

$$=\frac{1}{2}\frac{\sinh\left(n+\frac{1}{2}\right)\theta-\sinh\frac{1}{2}\theta}{\sinh\frac{1}{2}\theta}$$

$$=\frac{\cosh\frac{1}{2}(n+1)\theta\sinh\frac{1}{2}n\theta}{\sinh\frac{1}{2}\theta}.$$

24 $\sinh\alpha+\sinh(\alpha+2\beta)+\sinh(\alpha+3\beta)+$

$$=\frac{1}{2}\{e^{\alpha}+e^{\alpha+2\beta}+e^{\alpha+3\beta}+\quad-e^{-\alpha}-e^{-\alpha-2\beta}-e^{-\alpha-3\beta}-\quad\}$$

$$=\frac{1}{2}\left\{e^{\alpha}\ \frac{1-e^{n\beta}}{1-e^{\beta}}-e^{-\alpha}\ \frac{1-e^{-n\beta}}{1-e^{-\beta}}\right\}$$

$$=\frac{1}{2}\left\{\frac{e^{\alpha-\frac{1}{2}\beta}-e^{\alpha+\left(n+\frac{1}{2}\right)\beta}}{e^{-\frac{1}{2}\beta}-e^{\frac{1}{2}\beta}}-\frac{e^{-\alpha+\frac{1}{2}\beta}-e^{-\alpha-\left(n-\frac{1}{2}\right)\beta}}{e^{\frac{1}{3}\beta}-e^{-\frac{1}{2}\beta}}\right\}$$

$$=\frac{1}{2}\frac{e^{\alpha+\left(n-\frac{1}{2}\right)\beta}+e^{-\alpha-\left(n-\frac{1}{2}\right)\beta}-e^{\alpha-\frac{1}{2}\beta}-e^{-\alpha+\frac{1}{2}\beta}}{e^{\frac{1}{2}\beta}-e^{-\frac{1}{2}\beta}}$$

$$=\frac{1}{2}\frac{\cosh\left\{\alpha+\left(n-\frac{1}{2}\right)\beta\right\}-\cosh\left(\alpha-\frac{1}{2}\beta\right)}{\sinh\frac{1}{2}\beta}$$

$$=\frac{\sinh\frac{1}{2}(2\alpha+\overline{n-1}\beta)\sinh\frac{1}{2}n\beta}{\sinh\frac{1}{2}\beta}$$

25 $\sinh\alpha + \frac{1}{\underline{|2}}\sinh 2\alpha + \frac{1}{\underline{|3}}\sinh 3\alpha +$

$$=\frac{1}{2}\left\{e^{\alpha} + \frac{1}{\underline{|2}}\ e^{2\alpha} + \frac{1}{\underline{|3}}e^{3\alpha} + \quad - e^{-\alpha} - \frac{1}{\underline{|2}}e^{-2\alpha} - \frac{1}{\underline{|3}}e^{-3\alpha} - \quad\right\}$$

$$=\frac{1}{2}\{e^{e^{\alpha}} - e^{e^{-\alpha}}\}$$

26 $x\cosh\alpha + \frac{x^2}{\underline{|2}}\cosh 2\alpha + \frac{x^3}{\underline{|3}}\cosh 3\alpha +$

$$=\frac{1}{2}\left\{xe^{\alpha} + \frac{x^2}{\underline{|2}}e^{2\alpha} + \frac{x^3}{\underline{|3}}e^{3\alpha} + \quad + xe^{-\alpha} + \frac{x^2}{\underline{|2}}e^{-2\alpha} + \frac{x^3}{\underline{|3}}e^{-3\alpha} + \quad\right\}$$

$$=\frac{1}{2}\{e^{xe^{\alpha}} + e^{xe^{-\alpha}} - 2\}$$

27 $\cosh\theta + \frac{\sin\theta}{1}\cosh 2\theta + \frac{\sin^2\theta}{\underline{|2}}\cosh 3\theta +$

$$=\frac{1}{2}\left\{e^{\theta} + \frac{\sin\theta}{1}\ e^{2\theta} + \frac{\sin^2\theta}{\underline{|2}}e^{3\theta} + \right.$$

$$\left. + e^{-\theta} + \frac{\sin\theta}{1}e^{-2\theta} + \frac{\sin^2\theta}{\underline{|2}}e^{-3\theta} + \quad\right\}$$

$$=\frac{1}{2}\{e^{\theta} . e^{\sin\theta\ e^{\theta}} + e^{-\theta} . e^{\sin\theta\ e^{-\theta}}\}$$

$$=\frac{1}{2}\{e^{\theta + \sin\theta\ e^{\theta}} + e^{-\theta + \sin\theta\ e^{-\theta}}\}$$

28. $\cosh\theta - \frac{1}{2}\cosh 2\theta + \frac{1}{3}\cosh 3\theta -$

$$=\frac{1}{2}\left\{e^{\theta} - \frac{1}{2}e^{2\theta} + \frac{1}{3}e^{3\theta} - \quad + e^{-\theta} - \frac{1}{2}e^{2\theta} + \frac{1}{3}e^{3\theta} - \quad\right\}$$

$$=\frac{1}{2}\{\log(1 + e^{\theta}) + \log(1 + e^{-\theta})\}$$

$$=\frac{1}{2}\log(2 + e^{\theta} + e^{-\theta})$$

$$=\frac{1}{2}\log(e^{\frac{1}{2}\theta} + e^{-\frac{1}{2}\theta})^2$$

$$=\log\left(2\cosh\frac{1}{2}\theta\right)$$

29 From Art. 345 we have

$$e^{\theta}-e^{-\theta}=2\theta\left(1+\frac{\theta^2}{\pi^2}\right)\left(1+\frac{\theta^2}{2^2\pi^2}\right)$$

or
$$\sinh\theta=\theta\left(1+\frac{\theta^2}{\pi^2}\right)\left(1+\frac{\theta^2}{2^2\pi^2}\right)$$

30 From Art. 345 we have

$$e^{\theta}+e^{-\theta}=2\left(1+\frac{4\theta^2}{\pi^2}\right)\left(1+\frac{4\theta^2}{3^2\pi^2}\right)$$

or
$$\cosh\theta=\left(1+\frac{4\theta^2}{\pi^2}\right)\left(1+\frac{4\theta^2}{3^2\pi^2}\right)$$

31 $\sin x=x\left(1-\frac{x^2}{\pi^2}\right)\quad\left(1-\frac{x^2}{r^2\pi^2}\right)$.

In this equation put $x=\phi\pm\theta\sqrt{-1}$ and multiply the two results

$$\sin(\phi+\theta\sqrt{-1})\sin(\phi-\theta\sqrt{-1})=\sin^2\phi-\sin^2(\theta\sqrt{-1})$$
$$=\sin^2\phi+\sinh^2\theta=\frac{1}{2}\{2\sin^2\phi-1+1+2\sinh^2\theta\}$$
$$=\frac{1}{2}(\cosh 2\theta-\cos 2\phi)$$

Also $\left\{1-\frac{(\phi+\theta\sqrt{-1})^2}{r^2\pi^2}\right\}\left\{1-\frac{(\phi-\theta\sqrt{-1})^2}{r^2\pi^2}\right\}$

$$=\left\{1-\frac{\phi+\theta\sqrt{-1}}{r\pi}\right\}\left\{1+\frac{\phi+\theta\sqrt{-1}}{r\pi}\right\}\left\{1-\frac{\phi-\theta\sqrt{-1}}{r\pi}\right\}\left\{1+\frac{\phi-\theta\sqrt{-1}}{r\pi}\right\}$$
$$=\frac{(r\pi-\phi)^2+\theta^2}{r^2\pi^2}\times\frac{(r\pi+\phi)^2+\theta^2}{r^2\pi^2}.$$

And
$$(\phi+\theta\sqrt{-1})(\phi-\theta\sqrt{-1})=\phi^2+\theta^2$$

Hence
$$\cosh 2\theta-\cos 2\phi=2(\phi^2+\theta^2)\prod_{r=1}^{r=\infty}\frac{(r\pi\pm\phi)^2+\theta^2}{r^2\pi^2}$$

32 $\cos x=\left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{3^2\pi^2}\right)\quad\left(1-\frac{4x^2}{(2r-1)^2\pi^2}\right)$

In this equation put $x=\phi\pm\theta\sqrt{-1}$ and multiply the two results

$$\cos(\phi+\theta\sqrt{-1})\cos(\phi-\theta\sqrt{-1})=\cos^2\phi\cosh^2\theta+\sin^2\phi\sinh^2\theta$$
$$=\cosh^2\theta-\sin^2\phi\quad(\text{Ex. }9)$$
$$=\frac{1}{2}\{2\cosh^2\theta-1+1-2\sin^2\phi\}$$
$$=\frac{1}{2}(\cosh 2\theta+\cos 2\phi).$$

Also $\left\{1-\frac{4(\phi+\theta\sqrt{-1})^2}{(2r-1)^2\pi^2}\right\}\left\{1-\frac{4(\phi-\theta\sqrt{-1})^2}{(2r-1)^2\pi^2}\right\}$

$$=\left\{1-\frac{2(\phi+\theta\sqrt{-1})}{(2r-1)\pi}\right\}\left\{1-\frac{2(\phi-\theta\sqrt{-1})}{(2r-1)\pi}\right\}\left\{1+\frac{2(\phi+\theta\sqrt{-1})}{(2r-1)\pi}\right\}\left\{1+\frac{2(\phi-\theta\sqrt{-1})}{(2r-1)\pi}\right\}$$

$$=\frac{\{(2r-1)\pi-2\phi\}^2+4\theta^2}{(2r-1)^2\pi^2}\times\frac{\{(2r-1)\pi+2\phi\}^2+4\theta^2}{(2r-1)^2\pi^2}$$

Hence

$$\cosh 2\theta+\cos 2\phi=2\prod_{r=1}^{r=\infty}\frac{\{(2r-1)\pi\pm2\phi\}^2+4\theta^2}{(2r-1)^2\pi^2}$$

33 $\sin x=x\left(1-\frac{x^2}{1^2\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)$

In this equation put $x=\theta\pm\theta\sqrt{-1}$, so that $x^2=\pm2\theta^2\sqrt{-1}$,

$$\sin(\theta+\theta\sqrt{-1})=(\theta+\theta\sqrt{-1})\left(1-\frac{2\theta^2\sqrt{-1}}{1^2\pi^2}\right)\left(1-\frac{2\theta^2\sqrt{-1}}{2^2\pi^2}\right)$$

and $\sin(\theta-\theta\sqrt{-1})=(\theta-\theta\sqrt{-1})\left(1+\frac{2\theta^2\sqrt{-1}}{1^2\pi^2}\right)\left(1+\frac{2\theta^2\sqrt{-1}}{2^2\pi^2}\right)$

Multiply these results Therefore

$$\sin^2\theta-\sin^2(\theta\sqrt{-1})=2\theta^2\left(1+\frac{4\theta^4}{1^4\pi^4}\right)\left(1+\frac{4\theta^4}{2^4\pi^4}\right)$$

And $\sin^2\theta-\sin^2\theta\sqrt{-1}=\frac{1}{2}(\cosh 2\theta-\cos 2\theta)$;

$$\cosh 2\theta-\cos 2\theta=4\theta^2\left(1+\frac{4\theta^4}{1^4\pi^4}\right)\left(1+\frac{4\theta^4}{2^4\pi^4}\right)$$

34. For ϕ write $x+y\sqrt{-1}$

Then $\sin(\phi+ra)=\sin(x+ra+y\sqrt{-1})$

$$=\sin(x+ra)\cos(y\sqrt{-1})+\cos(x+ra)\sin(y\sqrt{-1})$$
$$=\sin(x+ra)\cosh y+\sqrt{-1}\cos(x+ra)\sinh y$$

And $\sin n\phi=\sin nx\cosh ny+\sqrt{-1}\cos nx\sinh ny$

Hence

$\sin nx\cosh ny+\sqrt{-1}\cos nx\sinh ny$

$$=(\sin x\cosh y+\sqrt{-1}\cos x\sinh y)$$
$$\{\sin(x+a)\cosh y+\sqrt{-1}\cos(x+a)\sinh y\}\ .$$

Hence by Ch xix, Example 19,

$$\tan^{-1}\left(\frac{\cos nx \sinh ny}{\sin nx \cosh ny}\right)=\tan^{-1}\frac{\cos x \sinh y}{\sin x \cosh y}+\tan^{-1}\frac{\cos(x+a)\sinh y}{\sin(x+a)\cosh y}+ \quad ,$$

$$\tan^{-1}(\cot nx \tanh ny)=\tan^{-1}(\cot x \tanh y)$$
$$+\tan^{-1}\{\cot(x+a)\tanh y\}+$$

35 From Example 25, Ch xxiii,

$$\cot y=\frac{1}{y}-2y\left(\frac{1}{\pi^2-y^2}+\frac{1}{2^2\pi^2-y^2}+ \quad\right),$$

$$\frac{1}{\pi^2-y^2}+\frac{1}{2^2\pi^2-y^2}+\frac{1}{3^2\pi^2-y^2}+ \quad =\frac{1}{2y^2}-\frac{1}{2y}\cot y$$

For y write $\frac{\pi x}{\sqrt{2}}(1\pm\sqrt{-1})$, then $y^2=\pm\pi^2x^2\sqrt{-1}$ We obtain the two results,

$$\frac{1}{1^2-x^2\sqrt{-1}}+\frac{1}{2^2-x^2\sqrt{-1}}+$$
$$=\frac{1}{2x^2\sqrt{-1}}-\frac{\pi^2}{\pi x\sqrt{2}(1+\sqrt{-1})}\cot\left\{\frac{\pi x}{\sqrt{2}}(1+\sqrt{-1})\right\}$$
$$=-\frac{\sqrt{-1}}{2x^2}-\frac{\pi(1-\sqrt{-1})}{2x\sqrt{2}}\cot\left\{\frac{\pi x}{\sqrt{2}}(1+\sqrt{-1})\right\},$$

$$\frac{1}{1^2+x^2\sqrt{-1}}+\frac{1}{2^2+x^2\sqrt{-1}}+$$
$$=+\frac{\sqrt{-1}}{2x^2}-\frac{\pi(1+\sqrt{-1})}{2x\sqrt{2}}\cot\left\{\frac{\pi x}{\sqrt{2}}(1-\sqrt{-1})\right\}$$

By subtraction,

$$2x^2\sqrt{-1}\sum_1^\infty\frac{1}{n^4+x^4}=-\frac{\sqrt{-1}}{x^2}-\frac{\pi(1-\sqrt{-1})}{2x\sqrt{2}}\cot\left\{\frac{\pi x}{\sqrt{2}}(1+\sqrt{-1})\right\}$$
$$+\frac{\pi(1+\sqrt{-1})}{2x\sqrt{2}}\cot\left\{\frac{\pi x}{\sqrt{2}}(1-\sqrt{-1})\right\}$$

By Example 15,

$$\cot\frac{\pi x}{\sqrt{2}}(1\pm\sqrt{-1})=-\frac{\sin(\pi x\sqrt{2})\mp\sqrt{-1}\sinh(\pi x\sqrt{2})}{\cos(\pi x\sqrt{2})-\cosh(\pi x\sqrt{2})}.$$

Hence the above expression is

$$=-\frac{\sqrt{-1}}{x^2}+\frac{\pi}{2x\sqrt{2}}\;\frac{1}{\cos(\pi x\sqrt{2})-\cosh(\pi x\sqrt{2})}$$
$$\{(1-\sqrt{-1})(\sin\pi x\sqrt{2}-\sqrt{-1}\sinh\pi x\sqrt{2})$$
$$-(1+\sqrt{-1})(\sin\pi x\sqrt{2}+\sqrt{-1}\sinh\pi x\sqrt{2})\}$$

$$=-\frac{\sqrt{-1}}{x^2}+\frac{\pi}{x\sqrt{2}}\cdot\frac{\sinh(\pi x\sqrt{2})+\sin(\pi x\sqrt{2})}{\cosh(\pi x\sqrt{2})-\cos(\pi x\sqrt{2})}\cdot\sqrt{-1}$$

Dividing by $2x^2\sqrt{-1}$ we obtain the required result

Otherwise In Example 33 put $2\theta=\pi x\sqrt{2}$, take logarithms and differentiate with respect to x The result follows at once

36 Let A_0 be the point from which x is measured; A_1, A_2, the points of division to the right of A_0, B_1, B_2, the points of division to the left of A_0 Then

$$PA_r^2=y^2+(ra-x)^2,$$

$$PB_r=y^2+(ra+x)^2$$

We require the value of

$$\Sigma\left[\frac{1}{y^2+(ra-x)^2}+\frac{1}{y^2+(ra+x)^2}\right]$$

for all positive values of r

This may be obtained from Example 31 by taking logarithms and differentiating with respect to θ Instead of differentiating we may take logarithms, write $\theta+h$ for θ, subtract, expand in powers of h and equate the coefficients of h Thus

$$\log\frac{\cosh(2\theta+2h)-\cos 2\phi}{\cosh 2\theta-\cos 2\phi}=\log\frac{(\theta+h)^2+\phi^2}{\theta^2+\phi^2}+\Sigma\log\frac{(r\pi\pm\phi)^2+(\theta+h)^2}{(r\pi\pm\phi)^2+\theta^2},$$

$$\cosh(2\theta+2h)=\frac{1}{2}(e^{2\theta+2h}+e^{-2\theta-2h})$$

$$=\frac{1}{2}\{e^{2\theta}(1+2h+\quad)+e^{-2\theta}(1-2h+\quad)\}$$

$$=\cosh 2\theta+2h\sinh 2\theta+$$

$$\therefore\log\frac{\cosh(2\theta+2h)-\cos 2\phi}{\cosh 2\theta-\cos 2\phi}=\log\left(1+2h\frac{\sinh 2\theta}{\cosh 2\theta-\cos 2\phi}+\quad\right)$$

$$=+2h\cdot\frac{\sinh 2\theta}{\cosh 2\theta-\cos 2\phi}+\quad,$$

$$\log\frac{(r\pi\pm\phi)^2+(\theta+h)^2}{(r\pi\pm\phi)^2+\theta^2}=\log\left(1+2h\ \frac{\theta}{(r\pi\pm\phi)^2+\theta^2}+\quad\right)$$

$$=2h\ \frac{\theta}{(r\pi\pm\phi)^2+\theta^2}+$$

Hence
$$\frac{\sinh 2\theta}{\cosh 2\theta-\cos 2\phi}=\frac{\theta}{\theta^2+\phi^2}+\theta\Sigma\frac{1}{(r\pi\pm\phi)^2+\theta^2}$$

For θ write $\frac{\pi y}{a}$, and for ϕ, $\frac{\pi x}{a}$, therefore

$$\frac{\sinh\frac{2\pi y}{a}}{\cosh\frac{2\pi y}{a}-\cos\frac{2\pi x}{a}}=\frac{ay}{\pi}\left\{\frac{1}{x^2+y^2}+\Sigma\frac{1}{(ra\pm x)^2+y^2}\right\}$$

$$=\frac{ay}{\pi}\left\{\frac{1}{PA_0^2}+\frac{1}{PA_1^2}+\frac{1}{PB_1^2}+\quad\right\}$$

$$\Sigma\frac{1}{PA^2}=\frac{\pi}{ay}\;\frac{\sinh\frac{2\pi y}{a}}{\cosh\frac{2\pi y}{a}-\cos\frac{2\pi x}{a}}$$

37 $\left(1+\frac{\alpha}{n\pi}\right)^4+\left(\frac{\beta}{n\pi}\right)^4$

$$=\left\{\left(1+\frac{\alpha}{n\pi}\right)^2+\left(\frac{\beta}{n\pi}\right)^2\sqrt{-1}\right\}\left\{\left(1+\frac{\alpha}{n\pi}\right)^2-\left(\frac{\beta}{n\pi}\right)^2\sqrt{-1}\right\}$$

$$=\left\{1+\frac{\alpha}{n\pi}+\frac{\beta}{n\pi}\;\frac{1-\sqrt{-1}}{\sqrt{2}}\right\}\left\{1+\frac{\alpha}{n\pi}-\frac{\beta}{n\pi}\;\frac{1-\sqrt{-1}}{\sqrt{2}}\right\}$$

$$\left\{1+\frac{\alpha}{n\pi}+\frac{\beta}{n\pi}\;\frac{1+\sqrt{-1}}{\sqrt{2}}\right\}\left\{1+\frac{\alpha}{n\pi}-\frac{\beta}{n\pi}.\;\frac{1+\sqrt{-1}}{\sqrt{2}}\right\}$$

With each factor take the corresponding factor with $-n$, and the continued product is equal to

$$P\left[1-\frac{1}{n^2\pi^2}\left\{\alpha+\beta\frac{1-\sqrt{-1}}{\sqrt{2}}\right\}^2\right]\left[1-\frac{1}{n^2\pi^2}\left\{\alpha-\beta\frac{1-\sqrt{-1}}{\sqrt{2}}\right\}^2\right]$$

$$\left[1-\frac{1}{n^2\pi^2}\left\{\alpha+\beta\frac{1+\sqrt{-1}}{\sqrt{2}}\right\}^2\right]\times\left[1-\frac{1}{n^2\pi^2}\left\{\alpha-\beta\frac{1+\sqrt{-1}}{\sqrt{2}}\right\}^2\right]$$

$$=\sin\left(\alpha+\beta\;\frac{1-\sqrt{-1}}{\sqrt{2}}\right)\sin\left(\alpha-\beta\frac{1-\sqrt{-1}}{\sqrt{2}}\right)$$

$$\sin\left(\alpha+\beta\;\frac{+\sqrt{-1}}{\sqrt{2}}\right)\sin\left(\alpha-\beta\frac{1+\sqrt{-1}}{\sqrt{2}}\right)$$

$$-\left(\alpha+\beta\;\frac{1-\sqrt{-1}}{\sqrt{2}}\right)\left(\alpha-\beta\frac{1-\sqrt{-1}}{\sqrt{2}}\right)\left(\alpha+\beta\frac{1+\sqrt{-1}}{\sqrt{2}}\right)\left(\alpha-\beta\;\frac{1+\sqrt{-1}}{\sqrt{2}}\right)$$

$$=\left[\sin^2\alpha-\sin^2\frac{\beta(1-\sqrt{-1})}{\sqrt{2}}\right]\left[\sin^2\alpha-\sin^2\frac{\beta(1+\sqrt{-1})}{\sqrt{2}}\right]-(\alpha^4+\beta^4)$$

$$=[-\cos 2\alpha+\cos\{\beta(1-\sqrt{-1})\sqrt{2}\}][-\cos 2\alpha+\cos\{\beta(1+\sqrt{-1})\sqrt{2}\}]$$
$$-4(\alpha^4+\beta^4)$$

$$=[\cos^2 2\alpha-2\cos 2\alpha\cos\beta\sqrt{2}\cos(\sqrt{2}\sqrt{-1}\beta)+\cos^2\beta\sqrt{2}-\sin^2(\beta\sqrt{2}\sqrt{-1})]$$
$$-4(\alpha^4+\beta^4)$$

$$=[\cos^2 2\alpha-2\cos 2\alpha\cos\beta\sqrt{2}\cosh\beta\sqrt{2}+\cos^2\beta\sqrt{2}+\sinh^2\beta\sqrt{2}]-4(\alpha^4+\beta^4)$$

MISCELLANEOUS EXAMPLES

1 Let x denote the number of degrees in the unit Then $3 \quad x \quad 15 \quad 1$
Hence $x=\frac{3}{15}=20$ The measure of a right angle will be $\frac{90}{20}$, that is $4\frac{1}{2}$

2 Let x denote the circular measure of the larger angle, y that of the smaller angle Then, since the circular measure of 1^0 is $\frac{\pi}{180}$, we have $x-y=\frac{\pi}{180}$, $x+y=1$ Hence $x=\frac{1}{2}\left(1+\frac{\pi}{180}\right)$, $y=\frac{1}{2}\left(1-\frac{\pi}{180}\right)$

3 Here $\tan x+\frac{ab}{\tan x}=a+b$, therefore $\tan^2 x-(a+b)\tan x+ab=0$

By solving this quadratic equation we obtain $\tan x=a$, or $\tan x=b$

4 Here $\sin(2\theta+\theta)=\sin\theta\cos 2\theta$, that is

$$\sin 2\theta\cos\theta+\cos 2\theta\sin\theta=\sin\theta\cos 2\theta,$$

therefore $\sin 2\theta\cos\theta=0$, that is $2\sin\theta\cos^2\theta=0$

If $\cos\theta=0$ we have θ an odd multiple of $\frac{\pi}{2}$, and if $\sin\theta=0$ we have θ an even multiple of $\frac{\pi}{2}$ hence all the solutions are comprised in $\theta=n\frac{\pi}{2}$, where n is zero or an integer

5 Let $2A$ denote the whole angle, and $A+x$ one of the two unequal parts, then $A-x$ denotes the other Hence we have to shew that

$$\sin(A+x)\sin(A-x)+\sin^2 x=\sin^2 A,$$

and this is obvious by Art 83

6 $(\sec\theta\sec\phi+\tan\theta\tan\phi)^2-(\tan\theta\sec\phi+\sec\theta\tan\phi)^2$

$$=\sec^2\theta\sec^2\phi+\tan^2\theta\tan^2\phi-\tan^2\theta\sec^2\phi-\sec^2\theta\tan^2\phi$$
$$=\sec^2\phi(\sec^2\theta-\tan^2\theta)-\tan^2\phi(\sec^2\theta-\tan^2\theta)$$
$$=\sec^2\phi-\tan^2\phi=1$$

$$2(1+\tan^2\theta\tan^2\phi)-\sec^2\theta\sec^2\phi=\frac{2(\cos^2\theta\cos^2\phi+\sin^2\phi\sin^2\theta)-1}{\cos^2\theta\cos^2\phi}$$

$$=\frac{(1+\cos 2\theta)\cos^2\phi+(1-\cos 2\theta)\sin^2\phi-1}{\cos^2\theta\cos^2\phi}$$

$$=\frac{\cos 2\theta(\cos^2\phi-\sin^2\phi)}{\cos^2\theta\cos^2\phi}=\frac{\cos 2\theta\cos 2\phi}{\cos^2\theta\cos^2\phi}$$

And $$1\div\frac{\cos 2\theta\cos 2\phi}{\cos^2\theta\cos^2\phi}=\frac{\cos^2\theta\cos^2\phi}{\cos 2\theta\cos 2\phi}=\frac{\sec 2\theta\sec 2\phi}{\sec^2\theta\sec^2\phi}$$

7 Since $A+B+C=360^{\circ}$, we have $\cos C=\cos(A+B)$.

Thus $1-\cos^2 A-\cos^2 B-\cos^2 C+2\cos A\cos B\cos C$

$$=1-\cos^2 A-\cos^2 B+\cos C(2\cos A\cos B-\cos C)$$
$$=1-\cos^2 A-\cos^2 B+\cos(A+B)(\cos A\cos B+\sin A\sin B)$$
$$=1-\cos^2 A-\cos^2 B+(\cos A\cos B-\sin A\sin B)(\cos A\cos B+\sin A\sin B)$$
$$=1-\cos^2 A-\cos^2 B+\cos^2 A\cos^2 B-\sin^2 A\sin^2 B$$
$$=1-\cos^2 A-\cos^2 B+\cos^2 A\cos^2 B-(1-\cos^2 A)(1-\cos^2 B)$$
$$=0$$

8
$$\sin A=\frac{3}{5},\quad \text{therefore } \cos A=\frac{4}{5}.$$
$$\sin B=\frac{12}{13},\quad \text{therefore } \cos B=\frac{5}{13}$$
$$\sin C=\frac{7}{25},\quad \text{therefore } \cos C=\frac{24}{25}$$

Hence we obtain $\sin(A+B)=\dfrac{63}{65}$, $\cos(A+B)=-\dfrac{16}{65}$;

then
$$\sin(A+B+C)=\frac{63\times 24-7\times 16}{25\times 65}=\frac{1400}{25\times 65}=\frac{56}{65}$$

9 $x=r\left(\sin\dfrac{\theta}{2}\cos\dfrac{\alpha}{2}-\cos\dfrac{\theta}{2}\sin\dfrac{\alpha}{2}\right)$, $y=r\left(\sin\dfrac{\theta}{2}\cos\dfrac{\alpha}{2}+\cos\dfrac{\theta}{2}\sin\dfrac{\alpha}{2}\right)$.

From these we obtain $\sin\dfrac{\theta}{2}=\dfrac{x+y}{2r\cos\dfrac{\alpha}{2}}$, $\cos\dfrac{\theta}{2}=\dfrac{y-x}{2r\sin\dfrac{\alpha}{2}}$.

Square and add, thus $1=\dfrac{1}{4r^2}\left\{\dfrac{(x+y)^2}{\cos^2\dfrac{\alpha}{2}}+\dfrac{(y-x)^2}{\sin^2\dfrac{\alpha}{2}}\right\}$,

therefore
$$4r^2\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}=(x+y)^2\sin^2\frac{\alpha}{2}+(y-x)^2\cos^2\frac{\alpha}{2},$$

that is $r^2\sin^2\alpha=x^2+y^2-2xy\left(\cos^2\dfrac{\alpha}{2}-\sin^2\dfrac{\alpha}{2}\right)=x^2+y^2-2xy\cos\alpha$

10 Here $\dfrac{\sin(\theta+\phi)}{\sin(\theta-\phi)}=\dfrac{a+b}{a-b}$, therefore $\dfrac{\sin(\theta+\phi)+\sin(\theta-\phi)}{\sin(\theta+\phi)-\sin(\theta-\phi)}=\dfrac{a}{b}$,

that is
$$\frac{\sin\theta\cos\phi}{\cos\theta\sin\phi}=\frac{a}{b};\ \text{so that } a\tan\phi=b\tan\theta$$

Hence
$$\frac{a\tan\dfrac{\phi}{2}}{1-\tan^2\dfrac{\phi}{2}}=\frac{b\tan\dfrac{\theta}{2}}{1-\tan^2\dfrac{\theta}{2}},$$

therefore $$a\tan\frac{\phi}{2}-b\tan\frac{\theta}{2}=\tan\frac{\theta}{2}\tan\frac{\phi}{2}\left(a\tan\frac{\theta}{2}-b\tan\frac{\phi}{2}\right)$$

$$=c\tan\frac{\theta}{2}\tan\frac{\phi}{2},$$

therefore $$a\tan\frac{\phi}{2}=\tan\frac{\theta}{2}\left(b+c\tan\frac{\phi}{2}\right).$$

Substitute for $\tan\frac{\theta}{2}$ from the second of the given equations, and we obtain

$$a^2\tan\frac{\phi}{2}=\left(b+c\tan\frac{\phi}{2}\right)\left(c+b\tan\frac{\phi}{2}\right)$$

11 Let x denote the number of degrees in one angle, then $90-x$ denotes the number of degrees in the other angle, and consequently $\frac{10}{9}(90-x)$ the number of grades Hence $x=\frac{3}{10}\times\frac{10}{9}(90-x)=\frac{1}{3}(90-x)$ Therefore $4x=90$, and $x=22\frac{1}{2}$

12 Let the circular measure of an angle be $\frac{n\pi}{20}$, then the number of degrees in it is $\frac{n\pi}{20}\,\frac{180}{\pi}$, that is $9n$, and the number of grades is $\frac{n\pi}{20}\,\frac{200}{\pi}$, that is $10n$

13 Here $(\sin\alpha\cos\beta+\cos\alpha\sin\beta)\cos\gamma=(\sin\alpha\cos\gamma+\cos\alpha\sin\gamma)\cos\beta$,

therefore $$\cos\alpha(\sin\beta\cos\gamma-\sin\gamma\cos\beta)=0,$$

that is $$\cos\alpha\sin(\beta-\gamma)=0$$

Either then $\cos\alpha=0$, so that α is an odd multiple of $\frac{\pi}{2}$, or $\sin(\beta-\gamma)=0$, so that $\beta-\gamma$ is a multiple of π

14 $\sin 4A(\tan^4 A+2\tan^2 A+1)=\sin 4A(\tan^2 A+1)^2$

$$=\frac{\sin 4A}{\cos^4 A}=\frac{2\sin 2A\cos 2A}{\cos^4 A}=\frac{4\sin A\cos 2A}{\cos^3 A}$$

$$=\frac{4\sin A(\cos^2 A-\sin^2 A)}{\cos^3 A}$$

And $$4\tan^3 A-4\tan A=4\tan A(\tan^2 A-1)$$

$$=\frac{4\sin A(\sin^2 A-\cos^2 A)}{\cos^3 A}.$$

Therefore $$\sin 4A(\tan^4 A+2\tan^2 A+1)+4\tan^3 A-4\tan A=0$$

15 By Art. 83, $\sin^2 24^\circ - \sin^2 6^\circ = \sin(24^\circ + 6^\circ)\sin(24^\circ - 6^\circ)$
$$= \sin 30^\circ \sin 18^\circ.$$

Also $\sin 30^\circ = \frac{1}{2}$, and $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$

16 The given expression is
$$\sin A(\sin A + \cos B \sin C + \cos C \sin B)$$
$$+ \sin B(\sin B + \cos C \sin A + \cos A \sin C)$$
$$+ \sin C(\sin C + \cos A \sin B + \cos B \sin A),$$

that is $\sin A\{\sin A + \sin(B+C)\} + \sin B\{\sin B + \sin(C+A)\}$
$$+ \sin C\{\sin C + \sin(A+B)\}$$

Now since $A+B+C = 360^\circ$, we have
$$\sin(B+C) = -\sin A, \quad \sin(C+A) = -\sin B, \quad \sin(A+B) = -\sin C$$
thus the whole expression vanishes

17 We have
$$\frac{\cos\alpha\cos\gamma}{a} + \frac{\sin\alpha\sin\gamma}{b} = \frac{1}{c}, \quad \text{and} \quad \frac{\cos\beta\cos\gamma}{a} + \frac{\sin\beta\sin\gamma}{b} = \frac{1}{c}$$

From these equations we find $\cos\gamma$ and $\sin\gamma$ We get

$$\cos\gamma = \frac{a(\sin\beta - \sin\alpha)}{c(\cos\alpha\sin\beta - \cos\beta\sin\alpha)} = \frac{2a\sin\frac{1}{2}(\beta-\alpha)\cos\frac{1}{2}(\beta+\alpha)}{c\sin(\beta-\alpha)}$$
$$= \frac{a\cos\frac{1}{2}(\beta+\alpha)}{c\cos\frac{1}{2}(\beta-\alpha)},$$

$$\sin\gamma = \frac{b(\cos\alpha - \cos\beta)}{c(\cos\alpha\sin\beta - \cos\beta\sin\alpha)} = \frac{2b\sin\frac{1}{2}(\beta+\alpha)\sin\frac{1}{2}(\beta-\alpha)}{c\sin(\beta-\alpha)}$$
$$= \frac{b\sin\frac{1}{2}(\beta+\alpha)}{c\cos\frac{1}{2}(\beta-\alpha)}$$

Square, and add, thus
$$1 = \frac{a^2\cos^2\frac{1}{2}(\beta+\alpha) + b^2\sin^2\frac{1}{2}(\beta+\alpha)}{c^2\cos^2\frac{1}{2}(\beta-\alpha)},$$

therefore $c^2\{1+\cos(\beta-\alpha)\} = a^2\{1+\cos(\beta+\alpha)\} + b^2\{1-\cos(\beta+\alpha)\}$,

therefore $(b^2+c^2-a^2)\cos\alpha\cos\beta + (a^2+c^2-b^2)\sin\alpha\sin\beta = a^2+b^2-c^2$

18 $$\sin A = \frac{1}{3};\ \text{therefore}\ \cos 2A = 1 - \frac{2}{9} = \frac{7}{9},$$

and $$\sin 2A = \sqrt{\left(1 - \frac{49}{81}\right)} = \frac{\sqrt{32}}{9}$$

$$\sin B = \frac{1}{2},\ \text{therefore}\ \cos 2B = 1 - \frac{1}{2} = \frac{1}{2},$$

and $$\sin 2B = \sqrt{\left(1 - \frac{1}{4}\right)} = \frac{\sqrt{3}}{2}.$$

Hence $$\sin(2A + 2B) = \frac{\sqrt{32} + 7\sqrt{3}}{18} = \frac{4\sqrt{2} + 7\sqrt{3}}{18}$$

19 $$\cos 4x + \cos 2x + \cos x = 0,$$

therefore $$2\cos 3x \cos x + \cos x = 0,$$

therefore either $\cos x = 0$ or $2\cos 3x + 1 = 0$

If $\cos x = 0$, then $x = (2n+1)\dfrac{\pi}{2}$

If $\cos 3x = -\dfrac{1}{2}$, then $3x = 2n\pi \pm \dfrac{2\pi}{3}$.

20 $$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2};$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$= -2\cos\frac{A+B}{2}\cos\frac{C-D}{2},\ \text{by Art 48}$$

Hence, by addition,

$$\cos A + \cos B + \cos C + \cos D = 2\cos\frac{A+B}{2}\left\{\cos\frac{A-B}{2} - \cos\frac{C-D}{2}\right\}$$

$$= 4\cos\frac{A+B}{2}\sin\frac{A+C-B-D}{4}\sin\frac{C+B-A-D}{4}$$

Also $$\sin\frac{A+C-B-D}{4} = \sin\frac{2A+2C-360^\circ}{4} = \sin\left(\frac{A+C}{2} - 90^\circ\right) = -\cos\frac{A+C}{2},$$

and in like manner $$\sin\frac{C+B-A-D}{4} = -\cos\frac{B+C}{2}$$

Thus we obtain finally $4\cos\dfrac{A+B}{2}\cos\dfrac{B+C}{2}\cos\dfrac{C+A}{2}$.

21 Suppose that the smaller unit contains x degrees, and therefore the larger unit $x+10$ degrees Let n denote the number of degrees in the angle measured; then $\frac{n}{x}$ is to $\frac{n}{x+10}$ as 3 is to 2 Therefore $\frac{2}{x}=\frac{3}{x+10}$, whence $x=20$

22 $\sin^2 x+\sin x=1$ Solving this quadratic in the ordinary way we obtain $\sin x=\frac{-1\pm\sqrt{5}}{2}$, the upper sign must be taken, as the lower would make $\sin x$ numerically greater than unity

Thus $$\sin^2 x=\frac{6-2\sqrt{5}}{4},$$

therefore $$\cos^2 x=1-\frac{6-2\sqrt{5}}{4}=\frac{-2+2\sqrt{5}}{4}=\frac{-1+\sqrt{5}}{2},$$

therefore $$\cos^4 x=\frac{6-2\sqrt{5}}{4}=\frac{3-\sqrt{5}}{2},$$

therefore $$\cos^2 x+\cos^4 x=1$$

Or thus $$\sin x=1-\sin^2 x=\cos^2 x, \text{ square},$$

therefore $$\sin^2 x=\cos^4 x, \text{ that is } 1-\cos^2 x=\cos^4 x;$$

therefore $$1=\cos^2 x+\cos^4 x$$

23 Here $$\tan^2 x+\frac{1}{\tan^2 x}=2,$$

therefore $$\tan^4 x-2\tan^2 x+1=0,$$

hence $$\tan^2 x=1, \text{ therefore } \tan x=\pm 1,$$

therefore $$x=n\pi\pm\frac{\pi}{4}$$

24 We have $a\sin\theta+b\cos\theta=c$, $\frac{a\cos\theta+b\sin\theta}{\sin\theta\cos\theta}=c$,

hence $$(a\sin\theta+b\cos\theta)(a\cos\theta+b\sin\theta)=c^2\sin\theta\cos\theta,$$

therefore $$(a^2+b^2)\sin\theta\cos\theta+ab=c^2\sin\theta\cos\theta,$$

therefore $$\sin 2\theta\,(c^2-a^2-b^2)=2ab$$

25 $$\cos^2(A+B)+\cos^2(A-B)=\frac{1+\cos(2A+2B)}{2}+\frac{1+\cos(2A-2B)}{2}$$
$$=1+\cos 2A\cos 2B,$$

therefore $$\cos^2(A+B)+\cos^2(A-B)-\cos 2A\cos 2B=1.$$

26 $$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}=\frac{\frac{3}{2}\tan B-\tan B}{1+\frac{3}{2}\tan^2 B}$$

$$=\frac{\tan B}{2+3\tan^2 B}=\frac{\sin B\cos B}{2\cos^2 B+3\sin^2 B}=\frac{\sin 2B}{2(1+\cos 2B)+3(1-\cos 2B)}$$

$$=\frac{\sin 2B}{5-\cos 2B}$$

27 $$\sin\frac{n+1}{2}\theta+\sin\frac{n-1}{2}\theta=\sin\theta,$$

therefore $$2\sin\frac{n\theta}{2}\cos\frac{\theta}{2}=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

Thus either $\cos\frac{\theta}{2}=0$, or $\sin\frac{n\theta}{2}=\sin\frac{\theta}{2}$

From the former we have $\frac{\theta}{2}=(2m+1)\frac{\pi}{2}$ All the solutions of the latter are comprised in $\frac{n\theta}{2}=m\pi+(-1)^m\frac{\theta}{2}$, where m is zero or an integer

28 Here $$\tan(2\alpha-3\beta)=\tan\left(\frac{\pi}{2}-3\alpha+2\beta\right),$$

and $$\tan(2\alpha+3\beta)=\tan\left(\frac{\pi}{2}-3\alpha-2\beta\right)$$

Hence all possible solutions are comprised in

$$2\alpha-3\beta=m\pi+\frac{\pi}{2}-3\alpha+2\beta,\quad\text{and}\quad 2\alpha+3\beta=n\pi+\frac{\pi}{2}-3\alpha-2\beta,$$

where m and n are zero or integers

From these we obtain $\alpha=(m+n+1)\frac{\pi}{10}$, $\beta=(n-m)\frac{\pi}{10}$,

so that α and β are multiples of $\frac{\pi}{10}$.

29 Here $$\frac{\sin(\alpha+x)\sin(\alpha-x)}{\cos(\alpha+x)\cos(\alpha-x)}=\frac{1-2\cos 2\alpha}{1+2\cos 2\alpha},$$

therefore, by Art 83, $$\frac{\sin^2\alpha-\sin^2 x}{\cos^2\alpha-\sin^2 x}=\frac{1-2\cos 2\alpha}{1+2\cos 2\alpha},$$

therefore $$4\cos 2\alpha\sin^2 x=\sin^2\alpha(1+2\cos 2\alpha)-\cos^2\alpha(1-2\cos 2\alpha)$$
$$=-\cos 2\alpha+2\cos 2\alpha=\cos 2\alpha,$$

therefore $\sin^2 x=\frac{1}{4}$, therefore $\sin x=\pm\frac{1}{2}$, therefore $x=n\pi\pm\frac{\pi}{6}$.

30
$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2},$$

$$\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$=2\sin\frac{A+B}{2}\cos\frac{C-D}{2}, \text{ by Art 48}$$

Hence, by addition,

$$\sin A+\sin B+\sin C+\sin D=2\sin\frac{A+B}{2}\left\{\cos\frac{A-B}{2}+\cos\frac{C-D}{2}\right\}$$

$$=4\sin\frac{A+B}{2}\cos\frac{A+C-B-D}{4}\cos\frac{A+D-B-C}{4}$$

Then, as in Example 20, we can shew that

$$\cos\frac{A+C-B-D}{4}=\sin\frac{A+C}{2}, \text{ and } \cos\frac{A+D-B-C}{4}=\sin\frac{B+C}{2}$$

31 The first angle contains 60 degrees, the second angle contains $\frac{9}{10}\times 50$ degrees, that is 45 degrees, the third angle contains $\frac{3\pi}{4}\times\frac{180}{\pi}$ degrees, that is 135 degrees Therefore the fourth angle must contain $360-60-45-135$ degrees, that is 120 degrees

32
$$\sin A=\sqrt{1-\left(\frac{40}{41}\right)^2}=\frac{\sqrt{(41-40)(41+40)}}{41}=\frac{\sqrt{81}}{41}=\frac{9}{41},$$

$$\sin B=\sqrt{1-\left(\frac{60}{61}\right)^2}=\frac{\sqrt{(61-60)(61+60)}}{61}=\frac{\sqrt{121}}{61}=\frac{11}{61}$$

$$\cos(A-B)=\cos A\cos B+\sin A\sin B=\frac{40\times 60+9\times 11}{41\times 61}=\frac{2499}{2501},$$

thus
$$1-2\sin^2\frac{1}{2}(A-B)=\frac{2499}{2501},$$

therefore
$$2\sin^2\frac{1}{2}(A-B)=\frac{2}{41\times 61},$$

therefore
$$\sin^2\frac{1}{2}(A-B)=\frac{1}{41\times 61}.$$

33 Here
$$3\sin\theta-4\sin^3\theta=8\sin^3\theta,$$

therefore
$$3\sin\theta=12\sin^3\theta,$$

therefore either
$$\sin\theta=0, \text{ or } \sin^2\theta=\frac{1}{4}.$$

the former gives $\theta=n\pi$, the latter gives $\theta=n\pi\pm\frac{\pi}{6}$.

34 Divide the first by the second; thus we get

$$a\cos\theta-b\cos\phi=\frac{c^2}{r};$$

therefore $\qquad \cos\theta=\frac{1}{2a}\left(r+\frac{c^2}{r}\right), \quad \cos\phi=\frac{1}{2b}\left(r-\frac{c^2}{r}\right).$

Now $\dfrac{a^2\sin^2\theta}{\cos^2\theta}=\dfrac{b^2\sin^2\phi}{\cos^2\phi}$; therefore $a^2(\sec^2\theta-1)=b^2(\sec^2\phi-1)$,

therefore $\qquad a^2\left\{\frac{4r^2a^2}{(r^2+c^2)^2}-1\right\}=b^2\left\{\frac{4r^2b^2}{(r^2-c^2)^2}-1\right\}.$

35 Here $\tan A+\tan C=2\tan B$, and $\dfrac{1}{\tan A}+\dfrac{1}{\tan D}=\dfrac{2}{\tan B}$,

therefore $\dfrac{\tan C}{\tan D}=(2\tan B-\tan A)\left(\dfrac{2}{\tan B}-\dfrac{1}{\tan A}\right)=5-2\left(\dfrac{\tan B}{\tan A}+\dfrac{\tan A}{\tan B}\right)$

$$=5-2\left(\frac{\sin B\cos A}{\cos B\sin A}+\frac{\sin A\cos B}{\cos A\sin B}\right)=5-2\,\frac{\sin^2 B\cos^2 A+\sin^2 A\cos^2 B}{\sin A\cos A\sin B\cos B}$$

$$=1-\frac{2(\sin A\cos B-\cos A\sin B)^2}{\sin A\cos A\sin B\cos B}=1-\frac{8\sin^2(A-B)}{\sin 2A\sin 2B}.$$

36 Here $\cos x(1-\cos^2 x)=\sin\alpha(1-\sin^2\alpha)$, therefore

$\cos x-\sin\alpha=\cos^3 x-\sin^3\alpha=(\cos x-\sin\alpha)(\cos^2 x+\cos x\sin\alpha+\sin^2\alpha)$;

therefore either $\cos x-\sin\alpha=0$, or $1=\cos^2 x+\cos x\sin\alpha+\sin^2\alpha$

The latter gives $\qquad \cos^2 x+\cos x\sin\alpha=\cos^2\alpha$,

by solving this quadratic equation we obtain

$$\cos x=\frac{-\sin\alpha\pm\sqrt{(\sin^2\alpha+4\cos^2\alpha)}}{2},$$

it will be found that only one of these values is numerically less than unity, namely, the numerically less of the two

37 We have $\quad 2\cos\theta-1=\dfrac{4\cos^2\theta-1}{2\cos\theta+1}=\dfrac{2\cos 2\theta+1}{2\cos\theta+1}$,

$$2\cos 2\theta-1=\frac{4\cos^2 2\theta-1}{2\cos 2\theta+1}=\frac{2\cos 4\theta+1}{2\cos 2\theta+1},$$

and so on, which we use down to

$$2\cos 2^{n-1}\theta-1=\frac{4\cos^2 2^{n-1}\theta-1}{2\cos 2^{n-1}\theta+1}=\frac{2\cos 2^n\theta+1}{2\cos 2^{n-1}\theta+1}$$

Multiply these expressions together, then by cancelling we obtain the required result

38 Here $$\tan(\pi\cot\theta)=\tan\left(\frac{\pi}{2}-\pi\tan\theta\right),$$

hence all possible solutions are comprised in the formula

$$\pi\cot\theta=n\pi+\frac{\pi}{2}-\pi\tan\theta,$$

thus $$\tan^2\theta-\left(n+\frac{1}{2}\right)\tan\theta+1=0,$$

by solving this quadratic equation we obtain the value of $\tan\theta$

39 $\cos^2 A+\cos^2 B+\cos^2 C-2\cos A\cos B\cos C-1$

$$=(\cos A-\cos B\cos C)^2+\cos^2 B+\cos^2 C-1-\cos^2 B\cos^2 C$$
$$=(\cos A-\cos B\cos C)^2-(1-\cos^2 B)(1-\cos^2 C)$$
$$=(\cos A-\cos B\cos C)^2-\sin^2 B\sin^2 C$$
$$=(\cos A-\cos B\cos C+\sin B\sin C)(\cos A-\cos B\cos C-\sin B\sin C)$$
$$=\{\cos A-\cos(B+C)\}\{\cos A-\cos(B-C)\}$$
$$=4\sin\frac{A+B+C}{2}\sin\frac{B+C-A}{2}\sin\frac{A+B-C}{2}\sin\frac{B-C-A}{2}$$
$$=-4\sin\frac{A+B+C}{2}\sin\frac{B+C-A}{2}\sin\frac{A+C-B}{2}\sin\frac{A+B-C}{2}$$

40 $$\sin A-\sin B=2\sin\frac{A-B}{2}\cos\frac{A+B}{2},$$
$$\sin C-\sin D=2\sin\frac{C-D}{2}\cos\frac{C+D}{2}$$
$$=-2\sin\frac{C-D}{2}\cos\frac{A+B}{2},\text{ by Art 48}$$

Hence, by addition,

$$\sin A-\sin B+\sin C-\sin D=2\cos\frac{A+B}{2}\left\{\sin\frac{A-B}{2}-\sin\frac{C-D}{2}\right\}$$
$$=4\cos\frac{A+B}{2}\sin\frac{A+D-B-C}{4}\cos\frac{A+C-B-D}{4}$$

Then, as in Example 20, we can shew that

$$\sin\frac{A+D-B-C}{4}=\cos\frac{B+C}{2},\text{ and }\cos\frac{A+C-B-D}{4}=\sin\frac{A+C}{2}$$

41. The angle described is $\frac{25}{60}$ of four right angles, the number of degrees $=\frac{5}{12}\times 360=150$, the number of grades $=\frac{5}{12}\times 400=166\frac{2}{3}$, the circular measure $=\frac{5}{12}\times 2\pi=\frac{5\pi}{6}$

42 $$\cos\theta+\sin\theta=\sqrt{2}\left(\frac{\cos\theta}{\sqrt{2}}+\frac{\sin\theta}{\sqrt{2}}\right)=\sqrt{2}.\cos\left(\theta-\frac{\pi}{4}\right);$$

similarly $$\cos 2\theta+\sin 2\theta=\sqrt{2}\cos\left(2\theta-\frac{\pi}{4}\right).$$

$$\text{the product}=2\cos\left(\theta-\frac{\pi}{4}\right)\cos\left(2\theta-\frac{\pi}{4}\right)=\cos\theta+\cos\left(3\theta-\frac{\pi}{2}\right)$$

43 $$\operatorname{cosec}2A(\operatorname{cosec}A+\operatorname{cosec}3A)=\frac{1}{\sin 2A}\cdot\frac{\sin A+\sin 3A}{\sin A\sin 3A}$$

$$=\frac{1}{\sin 2A}\,\frac{2\sin 2A\cos A}{\sin A\sin 3A}=\frac{2\cos A}{\sin A\sin 3A},$$

and $$\operatorname{cosec}A(\cot A-\cot 3A)=\frac{1}{\sin A}\left(\frac{\cos A}{\sin A}-\frac{\cos 3A}{\sin 3A}\right)$$

$$=\frac{1}{\sin A}\cdot\frac{\sin 3A\cos A-\cos 3A\sin A}{\sin A\sin 3A}=\frac{1}{\sin A}\,\frac{\sin(3A-A)}{\sin A\sin 3A}$$

$$=\frac{\sin 2A}{\sin^2 A\sin 3A}=\frac{2\sin A\cos A}{\sin^2 A\sin 3A}=\frac{2\cos A}{\sin A\sin 3A};$$

thus the proposed expressions are equal.

44. $$\sec^2\frac{1}{2}A\sec A\,\frac{\cot^2\frac{1}{2}A-\cot^2\frac{3}{2}A}{1+\cot^2\frac{3}{2}A}$$

$$=\frac{1}{\cos^2\frac{1}{2}A}\,\frac{1}{\cos A}\,\frac{\cos^2\frac{1}{2}A\sin^2\frac{3}{2}A-\cos^2\frac{3}{2}A\sin^2\frac{1}{2}A}{\sin^2\frac{1}{2}A\left(\cos^2\frac{3A}{2}+\sin^2\frac{3A}{2}\right)}.$$

$$=\frac{\left(\cos\frac{1}{2}A\sin\frac{3}{2}A-\cos\frac{3}{2}A\sin\frac{1}{2}A\right)\left(\cos\frac{1}{2}A\sin\frac{3}{2}A+\cos\frac{3}{2}A\sin\frac{1}{2}A\right)}{\sin^2\frac{1}{2}A\cos^2\frac{1}{2}A\cos A}$$

$$=\frac{\sin\left(\frac{3A}{2}-\frac{A}{2}\right)\sin\left(\frac{3A}{2}+\frac{A}{2}\right)}{\sin^2\frac{1}{2}A\cos^2\frac{1}{2}A\cos A}=\frac{\sin A\sin 2A}{\sin^2\frac{1}{2}A\cos^2\frac{1}{2}A\cos A}$$

$$=\frac{2\sin^2 A}{\sin^2\frac{1}{2}A\cos^2\frac{1}{2}A}=\frac{2\left(2\sin\frac{1}{2}A\cos\frac{1}{2}A\right)^2}{\sin^2\frac{1}{2}A\cos^2\frac{1}{2}A}=8.$$

45 $\sec A + \operatorname{cosec} A\,(1+\sec A) = \dfrac{1}{\cos A} + \dfrac{1}{\sin A}\left(1+\dfrac{1}{\cos A}\right)$

$$= \frac{1+\cos A+\sin A}{\cos A \sin A} = \frac{2\cos^2\frac{1}{2}A + 2\sin\frac{1}{2}A\cos\frac{1}{2}A}{2\sin\frac{1}{2}A\cos\frac{1}{2}A\cos A}$$

$$= \frac{\cos\frac{1}{2}A+\sin\frac{1}{2}A}{\sin\frac{1}{2}A\cos A} = \frac{\cos\frac{1}{2}A}{\cos A}\left(\sec\frac{1}{2}A + \operatorname{cosec}\frac{1}{2}A\right),$$

$$1-\tan^2\frac{1}{2}A = \frac{\cos^2\frac{1}{2}A - \sin^2\frac{1}{2}A}{\cos^2\frac{1}{2}A} = \frac{\cos A}{\cos^2\frac{1}{2}A};$$

$$1-\tan^2\frac{1}{4}A = \frac{\cos^2\frac{1}{4}A - \sin^2\frac{1}{4}A}{\cos^2\frac{1}{4}A} = \frac{\cos\frac{1}{2}A}{\cos^2\frac{1}{4}A}$$

Hence by multiplication we obtain the required result

46 Put c^2 for $a^4+\dfrac{a^2b^2}{a^2-1}$ thus

$$\cos\theta = \frac{a-b}{c}, \quad \text{and} \quad \cos\phi = \frac{a+b}{c}$$

from these we obtain

$$\sin\theta = \frac{a(a^2-1)+b}{c\sqrt{(a^2-1)}}, \quad \text{and} \quad \sin\phi = \frac{a(a^2-1)-b}{c\sqrt{(a^2-1)}}.$$

Next we obtain $\qquad \cos(\theta-\phi) = \dfrac{a^4-a^2-b^2}{a^4-a^2+b^2},$

and thus $\qquad \tan^2\dfrac{1}{2}(\theta-\phi) = \dfrac{1-\cos(\theta-\phi)}{1+\cos(\theta-\phi)} = \dfrac{b^2}{a^4-a^2}.$

47 $\dfrac{a}{b} = \dfrac{\sin(\phi-\theta)}{\sin(\phi+\theta)}$ (1), $\quad \dfrac{c}{x} = \cos(\phi-\theta)$ (2), $\quad \dfrac{b}{x} = \dfrac{\sin\theta}{\sin\phi}$ (3)

From (1) we get $\qquad \dfrac{b+a}{b-a} = \dfrac{\sin\phi\cos\theta}{\sin\theta\cos\phi} = \dfrac{x\cos\theta}{b\cos\phi}$, by (3)

By (2) we have $\qquad \dfrac{c}{x} = \cos\phi\cos\theta + \sin\phi\sin\theta$

$$= \frac{b}{x}\cdot\frac{b+a}{b-a}\cos^2\phi + \frac{b}{x}\sin^2\phi,$$

hence we get $$\cos^2\phi=\frac{(c-b)(b-a)}{2ab},$$

therefore $$\cos^2\theta=\frac{(c-b)(a+b)^2}{2ab(b-a)}\,\frac{b^2}{x^2}.$$

But from (3) we have $\dfrac{b^2}{x^2}(1-\cos^2\phi)=1-\cos^2\theta$,

so that $$\frac{b^2}{x^2}-1=\frac{b^2}{x^2}\left\{\frac{(c-b)(b-a)}{2ab}-\frac{(c-b)(a+b)^2}{2ab(b-a)}\right\},$$

or $$x^2-b^2=-\frac{b^2(c-b)\,4ab}{2ab(b-a)}=\frac{2b^2(c-b)}{b-a},$$

therefore $$x^2(b-a)=b^2(2c-a-b).$$

48. Put $2\cos^2\dfrac{x}{2}-1$ for $\cos x$, thus we obtain the quadratic equation

$$2\cos^2\frac{x}{2}\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right)-\sin\beta\cos\frac{x}{2}=\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right).$$

By solving this we have

$$\cos\frac{x}{2}=\frac{\sin\beta\pm\sqrt{\left\{\sin^2\beta+8\cos^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right\}}}{4\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right)},$$

and $$\sin^2\beta+8\cos^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)=\sin^2\beta+4\left\{1+\cos\left(\frac{\pi}{2}-\beta\right)\right\}$$
$$=\sin^2\beta+4+4\sin\beta.$$

Hence $$\cos\frac{x}{2}=\frac{\sin\beta\pm(2+\sin\beta)}{4\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}.$$

Take the upper sign; then $\cos\dfrac{x}{2}=\dfrac{1+\sin\beta}{2\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}$

$$=\frac{1+\cos\left(\frac{\pi}{2}-\beta\right)}{2\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}=\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}{2\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}=\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right).$$

Take the lower sign, then $\cos\dfrac{x}{2}=-\dfrac{1}{2\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}$.

49. Write the first equation in the form

$$\cos\left(\frac{\pi}{2}-2\theta-2\phi\right)=\cos(\theta+3\phi),$$

hence all possible solutions are comprised in

$$\frac{\pi}{2}-2\theta-2\phi=2m\pi\pm(\theta+3\phi)$$

If we take the upper sign we have

$$3\theta+5\phi=\frac{\pi}{2}-2m\pi \quad . \quad (1)$$

If we take the lower sign we have

$$\phi-\theta=2m\pi-\frac{\pi}{2} \quad . \quad . \quad (2)$$

Again, write the second equation in the form

$$\cos\left(\frac{\pi}{2}-\phi-3\theta\right)=\cos(2\theta+2\phi),$$

hence all possible solutions are comprised in

$$\frac{\pi}{2}-\phi-3\theta=2n\pi\pm(2\theta+2\phi)$$

If we take the upper sign we have

$$5\theta+3\phi=\frac{\pi}{2}-2n\pi \quad . \quad . \quad (3)$$

If we take the lower sign we have

$$\phi-\theta=2n\pi-\frac{\pi}{2},$$

this agrees with (2)

Thus either (2) holds, or both (1) and (3) hold. From (1) and (3) we obtain

$$16\theta=(3m-5n)\,2\pi+\pi, \quad 16\phi=(3n-5m)\,2\pi+\pi$$

50. We have $\quad 1+\sec 2\theta=\dfrac{1+\cos 2\theta}{\cos 2\theta}=\dfrac{\sin 2\theta}{\cos 2\theta}\,\dfrac{1+\cos 2\theta}{\sin 2\theta}$

$$=\tan 2\theta\,\cot\theta=\frac{\tan 2\theta}{\tan\theta}.$$

Similarly $\quad 1+\sec 4\theta=\dfrac{\tan 4\theta}{\tan 2\theta},$

and so on, which we use down to

$$1+\sec 2^n\theta=\frac{\tan 2^n\theta}{\tan 2^{n-1}\theta}$$

Multiply these expressions together, then by cancelling we obtain the required result.

51 Here the circular measure of an angle is given equal to $\frac{9}{10}$, hence the number of degrees in it is $\frac{9}{10} \cdot \frac{180}{\pi}$, that is $\frac{162}{\pi}$.

52 $$\sin(A-B)+\sin(A+3B)=2\sin(A+B)\cos 2B,$$
therefore $$\{\sin(A-B)+\sin(A+3B)\}\sec 2B=2\sin(A+B)$$
$$\cos 2B-\cos 2A=2\sin(A-B)\sin(A+B),$$
therefore $$(\cos 2B-\cos 2A)\operatorname{cosec}(A-B)=2\sin(A+B).$$
thus the proposed expressions are equal

53 Here $$\frac{\sin\theta}{\cos\theta}(1+\sin^2\theta)=\frac{\sin\alpha}{\cos\alpha}(1+\cos^2\theta),$$
therefore $$\cos\alpha(\sin\theta+\sin^3\theta)=\sin\alpha(\cos\theta+\cos^3\theta),$$
therefore $$\cos\alpha\sin\theta+\cos\alpha\frac{3\sin\theta-\sin 3\theta}{4}=\sin\alpha\cos\theta+\sin\alpha\frac{3\cos\theta+\cos 3\theta}{4},$$
therefore $$7(\sin\theta\cos\alpha-\cos\theta\sin\alpha)=\sin 3\theta\cos\alpha+\cos 3\theta\sin\alpha,$$
that is $$7\sin(\theta-\alpha)=\sin(3\theta+\alpha)$$

54 Here $$2\cos 4\theta\cos\theta+\sqrt{2}(\cos\theta+\sin\theta)\cos\theta=0,$$
therefore either $$\cos\theta=0 \text{ or } \cos 4\theta=-\frac{1}{\sqrt{2}}(\cos\theta+\sin\theta)$$

Take the former, then $\theta=(2n+1)\frac{\pi}{2}$.

Take the latter, thus $\cos 4\theta=\cos\left(\frac{3\pi}{4}+\theta\right)$,
therefore $$4\theta=2n\pi\pm\left(\frac{3\pi}{4}+\theta\right)$$

55 We have $$\sin\phi=\frac{n\sin\theta-m\cos\theta}{2m},$$
and $$n\sin 2\theta-m(1-2\sin^2\phi)=n,$$
therefore $$n\sin 2\theta+2m\left(\frac{n\sin\theta-m\cos\theta}{2m}\right)^2=m+n,$$
therefore $$2mn\sin 2\theta+(n\sin\theta-m\cos\theta)^2=2m(m+n),$$
therefore $$(n\sin\theta+m\cos\theta)^2=2m(m+n).$$

56 Substitute $\frac{3\cos\theta+\cos 3\theta}{4}$ for $\cos^3\theta$ and $\frac{3\sin\theta-\sin 3\theta}{4}$ for $\sin^3\theta$, thus the equation becomes

$$2\sin\left(\theta-\frac{\pi}{3}\right)\{3\cos\theta+\cos 3\theta\}+2\cos\left(\theta-\frac{\pi}{3}\right)\{3\sin\theta-\sin 3\theta\}$$

$$-6\sin\left(2\theta-\frac{\pi}{3}\right)=\sqrt{3},$$

that is $$2\sin\left(\theta-\frac{\pi}{3}\right)\cos 3\theta-2\cos\left(\theta-\frac{\pi}{3}\right)\sin 3\theta=\sqrt{3},$$

that is $$-\sin\left(3\theta-\theta+\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2},$$

thus $$\sin\left(2\theta+\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2},$$

therefore $$2\theta+\frac{\pi}{3}=n\pi+(-1)^n\frac{4\pi}{3}$$

57. $$\sin\theta\cos(\beta-\theta)=\frac{1}{2}\{\sin\beta+\sin(2\theta-\beta)\}.$$

The greatest value of $\sin(2\theta-\beta)$ is obviously when $2\theta-\beta=\frac{\pi}{2}$, so that $\theta=\frac{\beta}{2}+\frac{\pi}{4}$

58 $$\cos 55^0+\cos 65^0=2\cos 60^0\cos 5^0=\cos 5^0=-\cos 175^0,$$

therefore $$\cos 55^0+\cos 65^0+\cos 175^0=0$$

$$\cos 55^0\cos 65^0=\frac{1}{2}(\cos 10^0+\cos 120^0),$$

$$\cos 65^0\cos 175^0=\frac{1}{2}(\cos 110^0+\cos 240^0),$$

$$\cos 55^0\cos 175^0=\frac{1}{2}(\cos 120^0+\cos 230^0),$$

hence by addition we obtain $-\frac{3}{4}+\frac{1}{2}(\cos 10^0+\cos 110^0+\cos 230^0)$,

that is $$-\frac{3}{4}+\frac{1}{2}(\cos 10^0+\cos 110^0-\cos 50^0),$$

that is $$-\frac{3}{4}+\frac{1}{2}(2\cos 60^0\cos 50^0-\cos 50^0),\quad \text{that is } -\frac{3}{4}.$$

$$\cos 55^\circ \cos 65^\circ \cos 175^\circ = \frac{1}{2}(\cos 10^\circ + \cos 120^\circ)\cos 175^\circ$$

$$= \frac{1}{4}(\cos 165^\circ + \cos 185^\circ) + \frac{1}{4}(\cos 55^\circ + \cos 295^\circ)$$

$$= \frac{1}{4}(-\cos 15^\circ + \cos 175^\circ + \cos 55^\circ + \cos 65^\circ)$$

$$= -\frac{1}{4}\cos 15^\circ, \text{ by what has been already shown,}$$

$$= -\frac{1}{4}\cdot\frac{\sqrt{3}+1}{2\sqrt{2}} = -\frac{\sqrt{3}+1}{8\sqrt{2}}.$$

59 From $x\cos(\alpha+\beta) + \cos(\alpha-\beta) = x\cos(\beta+\gamma) + \cos(\beta-\gamma)$

we obtain
$$x = \frac{\cos(\beta-\gamma) - \cos(\alpha-\beta)}{\cos(\alpha+\beta) - \cos(\beta+\gamma)} = -\frac{\sin\left(\frac{\alpha+\gamma}{2} - \beta\right)}{\sin\left(\frac{\alpha+\gamma}{2} + \beta\right)}$$

Two other expressions for the value of x may be obtained, and thus we have

$$\frac{\sin\left(\frac{\alpha+\gamma}{2} - \beta\right)}{\sin\left(\frac{\alpha+\gamma}{2} + \beta\right)} = \frac{\sin\left(\frac{\beta+\alpha}{2} - \gamma\right)}{\sin\left(\frac{\beta+\alpha}{2} + \gamma\right)} = \frac{\sin\left(\frac{\gamma+\beta}{2} - \alpha\right)}{\sin\left(\frac{\gamma+\beta}{2} + \alpha\right)} = -x,$$

hence
$$\frac{\sin\frac{\alpha+\gamma}{2}\cos\beta}{\cos\frac{\alpha+\gamma}{2}\sin\beta} = \frac{1-x}{1+x}, \text{ that is } \frac{\tan\frac{\alpha+\gamma}{2}}{\tan\beta} = \frac{1-x}{1+x}$$

Similarly $\dfrac{\tan\frac{\beta+\alpha}{2}}{\tan\gamma}$ and $\dfrac{\tan\frac{\gamma+\beta}{2}}{\tan\alpha}$ are also equal to $\dfrac{1-x}{1-x}$.

60.
$$\cos A - \cos B = 2\sin\frac{A+B}{2}\sin\frac{B\ A}{2},$$

$$\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$$

$$= 2\sin\frac{A+B}{2}\sin\frac{D-C}{2}, \text{ by Art 48.}$$

Hence by addition,

$$\cos A - \cos B + \cos C - \cos D = 2 \sin \frac{A+B}{2} \left\{ \sin \frac{B-A}{2} + \sin \frac{D-C}{2} \right\}$$

$$= 4 \sin \frac{A+B}{2} \sin \frac{B+D-A-C}{4} \cos \frac{B+C-A-D}{4}.$$

Then, as in Example 20, we can shew that

$$\sin \frac{B+D-A-C}{4} = \cos \frac{A+C}{2}, \text{ and } \cos \frac{B+C-A-D}{4} = \sin \frac{B+C}{2}.$$

61 Let x denote the number of sides in the first regular polygon, and y the number of sides in the second All the angles of the first polygon are equal to $2x-4$ right angles, therefore each angle is equal to $\frac{2x-4}{x}$ right angles, and therefore contains $\frac{2x-4}{x}$ 90 degrees In the same way each angle of the second polygon contains $\frac{2y-4}{y}$ 100 grades Then, by supposition, we have $\frac{2x-4}{x}\,90 \quad \frac{2y-4}{y}\,100 \quad 3 \quad 5$,

therefore $$5\,\frac{2x-4}{x}\,90 = 3\,\frac{2y-4}{y}\,100,$$

therefore $$\frac{3(x-2)}{x} = \frac{2(y-2)}{y},$$

therefore $3y(x-2) = 2x(y-2)$, therefore $y(6-x) = 4x$ This formula shews that x cannot be greater than 5, for if $x=6$ we should have $y \times 0 = 24$, which is absurd, and if x is greater than 6 we should have a negative value for y, which is also absurd And x cannot be less than 3 Thus the only possible solutions are $x=3$, $x=4$, and $x=5$, which give respectively $y=4$, $y=8$, and $y=20$

62 Here $$\frac{1}{\cos^2 \frac{x}{2}} + \frac{1}{\sin^2 \frac{x}{2}} = \frac{16 \cos x}{\sin x},$$

therefore $$\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} = \frac{16 \cos x \cos^2 \frac{x}{2} \sin^2 \frac{x}{2}}{\sin x} = 8 \cos x \cos \frac{x}{2} \sin \frac{x}{2};$$

therefore $$1 = 4 \cos x \sin x = 2 \sin 2x,$$

therefore $$\sin 2x = \frac{1}{2}, \quad \text{therefore } 2x = n\pi + (-1)^n \frac{\pi}{6}.$$

63. Here $$m \sin 2\theta = n \sin \theta, \quad p \cos 2\theta = q \cos \theta;$$

from the first equation $2m \sin \theta \cos \theta = n \sin \theta$, therefore $\cos \theta = \frac{n}{2m}$

Substitute in the second equation, that is in $p(2\cos^2\theta - 1) = q \cos \theta$,

thus $$p\left\{2\left(\frac{n}{2m}\right)^2 - 1\right\} = \frac{qn}{2m}; \quad \text{therefore } p(n^2 - 2m^2) = qmn$$

64 We may write the equation thus

$$\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta = \frac{1}{\sqrt{2}}\cos\alpha - \frac{1}{\sqrt{2}}\sin\alpha;$$

therefore $$\cos\left(\theta+\frac{\pi}{4}\right) = \cos\left(\alpha+\frac{\pi}{4}\right);$$

therefore $$\theta+\frac{\pi}{4} = 2n\pi \pm \left(\alpha+\frac{\pi}{4}\right).$$

65. $\sin(A+C)\sin(A+D) - \sin(B+C)\sin(B+D)$

$$=\frac{1}{2}\{\cos(C-D)-\cos(2A+C+D)\} - \frac{1}{2}\{\cos(C-D)-\cos(2B+C+D)\}$$

$$=\frac{1}{2}\{\cos(2B+C+D)-\cos(2A+C+D)\}$$

$$=\sin(A-B)\sin(A+B+C+D)$$

Thus, if $\sin(A+B+C+D)$ vanishes, the difference between the two proposed expressions vanishes, and therefore the two expressions are equal

66 We have $\sin\phi = p - \sin\theta$, $\cos\phi = q - \cos\theta$, square and add,

thus $$1 = p^2+q^2-2p\sin\theta-2q\cos\theta+1;$$

therefore $$2p\sin\theta+2q\cos\theta = p^2+q^2.$$

Now assume that $\tan\alpha = \frac{q}{p}$, so that

$$\sin\alpha = \frac{q}{\sqrt{(p^2+q^2)}} \text{ and } \cos\alpha = \frac{p}{\sqrt{(p^2+q^2)}};$$

thus $$2\sqrt{(p^2+q^2)}\{\sin\theta\cos\alpha+\cos\theta\sin\alpha\} = p^2+q^2,$$

therefore $$\sin(\theta+\alpha) = \frac{\sqrt{(p^2+q^2)}}{2} = \sin\beta \text{ say},$$

therefore $$\theta+\alpha = n\pi+(-1)^n\beta$$

67. $$\cos\frac{\pi}{15}\cos\frac{4\pi}{15} = \frac{1}{2}\left(\cos\frac{\pi}{3}+\cos\frac{\pi}{5}\right) = \frac{1}{2}\left(\frac{1}{2}+\frac{\sqrt{5}+1}{4}\right) = \frac{3+\sqrt{5}}{8};$$

$$\cos\frac{2\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{2}\left(\cos\frac{\pi}{3}+\cos\frac{3\pi}{5}\right) = \frac{1}{2}\left(\frac{1}{2}-\frac{\sqrt{5}-1}{4}\right) = \frac{3-\sqrt{5}}{8};$$

$$\cos\frac{3\pi}{15}\cos\frac{6\pi}{15} = \frac{1}{2}\left(\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}\right) = \frac{1}{2}\left(\frac{\sqrt{5}+1}{4}-\frac{\sqrt{5}-1}{4}\right) = \frac{1}{4},$$

$$\cos\frac{5\pi}{15} = \frac{1}{2};$$

therefore $$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}$$

$$=\frac{3+\sqrt{5}}{8}\cdot\frac{3-\sqrt{5}}{8}\cdot\frac{1}{8}=\frac{4}{8^3}=\frac{1}{2^7}.$$

68. $a\sin^2\theta+b\sin\theta\cos\theta+c\cos^2\theta$

$$=\frac{1}{2}\{a(1-\cos 2\theta)+b\sin 2\theta+c(1+\cos 2\theta)\}$$

$$=\frac{1}{2}\{a+c+b\sin 2\theta-(a-c)\cos 2\theta\}.$$

Now let α be an angle such that $\tan\alpha=\frac{a-c}{b}$, so that $\cos\alpha=\frac{b}{\sqrt{b^2+(a-c)^2}}$, and $\sin\alpha=\frac{a-c}{\sqrt{b^2+(a-c)^2}}$. Then the above expression

$$=\frac{1}{2}(a+c)+\frac{1}{2}\sqrt{b^2+(a-c)^2}\{\sin 2\theta\cos\alpha-\cos 2\theta\sin\alpha\}$$

$$=\frac{1}{2}(a+c)+\frac{1}{2}\sqrt{b^2+(a-c)^2}\sin(2\theta-\alpha).$$

Then as $\sin(2\theta-\alpha)$ must lie between -1 and $+1$, we obtain the required result

69. $$\cos\left(\frac{2\pi}{3}+\alpha\right)+\cos\left(\frac{2\pi}{3}-\alpha\right)=2\cos\frac{2\pi}{3}\cos\alpha=-\cos\alpha,$$

therefore $$\cos\alpha+\cos\left(\frac{2\pi}{3}+\alpha\right)+\cos\left(\frac{2\pi}{3}-\alpha\right)=0$$

$$\cos\alpha\cos\left(\frac{2\pi}{3}+\alpha\right)=\frac{1}{2}\left\{\cos\frac{2\pi}{3}+\cos\left(\frac{2\pi}{3}+2\alpha\right)\right\}$$

$$\cos\alpha\cos\left(\frac{2\pi}{3}-\alpha\right)=\frac{1}{2}\left\{\cos\frac{2\pi}{3}+\cos\left(\frac{2\pi}{3}-2\alpha\right)\right\}$$

$$\cos\left(\frac{2\pi}{3}+\alpha\right)\cos\left(\frac{2\pi}{3}-\alpha\right)=\frac{1}{2}\left\{\cos\frac{4\pi}{3}+\cos 2\alpha\right\}.$$

Now $\cos 2\alpha+\cos\left(\frac{2\pi}{3}+2\alpha\right)+\cos\left(\frac{2\pi}{3}-2\alpha\right)$ is zero, in the manner already shewn, and $\cos\frac{2\pi}{3}$ and $\cos\frac{4\pi}{3}$ are each $-\frac{1}{2}$. thus the sum is $-\frac{3}{4}$.

$$\cos\alpha\cos\left(\frac{2\pi}{3}+\alpha\right)\cos\left(\frac{2\pi}{3}-\alpha\right)=\cos\alpha\left(\cos^2\alpha-\sin^2\frac{2\pi}{3}\right), \text{ by Art 83,}$$

$$=\cos\alpha\left(\cos^2\alpha-\frac{3}{4}\right)=\frac{1}{4}\cos\alpha\,(4\cos^2\alpha-3)=\frac{\cos 3\alpha}{4}.$$

70. $$\cos\frac{\alpha}{2^n}+\cos\frac{\beta}{2^n}=\frac{\cos^2\frac{\alpha}{2^n}-\cos^2\frac{\beta}{2^n}}{\cos\frac{\alpha}{2^n}-\cos\frac{\beta}{2^n}}=\frac{1}{2}\,\frac{\cos\frac{\alpha}{2^{n-1}}-\cos\frac{\beta}{2^{n-1}}}{\cos\frac{\alpha}{2^n}-\cos\frac{\beta}{2^n}};$$

similarly $$\cos\frac{\alpha}{2^{n-1}}+\cos\frac{\beta}{2^{n-1}}=\frac{1}{2}\,\frac{\cos\frac{\alpha}{2^{n-2}}-\cos\frac{\beta}{2^{n-2}}}{\cos\frac{\alpha}{2^{n-1}}-\cos\frac{\beta}{2^{n-1}}},$$

and we use a series of these transformations down to

$$\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}=\frac{1}{2}\,\frac{\cos\alpha-\cos\beta}{\cos\frac{1}{2}\alpha-\cos\frac{1}{2}\beta}.$$

Then by multiplication we obtain for the product

$$\frac{1}{2^n}\,\frac{\cos\alpha-\cos\beta}{\cos\frac{\alpha}{2^n}-\cos\frac{\beta}{2^n}}.$$

71. The angle $a^\circ b'$ is $\frac{60a+b}{60\times 90}$ of a right angle, the angle $p^\prime\, q^\backprime$ is $\frac{100p+q}{100\times 100}$ of a right angle Hence the excess of the former above the latter is $\left\{\frac{60a+b}{60\times 90}-\frac{100p+q}{100\times 100}\right\}$ of a right angle

72 Here $$1-\cos 2x+\sin^2 2x=2,$$

therefore $$1-\cos 2x+1-\cos^2 2x=2,$$

therefore $$\cos 2x\,(1+\cos 2x)=0.$$

Therefore either $\cos 2x=0$, or $1+\cos 2x=0$

If $\cos 2x=0$, we have $2x=2n\pi\pm\frac{\pi}{2}$, which may be written more simply as $2x=(2m+1)\frac{\pi}{2}$.

If $1+\cos 2x=0$ we have $\cos 2x=-1$, and therefore $2x=2n\pi\pm\pi$ which may be written more simply as $2x=(2m+1)\pi$.

73 $\tan A - \cot A = \frac{\sin A}{\cos A} - \frac{\cos A}{\sin A} = \frac{\sin^2 A - \cos^2 A}{\sin A \cos A} = \frac{2(\sin^2 A - \cos^2 A)}{2 \sin A \cos A}$

$$= -\frac{2 \cos 2A}{\sin 2A} = -2 \cot 2A.$$

Similarly $\qquad 2 \tan 2A - 2 \cot 2A = -4 \cot 4A,$

and $\qquad 4 \tan 4A - 4 \cot 4A = -8 \cot 8A$

Therefore by addition and cancelling

$$\tan A - \cot A + 2 \tan 2A + 4 \tan 4A = -8 \cot 8A;$$

therefore $\qquad \tan A + 2 \tan 2A + 4 \tan 4A + 8 \cot 8A = \cot A$

74 Here $2 \sin 3x \sin x = \sin x$, therefore either $\sin x = 0$ or $2 \sin 3x = 1$.

If $\sin x = 0$, then $x = n\pi$ If $\sin 3x = \frac{1}{2}$, then $3x = n\pi + (-1)^n \frac{\pi}{6}$

75 Let r denote the common ratio of the Geometrical Progression, so that $\tan B = r \tan A$, $\tan C = r \tan B$, $\tan D = r \tan C$,

therefore $\qquad \tan A \tan D = \tan B \tan C$

Now since $A + D = 360^\circ - B - C$, we have $\tan(A+D) = -\tan(B+C)$;

therefore $\qquad \frac{\tan A + \tan D}{1 - \tan A \tan D} = -\frac{\tan B + \tan C}{1 - \tan B \tan C}.$

Thus we must have either $\tan A \tan D = \tan B \tan C = 1$,

or else $\qquad \tan A + \tan D = -(\tan B + \tan C)$

The latter gives $\qquad (1+r^3) \tan A = -(r+r^2) \tan A,$

so that $\qquad 1 + r^3 + r + r^2 = 0$, that is $(1+r)(1+r^2) = 0$:

the only possible solution is $1 + r = 0$, so that $r = -1$

76 Let A, B, C denote the angles, then $A + B + C = 180^\circ$; and since the angles are in Arithmetical Progression $A + C = 2B$, thus $3B = 180^\circ$; therefore $B = 60^\circ$

Again we have $\frac{1}{\sin 2A} + \frac{1}{\sin 2C} = \frac{2}{\sin 2B} = \frac{4}{\sqrt{3}}$. Let x denote the common difference of the angles, so that $A = 60^\circ - x$, and $C = 60^\circ + x$ Then

$$\frac{\sin 2A + \sin 2C}{\sin 2A \sin 2C} = \frac{4}{\sqrt{3}}, \text{ therefore } \frac{2 \sin(A+C) \cos(A-C)}{\sin(120^\circ - 2x) \sin(120^\circ + 2x)} = \frac{4}{\sqrt{3}};$$

therefore $\frac{\sqrt{3} \cos 2x}{\sin^2 120^\circ - \sin^2 2x} = \frac{4}{\sqrt{3}}$, therefore

$$\cos 2x = \frac{4}{3}(\sin^2 120^\circ - \sin^2 2x) = \frac{4}{3}\left(\frac{3}{4} - 1 + \cos^2 2x\right) = -\frac{1}{3} + \frac{4}{3} \cos^2 2x$$

By solving this quadratic we obtain $\cos 2x = 1$, or $-\frac{1}{4}$. The latter must be taken: then $\cos^2 x = \frac{1}{2}\left(1 - \frac{1}{4}\right) = \frac{3}{8}$

77 $$\cos A + \cos 2A + \cos 3A = 2\cos 2A \cos A + \cos 2A$$

$$= \cos 2A\,(2\cos A + 1) = \cos 2A\left(2 - 4\sin^2\frac{1}{2}A + 1\right)$$

$$= \cos 2A\left(3 - 4\sin^2\frac{1}{2}A\right) = \frac{\cos 2A}{\sin\frac{1}{2}A}\left(3\sin\frac{1}{2}A - 4\sin^3\frac{1}{2}A\right)$$

$$= \frac{\cos 2A}{\sin\frac{1}{2}A}\sin\frac{3A}{2}.$$

78 Multiply the given expression out. The coefficient of x^2 is

$$-2\left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right),$$

by Example 77 this $= -\dfrac{2\cos\frac{4\pi}{7}\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}} = -\dfrac{1}{\sin\frac{\pi}{7}}\left(\sin\frac{7\pi}{7} - \sin\frac{\pi}{7}\right) = 1.$

The coefficient of x is

$$4\left(\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} + \cos\frac{2\pi}{7}\cos\frac{6\pi}{7} + \cos\frac{4\pi}{7}\cos\frac{6\pi}{7}\right),$$

this $= 2\left(\cos\frac{2\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{10\pi}{7}\right)$

$$= 2\left(\cos\frac{2\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{4\pi}{7}\right) + 2\left(\cos\frac{8\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{10\pi}{7}\right)$$

$$= 2\left(\cos\frac{2\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{4\pi}{7}\right) + 2\left(\cos\frac{8\pi}{7} + \cos\frac{16\pi}{7} + \cos\frac{24\pi}{7}\right)$$

The former expression $= -1$, as we have already shown. And by Example 77 the latter expression

$$= \frac{2\cos\frac{16\pi}{7}\sin\frac{12\pi}{7}}{\sin\frac{4\pi}{7}} = \frac{1}{\sin\frac{4\pi}{7}}\left(\sin 4\pi - \sin\frac{4\pi}{7}\right) = -1.$$

Hence the entire coefficient is -2.

The term independent of x is $-8\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}$; this

$$=-4\cos\frac{6\pi}{7}\left(\cos\frac{2\pi}{7}+\cos\frac{6\pi}{7}\right)$$

$$=-2\left(\cos\frac{8\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{12\pi}{7}+1\right)$$

$$=-\frac{2\cos\frac{8\pi}{7}\sin\frac{6\pi}{7}}{\sin\frac{2\pi}{7}}-2=-\frac{1}{\sin\frac{2\pi}{7}}\left(\sin 2\pi-\sin\frac{2\pi}{7}\right)-2$$

$$=1-2=-1$$

79 $\sin^3 A+\sin^3 B+\sin^3 C$

$$=\frac{1}{4}(3\sin A+3\sin B+3\sin C-\sin 3A-\sin 3B-\sin 3C).$$

Then by Example 32 of Chap VIII. we have

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

and
$$\sin 3A+\sin 3B+\sin 3C=-4\cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2}.$$

80 By solving the quadratic equation we obtain

$$\sin x=-b\pm\sqrt{(b^2-c)}$$

Hence b^2-c must not be negative, and $b\pm\sqrt{(b^2-c)}$ must not be greater than unity, in order that both values may be admissible

81 Let mx denote the number of sides in the first regular polygon, and nx the number of sides in the second Then, proceeding as in Example 61, we find that the number of degrees in an angle of the first polygon is $\frac{2mx-4}{mx}90$, and the number of grades in an angle of the second polygon is $\frac{2nx-4}{nx}100$ Therefore $\frac{2mx-4}{mx}90 : \frac{2nx-4}{nx}100 :: p : q$

Therefore $9q\frac{mx-2}{m}=10p\frac{nx-2}{n}$, therefore $x=\frac{2(9qn-10pm)}{mn(9q-10p)}$

Hence mx and nx are known

82 Here $2\cos 3x\cos 4x=\cos 4x$ Therefore either $\cos 4x=0$ or $2\cos 3x=1$

If $\cos 4x=0$ then $4x=(2n+1)\frac{\pi}{2}$

If $\cos 3x=\frac{1}{2}$ then $3x=2n\pi\pm\frac{\pi}{3}$.

83 From the first equation we have

$$x \sin(\alpha-\beta)\cos(\alpha+\beta)=y\sin(\alpha+\beta)\cos(\alpha-\beta),$$

therefore $$x(\sin 2\alpha-\sin 2\beta)=y(\sin 2\alpha+\sin 2\beta),$$

therefore $$(x-y)\sin 2\alpha=(x+y)\sin 2\beta$$

Thus $$\sin 2\alpha=\frac{(x+y)\sin 2\beta}{x-y},$$

and $$\cos 2\alpha=\frac{z-(x+y)\cos 2\beta}{x-y}.$$

Square and add, thus

$$1=\frac{(x+y)^2\sin^2 2\beta}{(x-y)^2}+\frac{\{z-(x+y)\cos 2\beta\}^2}{(x-y)^2}$$

Therefore $$(x-y)^2=(x+y)^2\sin^2 2\beta+\{z-(x+y)\cos 2\beta\}^2$$

$$=(x+y)^2+z^2-2z(x+y)\cos 2\beta,$$

therefore $$z^2+4xy=2z(x+y)\cos 2\beta$$

84 Let A denote the sum of x and y Suppose $x=\frac{A}{2}+z$, then $y=\frac{A}{2}-z$, and $\sin x\sin y=\sin\left(\frac{A}{2}-z\right)\sin\left(\frac{A}{2}+z\right)=\sin^2\frac{A}{2}-\sin^2 z.$ Now $\sin^2 z$ ranges between the values 0 and 1, hence $\sin x\sin y$ ranges between the values $\sin^2\frac{A}{2}$ and $-\cos^2\frac{A}{2}$: the former is always the greatest value *algebraically*.

85 We have $\sin\left(A+\frac{B}{2}\right)=\sin\left(\frac{A-C}{2}+\frac{A+B+C}{2}\right)=\cos\frac{A-C}{2},$

similarly $$\sin\left(B+\frac{C}{2}\right)=\cos\frac{B-A}{2},\quad \sin\left(C+\frac{A}{2}\right)=\cos\frac{C-B}{2}.$$

Then $$\cos\frac{A-C}{2}+\cos\frac{B-A}{2}=2\cos\frac{B-C}{4}\cos\frac{2A-B-C}{4},$$

and $$1+\cos\frac{C-B}{2}=2\cos^2\frac{B-C}{4};$$

therefore $$\cos\frac{A-C}{2}+\cos\frac{B-A}{2}+\cos\frac{C-B}{2}+1$$

$$=2\cos\frac{B-C}{4}\left\{\cos\frac{B-C}{4}+\cos\frac{2A-B-C}{4}\right\}$$

$$=4\cos\frac{B-C}{4}\cos\frac{A-C}{4}\cos\frac{A-B}{4}=4\cos\frac{A-B}{4}\cos\frac{B-C}{4}\cos\frac{C-A}{4}.$$

86 Bring the proposed expression to a common denominator, then the numerator

$$=2\cos\alpha\,(1-\cos^2\alpha)\cos B\cos C+2\cos\beta\,(1-\cos^2\beta)\cos C\cos A$$
$$+2\cos\gamma\,(1-\cos^2\gamma)\cos A\cos B+2\cos\alpha\cos\beta\cos\gamma$$
$$=2\cos\alpha\,(\cos^2\beta+\cos^2\gamma)\cos B\cos C+2\cos\beta\,(\cos^2\gamma+\cos^2\alpha)\cos C\cos A$$
$$+2\cos\gamma\,(\cos^2\alpha+\cos^2\beta)\cos A\cos B+2\cos\alpha\cos\beta\cos\gamma$$
$$=2\cos\alpha\cos\beta\,(\cos\alpha\cos A+\cos\beta\cos B)\cos C$$
$$+2\cos\beta\cos\gamma\,(\cos\beta\cos B+\cos\gamma\cos C)\cos A$$
$$+2\cos\gamma\cos\alpha\,(\cos\gamma\cos C+\cos\alpha\cos A)\cos B+2\cos\alpha\cos\beta\cos\gamma$$
$$=-2\cos\alpha\cos\beta\cos\gamma\cos^2 C-2\cos\alpha\cos\beta\cos\gamma\cos^2 A-2\cos\alpha\cos\beta\cos\gamma\cos^2 B$$
$$+2\cos\alpha\cos\beta\cos\gamma$$
$$=2\cos\alpha\cos\beta\cos\gamma\,(1-\cos^2 C-\cos^2 A-\cos^2 B)=0.$$

87. $$\sin^2 7\tfrac{1}{2}^\circ=\frac{1}{2}(1-\cos 15^\circ)=\frac{1}{2}\{1-\cos(45^\circ-30^\circ)\}=\frac{1}{2}\left\{1-\frac{\sqrt{3}+1}{2\sqrt{2}}\right\}$$
$$=\frac{2\sqrt{2}-\sqrt{3}-1}{4\sqrt{2}}=\frac{8-2\sqrt{6}-2\sqrt{2}}{16}$$

Now it will be found that

$$8-2\sqrt{6}-2\sqrt{2}=(2-\sqrt{2})(6+2\sqrt{2}-2\sqrt{3}-2\sqrt{6})=(2-\sqrt{2})(1+\sqrt{2}-\sqrt{3})^2,$$

therefore $$\sin 7\tfrac{1}{2}^\circ=\frac{1+\sqrt{2}-\sqrt{3}}{4}\sqrt{2-\sqrt{2}}$$

88 The second equation gives $$\frac{1+\tan\frac{\phi}{2}}{1-\tan\frac{\phi}{2}}=\frac{1+c}{1-c}\,\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}};$$

therefore $$\tan\frac{\phi}{2}=\frac{c+\tan\frac{\theta}{2}}{1+c\tan\frac{\theta}{2}},$$

The first equation gives $$\frac{2\tan\frac{\phi}{2}}{1-\tan^2\frac{\phi}{2}}=\frac{1+2c^2}{1-c^2}\,\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}},$$

therefore $$\frac{\left(c+\tan\frac{\theta}{2}\right)\left(1+c\tan\frac{\theta}{2}\right)}{\left(1+c\tan\frac{\theta}{2}\right)^2-\left(c+\tan\frac{\theta}{2}\right)^2}=\frac{1+2c^2}{1-c^2}\,\frac{\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}},$$

therefore $$\frac{c+(1+c^2)\tan\frac{\theta}{2}+c\tan^2\frac{\theta}{2}}{(1-c^2)\left(1-\tan^2\frac{\theta}{2}\right)}=\frac{(1+2c^2)\tan\frac{\theta}{2}}{(1-c^2)\left(1-\tan^2\frac{\theta}{2}\right)},$$

therefore either $1-\tan^2\frac{\theta}{2}=0$, or $c\tan^2\frac{\theta}{2}-c^2\tan\frac{\theta}{2}+c=0$

The former gives $\cos\theta=0$, the latter gives $1-c\sin\frac{\theta}{2}\cos\frac{\theta}{2}=0$, therefore $2-c\sin\theta=0$.

89 Put $2\cos^2 x-1$ for $\cos 2x$, then $2\cos^2 x+b\cos x+c-1=0$ By solving this quadratic equation we obtain

$$\cos x=\frac{-b\pm\sqrt{b^2-8(c-1)}}{4}.$$

Hence b^2+8-8c must not be negative, and $\sqrt{(b^2+8-8c)}-b$ must not be numerically greater than 4

90 Suppose θ_2 greater than θ_1, and each between 0 and γ Now

$$\sin\theta\,\{1+\sin(\gamma-\theta)\}=\sin\theta+\frac{1}{2}\sin\gamma\sin 2\theta-\cos\gamma\sin^2\theta$$

Put θ_1 and θ_2 in succession for θ, and subtract the second value of the expression from the first Thus we get

$$(\sin\theta_2-\sin\theta_1)\left\{1+\frac{1}{2}\sin\gamma\frac{\sin 2\theta_2-\sin 2\theta_1}{\sin\theta_2-\sin\theta_1}-(\sin\theta_2+\sin\theta_1)\cos\gamma\right\}$$

Now $(\sin\theta_2+\sin\theta_1)\cos\gamma$ is less than $2\sin\gamma\cos\gamma$, that is less than $\sin 2\gamma$, and therefore less than 1 Hence the preceding expression is necessarily positive, and this is what was to be proved

91 Let x be the number of sides in one regular polygon, and y the number of sides in another Then, as in Example 61, the number of degrees in an angle of the first polygon is $\frac{2x-4}{x}90$, and the number of grades in an angle of the second polygon is $\frac{2y-4}{y}100$ Hence we must have $\frac{2x-4}{x}90=\frac{2y-4}{y}100$, therefore $9y(x-2)=10x(y-2)$, therefore

$$x(20-y)=18y$$

We must then try in succession values of y from 3 to 19 inclusive, and ascertain in how many cases we obtain an integral value of x The admissible values will be found to be these

y	5	8	10	11	12	14	15	16	17	18	19
x	6	12	18	22	27	42	54	72	102	162	342

The cases in which the angles are expressed by integers are when

$$y=5,\ 8,\ 10 \text{ or } 16$$

92. We have $$\tan\gamma=\frac{1+\sin\alpha\sin\beta}{\cos\alpha\cos\beta},$$

and $$\cos 2\gamma=\frac{1-\tan^2\gamma}{1+\tan^2\gamma}=\frac{\cos^2\alpha\cos^2\beta-(1+\sin\alpha\sin\beta)^2}{\cos^2\alpha\cos^2\beta+(1+\sin\alpha\sin\beta)^2}.$$

The numerator

$$=(\cos\alpha\cos\beta+1+\sin\alpha\sin\beta)(\cos\alpha\cos\beta-1-\sin\alpha\sin\beta)$$
$$=-\{1+\cos(\alpha-\beta)\}\{1-\cos(\alpha+\beta)\},$$

and this cannot be positive, for $1+\cos(\alpha-\beta)$ and $1-\cos(\alpha+\beta)$ cannot be negative

93 Denote the angle by θ, then $\frac{\cos\theta}{\tan\theta}=\frac{3}{2}$, therefore $\cos^2\theta=\frac{3}{2}\sin\theta$, therefore $1-\sin^2\theta=\frac{3}{2}\sin\theta$ By solving this quadratic equation we get $\sin\theta=\frac{1}{2}$, or -2, the former is the only admissible value, and hence

$$\theta=n\pi+(-1)^n\frac{\pi}{6}.$$

94 Let A denote the sum of x and y Then

$$\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2}=2\sin\frac{A}{2}\cos\frac{x-y}{2},$$

and as $\cos\frac{x-y}{2}$ ranges between -1 and $+1$ the value of $\sin x+\sin y$ ranges between $-2\sin\frac{A}{2}$ and $2\sin\frac{A}{2}$ and the positive value out of these two is algebraically and arithmetically the greatest value of $\sin x+\sin y$

95 Here $\frac{\cos\theta}{\sqrt{2}}-\frac{\sin\theta}{\sqrt{2}}=1$, therefore $\cos\left(\theta+\frac{\pi}{4}\right)=1$; therefore

$$\theta+\frac{\pi}{4}=2n\pi$$

96 $$\sin^2A+\sin^2B+\sin^2C-2\sin A\sin B\sin C-1$$
$$=(\sin A-\sin B\sin C)^2+\sin^2B+\sin^2C-1-\sin^2B\sin^2C$$
$$=(\sin A-\sin B\sin C)^2-(1-\sin^2B)(1-\sin^2C)$$
$$=(\sin A-\sin B\sin C)^2-\cos^2B\cos^2C$$
$$=(\sin A-\sin B\sin C-\cos B\cos C)(\sin A-\sin B\sin C+\cos B\cos C)$$
$$=\{\sin A-\cos(B-C)\}\{\sin A+\cos(B+C)\}$$

$$=\left\{\cos\left(\frac{\pi}{2}-A\right)-\cos(B-C)\right\}\left\{\cos\left(\frac{\pi}{2}-A\right)+\cos(B+C)\right\}$$

$$=\text{ the product of } 4\sin\left(\frac{B-C-A}{2}+\frac{\pi}{4}\right)\sin\left(\frac{B-C+A}{2}-\frac{\pi}{4}\right)$$

$$\text{into }\cos\left(\frac{B+C-A}{2}+\frac{\pi}{4}\right)\cos\left(\frac{B+C+A}{2}-\frac{\pi}{4}\right)$$

Instead of $\sin\left(\frac{B-C-A}{2}+\frac{\pi}{4}\right)\sin\left(\frac{B-C+A}{2}-\frac{\pi}{4}\right)$ we may put

$$-\cos\left(\frac{A+C-B}{2}+\frac{\pi}{4}\right)\cos\left(\frac{A+B-C}{2}+\frac{\pi}{4}\right)$$

Thus the expression becomes the product of

$$-4\cos\left(\frac{A+B+C}{2}-\frac{\pi}{4}\right)\cos\left(\frac{B+C-A}{2}+\frac{\pi}{4}\right)$$

$$\text{into }\cos\left(\frac{A+C-B}{2}+\frac{\pi}{4}\right)\cos\left(\frac{A+B-C}{2}+\frac{\pi}{4}\right)$$

97 $\cos^2 7\frac{1}{2}^\circ=\frac{1}{2}(1+\cos 15^\circ)=\frac{1}{2}\{1+\cos(45^\circ-30^\circ)\}=\frac{1}{2}\left\{1+\frac{\sqrt{3}+1}{2\sqrt{2}}\right\}$

$$=\frac{2\sqrt{2}+\sqrt{3}+1}{4\sqrt{2}}=\frac{8+2\sqrt{6}+2\sqrt{2}}{16}$$

Now it will be found that

$$8+2\sqrt{6}+2\sqrt{2}=(2+\sqrt{2})(6-2\sqrt{2}-2\sqrt{3}+2\sqrt{6})=(2+\sqrt{2})(-1+\sqrt{2}+\sqrt{3})^2,$$

therefore $\qquad \cos 7\frac{1}{2}^\circ=\frac{-1+\sqrt{2}+\sqrt{3}}{4}\sqrt{2+\sqrt{2}}$

98. By addition $2a(\sin\theta+\cos\theta)=c(1+\sin 2\theta+\cos 2\theta)$

$$=2c\cos\theta(\sin\theta+\cos\theta),$$

therefore $\qquad a=c\cos\theta \qquad (1)$

Again, by subtraction, $2b(\sin\theta-\cos\theta)=c(1-\sin 2\theta-\cos 2\theta)$

$$=2c\sin\theta(\sin\theta-\cos\theta),$$

therefore $\qquad b=c\sin\theta \qquad (2)$

Square and add (1) and (2); thus $a^2+b^2=c^2$ This assumes that $\tan\theta$ is neither equal to 1 nor to -1

99 We have

$$A\cot\alpha(1-\cot\beta\cot\gamma)+B\cot\beta(1-\cot\alpha\cot\gamma)+C\cot\gamma(1-\cot\beta\cot\alpha)=0,$$

and

$$A\cot\alpha(\cot\beta+\cot\gamma)+B\cot\beta(\cot\gamma+\cot\alpha)+C\cot\gamma(\cot\alpha+\cot\beta)=0$$

These may be written

$$A\cos\alpha\cos(\beta+\gamma)+B\cos\beta\cos(\gamma+\alpha)+C\cos\gamma\cos(\alpha+\beta)=0,$$
$$A\cos\alpha\sin(\beta+\gamma)+B\cos\beta\sin(\gamma+\alpha)+C\cos\gamma\sin(\alpha+\beta)=0$$

Hence by Algebra, Art 385, we have

$$A=k\cos\beta\cos\gamma\{\cos(\gamma+\alpha)\sin(\alpha+\beta)-\cos(\alpha+\beta)\sin(\gamma+\alpha)\},$$
$$B=k\cos\gamma\cos\alpha\{\cos(\alpha+\beta)\sin(\beta+\gamma)-\cos(\beta+\gamma)\sin(\alpha+\beta)\},$$
$$C=k\cos\alpha\cos\beta\{\cos(\beta+\gamma)\sin(\gamma+\alpha)-\cos(\gamma+\alpha)\sin(\beta+\gamma)\},$$

where k is some constant

Thus $$A=k\cos\beta\cos\gamma\sin(\beta-\gamma),$$
$$B=k\cos\gamma\cos\alpha\sin(\gamma-\alpha),$$
$$C=k\cos\alpha\cos\beta\sin(\alpha-\beta)$$

Therefore $$A\sin 2\alpha+B\sin 2\beta+C\sin 2\gamma$$
$$=2k\cos\alpha\cos\beta\cos\gamma\{\sin\alpha\sin(\beta-\gamma)+\sin\beta\sin(\gamma-\alpha)+\sin\gamma\sin(\alpha-\beta)\}$$

The term within the brackets will be seen to vanish, since

$$\sin\alpha\sin(\beta-\gamma)=\frac{1}{2}\{\cos(\gamma+\alpha-\beta)-\cos(\alpha+\beta-\gamma)\},$$

$$\sin\beta\sin(\gamma-\alpha)=\frac{1}{2}\{\cos(\alpha+\beta-\gamma)-\cos(\beta+\gamma-\alpha)\},$$

and $$\sin\gamma\sin(\alpha-\beta)=\frac{1}{2}\{\cos(\beta+\gamma-\alpha)-\cos(\gamma+\alpha-\beta)\}$$

Or we might proceed thus let σ stand for $\alpha+\beta+\gamma$, then the two given relations may be written

$$A\cos\alpha\cos(\sigma-\alpha)+B\cos\beta\cos(\sigma-\beta)+C\cos\gamma\cos(\sigma-\gamma)=0,$$
$$A\cos\alpha\sin(\sigma-\alpha)+B\cos\beta\sin(\sigma-\beta)+C\cos\gamma\sin(\sigma-\gamma)=0,$$

therefore $$(A\cos^2\alpha+B\cos^2\beta+C\cos^2\gamma)\cos\sigma$$
$$=-(A\sin\alpha\cos\alpha+B\sin\beta\cos\beta+C\sin\gamma\cos\gamma)\sin\sigma \qquad (1)$$

And $$(A\cos^2\alpha+B\cos^2\beta+C\cos^2\gamma)\sin\sigma$$
$$=(A\sin\alpha\cos\alpha+B\sin\beta\cos\beta+C\sin\gamma\cos\gamma)\cos\sigma \qquad (2)$$

Multiply (1) by $\sin\sigma$ and (2) by $\cos\sigma$ and subtract thus

$$A\sin\alpha\cos\alpha+B\sin\beta\cos\beta+C\sin\gamma\cos\gamma=0,$$

which is the required result

Again, multiply (1) by $\cos\sigma$ and (2) by $\sin\sigma$ and add, then we obtain the additional result $A\cos^2\alpha+B\cos^2\beta+C\cos^2\gamma=0$

100. From the first equation

$$(\cos\theta - \cos\alpha\cos\beta)^2 = \sin^2\alpha\sin^2\beta\,(1 - c^2\sin^2\theta),$$

substitute $1 - \cos^2\theta$ for $\sin^2\theta$, thus

$$\cos^2\theta\,(1 - c^2\sin^2\alpha\sin^2\beta) - 2\cos\theta\cos\alpha\cos\beta$$
$$+ \cos^2\alpha\cos^2\beta - (1 - c^2)\sin^2\alpha\sin^2\beta = 0$$

The second equation leads to the same quadratic for finding $\cos\phi$ Hence we infer that $\cos\theta$ is one root of the quadratic and $\cos\phi$ the other Hence by the theory of quadratic equations, *Algebra*, Chapter XXII,

$$\cos\theta + \cos\phi = \frac{2\cos\alpha\cos\beta}{1 - c^2\sin^2\alpha\sin^2\beta},$$

$$\cos\theta\cos\phi = \frac{\cos^2\alpha\cos^2\beta - (1 - c^2)\sin^2\alpha\sin^2\beta}{1 - c^2\sin^2\alpha\sin^2\beta},$$

therefore $$1 + \cos\theta\cos\phi = \frac{1 - \sin^2\alpha\sin^2\beta + \cos^2\alpha\cos^2\beta}{1 - c^2\sin^2\alpha\sin^2\beta}$$

$$= \frac{\cos^2\alpha + \cos^2\beta}{1 - c^2\sin^2\alpha\sin^2\beta}$$

Then $$\sin^2\theta\sin^2\phi = (1 - \cos^2\theta)(1 - \cos^2\phi)$$

$$= (1 + \cos\theta\cos\phi)^2 - (\cos\theta + \cos\phi)^2$$

$$= \frac{(\cos^2\alpha - \cos^2\beta)^2}{(1 - c^2\sin^2\alpha\sin^2\beta)^2};$$

therefore $$\sin\theta\sin\phi = \pm\frac{\cos^2\alpha - \cos^2\beta}{1 - c^2\sin^2\alpha\sin^2\beta}.$$

And $$\tan\frac{\theta}{2}\tan\frac{\phi}{2} = \frac{1 - \cos\theta}{\sin\theta}\cdot\frac{1 - \cos\phi}{\sin\phi}$$

$$= \frac{1 - (\cos\theta + \cos\phi) + \cos\theta\cos\phi}{\sin\theta\sin\phi}$$

$$= \frac{(\cos\alpha - \cos\beta)^2}{\pm(\cos^2\alpha - \cos^2\beta)} = \pm\frac{\cos\alpha - \cos\beta}{\cos\alpha + \cos\beta}$$

101 $$\cos 11A + \cos 5A = 2\cos 8A\,\cos 3A,$$

$$3\cos 9A + 3\cos 7A = 6\cos 8A\,\cos A,$$

hence by addition we find that the proposed expression

$$= 2\cos 8A\,(\cos 3A + 3\cos A) = 8\cos 8A\,\cos^3 A$$

$$= 8\cos^3 A\,(2\cos^2 4A - 1) = 16\cos^3 A\left(\cos^2 4A - \frac{1}{2}\right)$$

$$= 16\cos^3 A\left(\cos^2 4A - \sin^2\frac{\pi}{4}\right) = 16\cos^3 A\cos\left(4A + \frac{\pi}{4}\right)\cos\left(4A - \frac{\pi}{4}\right);$$

see Art 83

102 Let the distance be denoted by x inches, then we must have $\frac{\frac{1}{2}}{x}$ = the tangent of a quarter of a degree As the tangent of a small angle is approximately equal to its circular measure we have approximately $\frac{1}{2x}=\frac{1}{4}\,\frac{\pi}{180}$, therefore $x=\frac{2\times 180}{\pi}=114\ 6$ nearly.

103 By Example 27 of Chapter VI this becomes $\sin 4\theta=1$, therefore

$$4\theta=(4n+1)\frac{\pi}{2}$$

104 Here $\quad c\sin\theta=a\sin\theta\cos\phi+a\cos\theta\sin\phi,$

substitute for $\sin\phi$ and $\cos\phi$, thus

$$c\sin\theta=a(\cos\theta-2m)\sin\theta+a\cos\theta\times\frac{b}{a}\sin\theta,$$

therefore $$c=a(\cos\theta-2m)+b\cos\theta,$$

therefore $$\cos\theta=\frac{c+2am}{a+b}.$$

Therefore $$\cos\phi=\frac{c+2am}{a+b}-2m=\frac{c-2bm}{a+b}.$$

But $a^2\sin^2\phi=b^2\sin^2\theta$, therefore

$$a^2-b^2=a^2\cos^2\phi-b^2\cos^2\theta=\frac{a^2(c-2bm)^2-b^2(c+2am)^2}{(a+b)^2}$$

$$=\frac{(a-b)c^2-4abcm}{a+b}$$

105 We have, by Art 252,

$$a=2R\sin A,\quad b=2R\sin B,\quad c=2R\sin C,$$

hence the proposed expression

$$=2R\{\sin A\sin(B-C)+\sin B\sin(C-A)+\sin C\sin(A-B)\}$$

$$=2R\{\sin(B+C)\sin(B-C)+\sin(C+A)\sin(C-A)$$
$$+\sin(A+B)\sin(A-B)\}$$

$$=2R\{\sin^2 B-\sin^2 C+\sin^2 C-\sin^2 A+\sin^2 A-\sin^2 B\}$$

$$=0$$

106 By Art 108 the proposed expression

$$=\frac{10+2\sqrt{5}}{16}\ \frac{10-2\sqrt{5}}{16}-\frac{\sqrt{5}+1}{4}\cdot\frac{\sqrt{5}-1}{4}=\frac{5}{16}-\frac{1}{4}=\frac{1}{16}.$$

107 From the triangle OAB we have

$$\frac{OA}{AB}=\frac{\sin OBA}{\sin AOB}=\frac{\cos A}{\sin C},$$

therefore $$x=\frac{c\cos A}{\sin C}=\frac{a\cos A}{\sin A}.$$

Similarly $$y=\frac{b\cos B}{\sin B},\quad\text{and } z=\frac{c\cos C}{\sin C}.$$

Hence we have only to show that

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C,$$

and this is known to be true by Art. 114

108 Let l denote the length of the pole The distance of the coping from the ground is $l\sin A$, and the distance of the sill from the ground is $l\sin B$, hence the distance from the coping to the sill $=l(\sin A-\sin B)$

The distance of the foot of the ladder from the wall is $l\cos A$ at first, and $l\cos B$ afterwards, therefore $a=l(\cos B-\cos A)$

Substitute for l in the former expression, and we obtain

$$\frac{a(\sin A-\sin B)}{\cos B-\cos A},\quad\text{that is}\quad\frac{a\cos\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A+B)},\quad\text{that is}\quad a\cot\frac{1}{2}(A+B)$$

109 Let r denote the radius Then the area of the sector PCB $=\frac{r^2}{2}\left(\frac{\pi}{2}-\theta\right)$, by Art 260,

and the area of the triangle $ACP=\frac{r^2}{2}\sin\left(\frac{\pi}{2}+\theta\right)$, by Art 247

The sum of these two areas by supposition is equal to half the area of the semicircle, thus

$$\frac{r^2}{2}\left(\frac{\pi}{2}-\theta\right)+\frac{r^2}{2}\cos\theta=\frac{\pi r^2}{4},$$

then by simplifying we obtain $\cos\theta=\theta$

110 By Art 257 we have $a=2R\sin 36^\circ$, and $a'=2R\sin 18^\circ$,

therefore $$a^2-a'^2=4R^2\left\{\frac{10-2\sqrt{5}}{16}-\frac{(\sqrt{5}-1)^2}{16}\right\}=\frac{4R^2\times 4}{16}=R^2$$

Also $$\frac{a}{r}=2\tan 36^\circ,\quad\text{and}\quad\frac{a'}{r'}=2\tan 18^\circ,$$

therefore $$\frac{a}{r}+\frac{a'}{r'}=2\left(\frac{\sin 36^\circ}{\cos 36^\circ}+\frac{\sin 18^\circ}{\cos 18^\circ}\right)=\frac{2\sin(36^\circ+18^\circ)}{\cos 36^\circ\cos 18^\circ}$$
$$=\frac{2\sin 54^\circ}{\cos 36^\circ\cos 18^\circ}=\frac{2}{\cos 18^\circ}=\frac{2R}{r'}.$$

111 $\cos^2\frac{A}{2}=\frac{1}{2}(1+\cos A)$,

$$\cos^4\frac{A}{2}=\frac{1}{4}(1+\cos A)^2=\frac{1}{4}(1+2\cos A+\cos^2 A)$$
$$=\frac{1}{4}\left\{1+2\cos A+\frac{1}{2}(1+\cos 2A)\right\}=\frac{3}{8}+\frac{1}{2}\cos A+\frac{1}{8}\cos 2A.$$

In this way the proposed expression becomes

$$\frac{9}{8}+\frac{1}{2}(\cos A+\cos B+\cos C)+\frac{1}{8}(\cos 2A+\cos 2B+\cos 2C)$$
$$-\frac{1}{2}(1+\cos A)(1+\cos B)-\frac{1}{2}(1+\cos B)(1+\cos C)-\frac{1}{2}(1+\cos C)(1+\cos A)$$
$$+\frac{1}{2}(1+\cos A)(1+\cos B)(1+\cos C)$$
$$=\frac{1}{8}+\frac{1}{8}(\cos 2A+\cos 2B+\cos 2C)+\frac{1}{2}\cos A\cos B\cos C$$
$$=\frac{1}{8}-\frac{1}{8}=0,$$ see Example VIII 18

112 We have $1+\tan^2\theta=\frac{1}{\cos^2\theta}=\frac{1}{1-\sin^2\theta}$; take the logarithms of both sides, thus

$$\log(1+\tan^2\theta)=\log\frac{1}{1-\sin^2\theta}=-\log(1-\sin^2\theta),$$

therefore, by Art 116,

$$\tan^2\theta-\frac{1}{2}\tan^4\theta+\frac{1}{3}\tan^6\theta-\dots=\sin^2\theta+\frac{1}{2}\sin^4\theta+\frac{1}{3}\sin^6\theta+\dots$$

The series are convergent, since $\tan^2\theta$ is supposed to be less than unity

113. Here $$2\sin\frac{3\theta}{2}\sin\frac{\theta}{2}=2\sin\frac{3\theta}{2}\cos\frac{3\theta}{2},$$

therefore either $$\sin\frac{3\theta}{2}=0,\quad\text{or}\quad\sin\frac{\theta}{2}=\cos\frac{3\theta}{2}.$$

If $\sin\frac{3\theta}{2}=0$, then $\frac{3\theta}{2}=n\pi$.

If $\sin\frac{\theta}{2}=\cos\frac{3\theta}{2}$, or $\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)=\cos\frac{3\theta}{2}$, then $\frac{\pi}{2}-\frac{\theta}{2}=2n\pi\pm\frac{3\theta}{2}$

114. We have

$$a=2R\sin A, \quad b=2R\sin B, \quad c=2R\sin C,$$

hence the proposed expression

$$=4R^2\{\sin A\sin(B-C)+\sin B\sin(C-A)+\sin C\sin(A-B)\},$$

and this is zero, as in the solution of Example 105.

115. We have $\qquad \dfrac{\cos 3\theta}{\sqrt{2}}+\dfrac{\sin 3\theta}{\sqrt{2}}=\dfrac{\cos\theta}{\sqrt{2}}+\dfrac{\sin\theta}{\sqrt{2}},$

therefore $\qquad \cos\left(3\theta-\dfrac{\pi}{4}\right)=\cos\left(\theta-\dfrac{\pi}{4}\right),$

therefore $\qquad 3\theta-\dfrac{\pi}{4}=2n\pi\pm\left(\theta-\dfrac{\pi}{4}\right).$

116. We have $\quad \tan^2 x=\dfrac{\sin(\alpha+x)\sin(\alpha-x)}{\cos(\alpha+x)\cos(\alpha-x)}=\dfrac{\sin^2\alpha-\sin^2 x}{\cos^2 x-\sin^2\alpha},$

therefore $\qquad \sin^2 x(\cos^2 x-\sin^2\alpha)=\cos^2 x(\sin^2\alpha-\sin^2 x),$

therefore $\qquad 2\sin^2 x\cos^2 x=\sin^2\alpha(\sin^2 x+\cos^2 x)=\sin^2\alpha,$

therefore $\qquad 4\sin^2 x\cos^2 x=2\sin^2\alpha,$

therefore $\qquad 2\sin x\cos x=\sqrt{2}\,\sin\alpha,$

therefore $\qquad \sin 2x=\sqrt{2}\,.\sin\alpha.$

117. Put x for $\tan A$, y for $\tan B$, z for $\tan C$, x' for $\tan A'$, y' for $\tan B'$, and z' for $\tan C'$, for the sake of abbreviation. Then we have given that

$$x^2x'=y^2y'=z^2z'=xyz \qquad (1),$$

and

$$\frac{1+x^2}{2x}+\frac{1+y^2}{2y}+\frac{1+z^2}{2z}=0 \qquad (2).$$

Now $\qquad \tan(A-A')=\dfrac{x-x'}{1+xx'}=\dfrac{x-\dfrac{yz}{x}}{1+yz}$, by (1), $=\dfrac{x^2-yz}{x(1+yz)}.$

But from (2) we have

$$x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}=0,$$

therefore $\qquad xyz(x+y+z)+xy+yz+zx=0,$

therefore $\qquad x^2-yz=x^2+xyz(x+y+z)+xy+zx$

$$=x(x+y+z)+xyz(x+y+z)=x(1+yz)(x+y+z),$$

therefore $\qquad \dfrac{x^2-yz}{x(1+yz)}=x+y+z.$

Thus $$\tan(A-A')=\tan A+\tan B+\tan C.$$

Similarly $\tan(B-B')$ and $\tan(C-C')$ may be shewn to be equal to the same expression.

118 Here $$\cos A=\frac{\cos 60^\circ}{\sin 36^\circ},$$

therefore $$\sin A=\frac{\sqrt{\sin^2 36^\circ-\cos^2 60^\circ}}{\sin 36^\circ},$$

and $$\sin^2 36-\cos^2 60^\circ=\frac{10-2\sqrt5}{16}-\frac14=\frac{6-2\sqrt5}{16}=\left(\frac{\sqrt5-1}{4}\right)^2,$$

therefore $$\sin A=\frac{\sqrt5-1}{4\sin 36^\circ}$$

Hence $$\tan A=\frac{\sqrt5-1}{4\cos 60^\circ}=\frac{\sqrt5-1}{2}.$$

Again, $$\cos B=\frac{\cos 36^\circ}{\sin 60^\circ},$$

therefore $$\sin B=\frac{\sqrt{\sin^2 60^\circ-\cos^2 36^\circ}}{\sin 60^\circ}=\frac{\sqrt{\sin^2 36^\circ-\cos^2 60^\circ}}{\sin 60^\circ}=\frac{\sqrt5-1}{4\sin 60^\circ}$$

Hence $$\tan B=\frac{\sqrt5-1}{4\cos 36^\circ}=\frac{\sqrt5-1}{\sqrt5+1}=\frac{(\sqrt5-1)^2}{(\sqrt5+1)(\sqrt5-1)}=\frac{6-2\sqrt5}{4}=\frac{3-\sqrt5}{2}$$

Therefore $$\tan A+\tan B=\frac{\sqrt5-1}{2}+\frac{3-\sqrt5}{2}=1,$$

therefore $\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}=1$, therefore $\sin(A+B)=\cos A\cos B=\cos C$, therefore $A+B=90^\circ-C$ is one solution

119 Let θ be the sun's altitude at the first observation, and $\theta+\alpha$ that at the second observation, then

$$h=a\tan\theta,\ \text{and}\ h=b\tan(\theta+\alpha)$$

Thus $$h=\frac{b(\tan\theta+\tan\alpha)}{1-\tan\theta\tan\alpha}=\frac{\frac{hb}{a}+b\tan\alpha}{1-\frac{h}{a}\tan\alpha},$$

therefore $$h\left(1-\frac{h}{a}\tan\alpha\right)=\frac{hb}{a}+b\tan\alpha,$$

therefore $$h^2\tan\alpha+h(b-a)+ab\tan\alpha=0,$$

therefore $$h^2+h(b-a)\cot\alpha+ab=0.$$

120 Let $$AP=b,\ BP=a,\quad AB=c$$

The diameter of the circle which touches the semicircle and also touches AP at its middle point is $\frac{c}{2}-\frac{c}{2}\sin PAB$, that is $\frac{c}{2}-\frac{c}{2}\frac{a}{c}$, that is $\frac{c-a}{2}$, therefore the radius of this circle is $\frac{c-a}{4}$

Similarly the radius of the circle which touches the semicircle and also touches BP at its middle point is $\frac{c-b}{4}$. We have then to shew that

$$\frac{(c-a)(c-b)}{16}=\frac{r^2}{8},\text{ that is }\frac{(c-a)(c-b)}{2}=r^2$$

But $r=\frac{S}{s}=\frac{ab}{a+b+c}=\frac{ab(a+b-c)}{(a+b)^2-c^2}=\frac{a+b-c}{2}$, since $c^2=a^2+b^2$, therefore

$$r^2=\frac{(a+b-c)^2}{4}=\frac{2c^2+2ab-2c(a+b)}{4}=\frac{(c-a)(c-b)}{2}.$$

121 $$\sin\alpha\sin(\beta-\gamma)\cos(\beta+\gamma-\alpha)$$
$$=\frac{1}{2}\{\cos(\alpha-\beta+\gamma)-\cos(\alpha+\beta-\gamma)\}\cos(\beta+\gamma-\alpha)$$
$$=\frac{1}{4}\{\cos 2\gamma+\cos 2(\alpha-\beta)-\cos 2\beta-\cos 2(\alpha-\gamma)\}$$

The other two terms may be transformed in a similar manner, and then it will be obvious that the sum is zero

122 $$\cot\theta=\frac{\cos\theta}{\sin\theta}=\frac{2\cos^2\theta}{2\sin\theta\cos\theta}=\frac{1+\cos 2\theta}{\sin 2\theta}$$
$$=\frac{1+\cos 2\theta}{\sqrt{(1-\cos^2 2\theta)}}=\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}$$

Hence, taking logarithms we have

$$\log\cot\theta=\frac{1}{2}\log\frac{1+\cos 2\theta}{1-\cos 2\theta}=\cos 2\theta+\frac{1}{3}(\cos 2\theta)^3+\frac{1}{5}(\cos 2\theta)^5+$$

123 We have shewn in the solution of Example VIII. 15 that

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2},$$

hence the expression on the right hand-side

$$=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\ \frac{8\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}{\cos A\cos B\cos C}$$
$$=\frac{8\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}}{\cos A\cos B\cos C}=\tan A\tan B\tan C$$

Again $\qquad \cot A - 2\cot 2A = \cot A - \dfrac{2(\cos^2 A - \sin^2 A)}{2\cos A\sin A} = \tan A;$

similarly $\qquad \cot B - 2\cot 2B = \tan B$, and $\cot C - 2\cot 2C = \tan C$,

thus the expression on the left-hand side

$$= \tan A + \tan B + \tan C = \tan A \tan B \tan C, \text{ by Art. 114.}$$

Thus the two expressions are equivalent

124 $\quad \sin A \sin B \sin (A - B) = \frac{1}{2}\{\cos (A - B) - \cos (A + B)\} \sin (A - B)$

$$= \frac{1}{4} \sin (2A - 2B) - \frac{1}{4}(\sin 2A - \sin 2B)$$

Transform the second and third terms in the same way, then by addition we obtain the required result

125 $\quad \dfrac{1}{a}\cos^2\dfrac{A}{2} + \dfrac{1}{b}\cos^2\dfrac{B}{2} + \dfrac{1}{c}\cos^2\dfrac{C}{2} = \dfrac{s(s-a)+s(s-b)+s(s-c)}{abc}$

$$= \frac{3s^2 - s(a+b+c)}{abc} = \frac{3s^2 - 2s^2}{abc} = \frac{s^2}{abc} = \frac{(a+b+c)^2}{4abc}.$$

126 Here $$\frac{1}{\cos\left(\frac{\pi}{4}+x\right)} + \frac{1}{\cos\left(\frac{\pi}{4}-x\right)} = 2\sqrt{2},$$

therefore $$\frac{\sqrt{2}}{\cos x - \sin x} + \frac{\sqrt{2}}{\cos x + \sin x} = 2\sqrt{2},$$

therefore $$\cos x = \cos^2 x - \sin^2 x = \cos 2x,$$

therefore $$2x = 2n\pi \pm x$$

127 Express the fractions with the common denominator

$$\sin (\alpha - \beta) \sin (\beta - \gamma) \sin (\gamma - \alpha)$$

then the numerator becomes

$$-\{\sin (\beta-\gamma) \sin (\theta-\alpha) + \sin (\gamma-\alpha) \sin (\theta-\beta) + \sin (\alpha-\beta) \sin (\theta-\gamma)\}$$

Now $\quad \sin (\beta-\gamma) \sin (\theta-\alpha) = \frac{1}{2}\cos (\theta-\alpha-\beta+\gamma) - \frac{1}{2}\cos (\theta-\alpha+\beta-\gamma),$

$$\sin (\gamma-\alpha) \sin (\theta-\beta) = \frac{1}{2}\cos (\theta-\beta+\alpha-\gamma) - \frac{1}{2}\cos (\theta-\beta+\gamma-\alpha),$$

$$\sin (\alpha-\beta) \sin (\theta-\gamma) = \frac{1}{2}\cos (\theta-\gamma+\beta-\alpha) - \frac{1}{2}\cos (\theta-\gamma+\alpha-\beta),$$

thus the sum of the expressions is zero

128 Let x denote the length of the pillar, h the height of the foot of the pillar above the horizontal plane, b the horizontal distance of the pillar from the first station Let θ be the angle subtended by the pillar. Then

$$\frac{h}{b}=\tan(\alpha-\theta),\quad \frac{h}{b+c}=\tan(\beta-\theta),\quad \frac{h+x}{b}=\tan\alpha,\quad \frac{h+x}{b+c}=\tan\beta$$

And from the fact that a circle would pass through the two stations and the top and the foot of the pillar we have $\alpha+\beta-\theta=\frac{\pi}{2}$ Thus

$$\frac{h}{b}=\cot\beta,\quad \frac{h+x}{b}=\tan\alpha;\ \text{therefore}$$

$$\frac{x}{b}=\tan\alpha-\cot\beta=-\frac{\cos(\alpha+\beta)}{\cos\alpha\sin\beta}.$$

Similarly $$\frac{x}{b+c}=\tan\beta-\cot\alpha=-\frac{\cos(\alpha+\beta)}{\cos\beta\sin\alpha}$$

Therefore $$\frac{c}{x}=\frac{\cos\alpha\sin\beta-\cos\beta\sin\alpha}{\cos(\alpha+\beta)}=\frac{\sin(\beta-\alpha)}{\cos(\beta+\alpha)};$$

therefore $$x=\frac{c\cos(\beta+\alpha)}{\sin(\beta-\alpha)}.$$

129 In every *right-angled* triangle $r=\frac{1}{2}(a+b-c)$, see the solution of Example 120 In the present case $2\sqrt{R^2-2Rr}=\sqrt{c(a+b-c)}$, and $R=\frac{c}{2}$ thus $4\left(\frac{c^2}{4}-cr\right)=c(a+b-c)$, therefore $c-4r=a+b-c$, therefore $4r=2c-a-b$, therefore $2(a+b-c)=2c-a-b$, therefore $a+b=\frac{4c}{3}$, and therefore $r=\frac{c}{6}$.

130. $$\cos^2\frac{1}{2}A+\cos^2\frac{1}{2}B+\cos^2\frac{1}{2}C=\frac{1}{2}(3+\cos A+\cos B+\cos C)$$

$$=\frac{1}{2}\left(4+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right),\ \text{by Art 114,}$$

$$=2+\frac{2S^2}{sabc}=2+\frac{2rS}{abc}=2+\frac{r}{2R}.$$

131 $$\frac{\sin(x+A)}{\sqrt{\sin 2A}}=\frac{\sin(x+B)}{\sqrt{\sin 2B}},$$

therefore $$\frac{\sin x\cos A+\cos x\sin A}{\sqrt{\sin 2A}}=\frac{\sin x\cos B+\cos x\sin B}{\sqrt{\sin 2B}},$$

therefore $$\frac{\cos A(\tan x+\tan A)}{\sqrt{\sin 2A}}=\frac{\cos B(\tan x+\tan B)}{\sqrt{\sin 2B}},$$

therefore $$\frac{\tan x+\tan A}{\sqrt{\tan A}}=\frac{\tan x+\tan B}{\sqrt{\tan B}},$$

therefore $$\tan x\left(\sqrt{\tan B}-\sqrt{\tan A}\right)=\sqrt{\tan A\tan B}\left(\sqrt{\tan B}-\sqrt{\tan A}\right),$$

therefore $$\tan x=\sqrt{\tan A\tan B}$$

132 $$\sin^2 2A+\cos 2A\cos 2B\cos 2C=1-\cos^2 2A+\cos 2A\cos 2B\cos 2C$$
$$=1+\cos 2A\{\cos 2B\cos 2C-\cos 2A\}$$
$$=1+\cos 2A\{\cos 2B\cos 2C-\cos(2B+2C)\}$$
$$=1+\cos 2A\sin 2B\sin 2C$$

Similarly
$$\sin^2 2B+\cos 2A\cos 2B\cos 2C=1+\cos 2B\sin 2A\sin 2C$$

Hence the proposed expression
$$=2+\sin 2C\{\sin 2C+\sin 2B\cos 2A+\sin 2A\cos 2B\}$$
$$=2+\sin 2C\{-\sin(2A+2B)+\sin(2A+2B)\}$$
$$=2$$

133 The series may be separated into two, namely
$$\log 2+\frac{1}{\underline{|2}}(\log 2)^2+\frac{1}{\underline{|3}}(\log 2)^3+$$
and $$2\log 2+\frac{1}{\underline{|2}}(2\log 2)^2+\frac{1}{\underline{|3}}(2\log 2)^3+$$
and is therefore equal to $e^{\log 2}-1+e^{2\log 2}-1$, that is to $e^{\log 2}-1+e^{\log 4}-1$, that is to $2-1+4-1$, that is to 4

134 $$\sin(A-B)\cos(C-B)\cos(A-C)$$
$$=\frac{1}{2}\sin(A-B)\{\cos(A+B-2C)+\cos(A-B)\}$$
$$=\frac{1}{4}\sin 2(A-C)+\frac{1}{4}\sin 2(C-B)+\frac{1}{4}\sin 2(A-B)$$

Transform the second and third terms in like manner, then by addition we obtain the required result

135 $$\frac{\sin(A-B)}{\sin(A+B)}=\frac{\sin(A+B)\sin(A-B)}{\sin^2(A+B)}=\frac{\sin^2 A-\sin^2 B}{\sin^2 C}, \text{ by Art 83,}$$
$$=\left(\frac{\sin A}{\sin C}\right)^2-\left(\frac{\sin B}{\sin C}\right)^2=\left(\frac{a}{c}\right)^2-\left(\frac{b}{c}\right)^2=\frac{a^2-b^2}{c^2}$$

136 Suppose the diagonal h of the quadrilateral to make an angle θ with the sides of the rectangle which pass through its extremities, then each of the other sides is equal to $h\sin\theta$ It will be seen from a diagram that

the diagonal k of the quadrilateral will make an angle $\frac{3\pi}{2}-(\theta+A)$ with the sides of the rectangle which pass through its extremities, then each of the other sides is equal to $k \sin\left(\frac{3\pi}{2}-\theta-A\right)$, that is to $-k\cos(\theta+A)$ Hence the area of the rectangle $=-hk\sin\theta\cos(\theta+A)=\frac{hk}{2}\{-\sin(2\theta+A)+\sin A\}$

The greatest value of this is when $2\theta+A=\frac{3\pi}{2}$, and is $\frac{hk}{2}(1+\sin A)$

137. Let h denote the height of the house, x the height of the wall, y the height of the church Then $x\cot a$ is the distance of the wall from the house, and $y\cot a$ is the distance of the church from the house By similar triangles $\frac{h}{x\cot a}=\frac{y}{y\cot a-x\cot a}$, therefore $h(y-x)=xy$

Also $\frac{y-h}{y\cot a}=\tan\beta$, therefore $y=\frac{h}{1-\cot a\tan\beta}=\frac{h\tan a}{\tan a-\tan\beta}$

Then
$$x=\frac{hy}{h+y}=\frac{h\tan a}{2\tan a-\tan\beta}$$

138.
$$a\cos^2\frac{1}{2}A+b\cos^2\frac{1}{2}B+c\cos^2\frac{1}{2}C$$
$$=\frac{1}{2}(a+b+c+a\cos A+b\cos B+c\cos C)$$
$$=s+\frac{1}{2}R(\sin 2A+\sin 2B+\sin 2C)$$
$$=s+2R\sin A\sin B\sin C, \text{ by Art 114},$$
$$=s+2R\frac{8S^3}{a^2b^2c^2}=s+\frac{4S^2}{abc}=s+\frac{S}{R}$$

139 Through C draw a plane parallel to the horizon, from A draw AP perpendicular to the intersection of this plane with that which contains A, B, and C, from B draw BQ perpendicular to the same intersection Let $ACP=\phi$, and $BCQ=\psi$, so that $\phi+\psi+\gamma=\pi$. Therefore

$$\cos\gamma=\sin\phi\sin\psi-\cos\phi\cos\psi$$

Now $AP=AC\sin\phi$, thus the perpendicular from A on the plane drawn through C parallel to the horizon $=AP\sin\theta=AC\sin\theta\sin\phi$, but this perpendicular also $=AC\sin a$, therefore

$$\sin a=\sin\theta\sin\phi$$

Similarly
$$\sin\beta=\sin\theta\sin\psi$$

Hence
$$\cos\gamma=\frac{\sin a\sin\beta}{\sin^2\theta}-\frac{\sqrt{(\sin^2\theta-\sin^2 a)(\sin^2\theta-\sin^2\beta)}}{\sin^2\theta},$$

therefore $(\cos\gamma \sin^2\theta - \sin\alpha \sin\beta)^2 = (\sin^2\theta - \sin^2\alpha)(\sin^2\theta - \sin^2\beta)$,

therefore $\cos^2\gamma \sin^2\theta - 2\cos\gamma \sin\alpha \sin\beta = \sin^2\theta - \sin^2\alpha - \sin^2\beta$,

therefore $\sin^2\theta \sin^2\gamma = \sin^2\alpha + \sin^2\beta - 2\sin\alpha \sin\beta \cos\gamma$

140. With the diagram of Art 248 we see that

$$a' = 2r \sin FOA = 2r \cos\frac{1}{2}A,$$

and we have similar values for b' and c'

Thus $$a'b'c' = 8r^3 \cos\frac{1}{2}A \cos\frac{1}{2}B \cos\frac{1}{2}C = 8r^3 \frac{sS}{abc};$$

therefore $$\frac{a'b'c'}{abc} = \frac{8r^2S^2}{a^2b^2c^2} = \frac{8r^2}{(4R)^2} = \frac{r^2}{2R^2}$$

141 Here $$\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} + \frac{2\tan\theta}{1 - \tan^2\theta} + \tan\theta = 0$$

Therefore either $\tan\theta = 0$ or $\dfrac{3 - \tan^2\theta}{1 - 3\tan^2\theta} + \dfrac{2}{1 - \tan^2\theta} + 1 = 0.$

The latter gives

$$(3 - \tan^2\theta)(1 - \tan^2\theta) + 2(1 - 3\tan^2\theta) + (1 - \tan^2\theta)(1 - 3\tan^2\theta) = 0,$$

therefore $$4\tan^4\theta - 14\tan^2\theta + 6 = 0.$$

By solving this quadratic we obtain $\tan^2\theta = 3$ or $\frac{1}{2}$

142 We may obtain the result by taking the values of the four cosines and raising them to the eighth power. Or we may proceed thus:

$$\cos^8\frac{\pi}{8} + \cos^8\frac{3\pi}{8} + \cos^8\frac{5\pi}{8} + \cos^8\frac{7\pi}{8}$$

$$= 2\left(\cos^8\frac{\pi}{8} + \cos^8\frac{3\pi}{8}\right) = 2\left(\cos^8\frac{\pi}{8} + \sin^8\frac{\pi}{8}\right)$$

$$= \frac{1}{32}\left(\cos\pi + 28\cos\frac{\pi}{2} + 35\right), \text{ by Example IX 13,}$$

$$= \frac{34}{32} = \frac{17}{16}$$

143 Here $\dfrac{b}{a} = \dfrac{\cos\phi}{\cos\theta}$, therefore $\dfrac{a+b}{a-b} = \dfrac{\cos\theta + \cos\phi}{\cos\theta - \cos\phi}$

$$= \frac{2\cos\frac{1}{2}(\phi+\theta)\cos\frac{1}{2}(\phi-\theta)}{2\sin\frac{1}{2}(\phi+\theta)\sin\frac{1}{2}(\phi-\theta)} = \cot\frac{1}{2}(\phi+\theta)\cot\frac{1}{2}(\phi-\theta)$$

144. $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)\sin\left(\frac{\pi}{2}-\frac{B}{2}\right)\sin\left(\frac{\pi}{2}-\frac{C}{2}\right)$,

and the sum of $\frac{\pi}{2}-\frac{A}{2}$, $\frac{\pi}{2}-\frac{B}{2}$, and $\frac{\pi}{2}-\frac{C}{2}$ is a fixed quantity, namely π

Hence proceeding as in Example XIII 40, we see that the proposed product is greatest when

$$\frac{\pi}{2}-\frac{A}{2}=\frac{\pi}{2}-\frac{B}{2}=\frac{\pi}{2}-\frac{C}{2},$$

that is when
$$\frac{A}{2}=\frac{B}{2}=\frac{C}{2}=\frac{\pi}{6};$$

and then the product
$$=\left(\frac{\sqrt{3}}{2}\right)^3=\frac{3\sqrt{3}}{8}$$

145 We have $\cot\frac{B}{2}=\sqrt{\frac{s(s-b)}{(s-a)(s-c)}}=\frac{s-b}{s-a}\cot\frac{A}{2}$,

similarly
$$\cot\frac{C}{2}=\frac{s-c}{s-a}\cot\frac{A}{2}$$

Hence $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\left(1+\frac{s-b}{s-a}+\frac{s-c}{s-a}\right)\cot\frac{A}{2}$

$$=\frac{3s-a-b-c}{s-a}\cot\frac{A}{2}=\frac{s}{s-a}\cot\frac{A}{2}=\frac{a+b+c}{b+c-a}\cot\frac{A}{2}$$

146. Let A denote the angle between the diagonals, then $C=\frac{1}{2}h\lambda\sin A$, and by the solution of Example 136 the area of the circumscribed rectangle is $-h\lambda\sin\theta\cos(A+\theta)$ And since the rectangle is to be a square, we have by the solution of Example 136

$$h\sin\theta=-\lambda\cos(A+\theta),$$

therefore
$$h=-\lambda(\cos A\cot\theta-\sin A),$$

therefore
$$\cot\theta=\frac{\lambda\sin A-h}{\lambda\cos A},$$

therefore $\sin^2\theta=\frac{\lambda^2\cos^2 A}{(\lambda\sin A-h)^2+\lambda^2\cos^2 A}=\frac{\lambda^2-\lambda^2\sin^2 A}{\lambda^2-2\lambda h\sin A+h^2}=\frac{\lambda^2-\frac{4C^2}{h^2}}{h^2+\lambda^2-4C}$.

And the area of the circumscribing square $=h^2\sin^2\theta=\frac{h^2\lambda^2-4C^2}{h^2+\lambda^2-4C}$

147. We may put the proposed expression in the form

$$L\sin^2\theta+M\sin\theta\cos\theta+N\cos^2\theta,$$

where L, M, N involve the angles α, β, γ and also x, y, z. moreover x, y, z occur only in the first power Now if we put $M=0$ and $L=N=$ the given

constant, the expression is equal to the given constant whatever θ may be So we have only to determine x, y, and z from the three simple equations $M=0$, and $L=N=$the given constant

As soon as we have thus shewn that such values of x, y, z as we require must exist, we can determine the values more simply For let C denote the given constant, put a for θ, then

$$x \sin(\alpha-\beta)\sin(\alpha-\gamma)=C$$

This finds x. Similarly, by putting β for θ we find y, and by putting γ for θ we find z

148 Let AB denote the side of the regular pentagon, P the middle point of the arc subtended by the side adjacent to AB at B Then the angle APB is the angle subtended at the circumference of the circle by the side of a regular pentagon inscribed in the circle, so that the angle$=\frac{\pi}{5}$. Similarly the angle $PAB=\frac{\pi}{10}$, and therefore the angle $ABP=\frac{7\pi}{10}$

Let r denote the radius of the circle, so that

$$AB=2r\sin\frac{\pi}{5},\quad PB=2r\sin\frac{\pi}{10},\text{ and } PA=2r\sin\frac{7\pi}{10}=2r\sin\frac{3\pi}{10}$$

Hence
$$PA-PB=2r\left\{\frac{\sqrt{5}+1}{4}-\frac{\sqrt{5}-1}{4}\right\}=r,$$

$$PA\ \ PB=\frac{4r^2(\sqrt{5}+1)(\sqrt{5}-1)}{16}=r^2,$$

$$PA^2+PB^2=4r^2\left\{\left(\frac{\sqrt{5}+1}{4}\right)^2+\left(\frac{\sqrt{5}-1}{4}\right)^2\right\}=3r^2$$

149 Suppose the tower to subtend an angle ϕ at the eye of the observer, let x be the length of the flag-staff then

$$\frac{a}{b}=\tan\phi,\quad \frac{a+x}{b}=\tan(\phi+\theta)=\frac{\tan\phi+\tan\theta}{1-\tan\phi\tan\theta}=\frac{a+b\tan\theta}{b-a\tan\theta},$$

therefore
$$\frac{x}{b}=\frac{a+b\tan\theta}{b-a\tan\theta}-\frac{a}{b}=\frac{(b^2+a^2)\tan\theta}{b(b-a\tan\theta)},$$

then if θ be very small we may put θ for $\tan\theta$, and neglect $a\tan\theta$ in comparison with b, so that $x=\frac{b^2+a^2}{b}\theta$ nearly

150 We have by Art 219

$$(s-a)^2\sin A+(s-b)^2\sin B+(s-c)^2\sin C$$

$$=r\left\{(s-a)\sin A\cot\frac{A}{2}+(s-b)\sin B\cot\frac{B}{2}+(s-c)\sin C\cot\frac{C}{2}\right\}$$

$$=2r\left\{(s-a)\cos^2\frac{A}{2}+(s-b)\cos^2\frac{B}{2}+(s-c)\cos^2\frac{C}{2}\right\},$$

and by Examples 130 and 138, this

$$=2r\left\{\left(2+\frac{r}{2R}\right)s-\left(s+\frac{S}{R}\right)\right\}=2r\left(s+\frac{S}{2R}-\frac{S}{R}\right)$$

$$=2r\left(s-\frac{S}{2R}\right)=2r\left(\frac{S}{r}-\frac{S}{2R}\right)=\frac{S(2R-r)}{R}$$

And $$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{Ss}{abc}=\frac{s}{4R},$$

so that $$4r(2R-r)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=4r(2R-r)\frac{s}{4R}=\frac{S(2R-r)}{R}$$

Thus the proposed expressions are equal

151 $$2\sin 7A\cos A=\sin 8A+\sin 6A,$$

therefore $2\sin 7A\cos A+16\sin A\cos^3 A=\sin 6A+\sin 8A+16\sin A\cos^3 A$

$$=\sin 6A+2\sin 4A\cos 4A+8\sin 2A\cos^2 A$$

$$=\sin 6A+4\sin 2A\cos 2A\cos 4A+8\sin 2A\cos^2 A$$

$$=\sin 6A+4\sin 2A\,(2\cos^2 A+\cos 2A\cos 4A)$$

$$=\sin 6A+4\sin 2A\,\{1+\cos 2A\,(1+\cos 4A)\}$$

$$=\sin 6A+4\sin 2A\,(1+2\cos^3 2A)$$

152 Let x denote the logarithm of 32 to the base $\sqrt[3]{4}$, then $32=(\sqrt[3]{4})^x$, that is $2^5=4^{\frac{x}{3}}=2^{\frac{2x}{3}}$, therefore $5=\frac{2x}{3}$, therefore $x=\frac{15}{2}$

Let x denote the logarithm of $81\sqrt[3]{3}$ to the base $\sqrt[3]{9}$, then $81\sqrt[3]{3}=(\sqrt[3]{9})^x$, that is $3^{4+\frac{1}{3}}=9^{\frac{x}{3}}=3^{\frac{2x}{3}}$, therefore $\frac{2x}{3}=4\frac{1}{3}=\frac{13}{3}$, therefore $x=\frac{13}{2}$

153. Here $$\frac{\sin(A+B)}{\cos(A+B)}=\frac{3\sin A}{\cos A},$$

therefore $\sin(A+B)\cos A-\cos(A+B)\sin A=2\sin A\cos(A+B)$;

therefore $$\sin(A+B-A)=2\sin A\cos(A+B),$$

that is $$\sin B=2\sin A\cos(A+B)=\sin(2A+B)-\sin B;$$

therefore $$2\sin B=\sin(2A+B),$$

therefore $$2\sin B\cos B=\sin(2A+B)\cos B,$$

therefore $$\sin 2B=\frac{1}{2}\{\sin(2A+2B)+\sin 2A\};$$

therefore $$2\sin 2B=\sin(2A+2B)+\sin 2A$$

154 $\frac{1}{a}\sin^2\frac{A}{2}+\frac{1}{b}\sin^2\frac{B}{2}+\frac{1}{c}\sin^2\frac{C}{2}$

$$=\frac{1}{abc}\{(s-b)(s-c)+(s-a)(s-c)+(s-a)(s-b)\}$$

$$=\frac{1}{abc}\{3s^2-2s(a+b+c)+ab+bc+ca\}$$

$$=\frac{1}{abc}\{ab+bc+ca-s^2\}$$

$$=\frac{1}{4abc}\{4ab+4bc+4ca-(a+b+c)^2\}$$

$$=\frac{1}{4abc}\{2ab+2bc+2ca-a^2-b^2-c^2\}$$

155 Let $ABCD$ denote the quadrilateral figure Let P, Q, R, S be taken in AB, BC, CD, DA respectively, such that

$$\frac{AP}{PB}=\frac{BQ}{QC}=\frac{CR}{RD}=\frac{DS}{SA}=\frac{m}{n}.$$

Then $$\frac{PB}{AB}=\frac{n}{m+n},\quad \frac{BQ}{BC}=\frac{m}{m+n},$$

and the area of the triangle $PBQ=\frac{1}{2}BP\ BQ\sin B=\frac{mn}{2(m+n)^2}AB\ BC\sin B$

$$=\frac{mn}{(m+n)^2}\text{ area of the triangle }ABC$$

Similarly the area of the triangle $RDS=\frac{mn}{(m+n)^2}$ area of the triangle ADC

Therefore the area of the triangles PBQ and $RDS=\frac{mnH}{(m+n)^2}$, where H denotes the area of the quadrilateral figure $ABCD$

In the same way we shew that the area of the triangles QCR and SAP $=\frac{mnH}{(m+n)^2}$

Thus the area of the four triangles PBQ, QCR, RDS, and $SAP=\frac{2mnH}{(m+n)^2}$ Therefore the area of the quadrilateral figure $PQRS$

$$=H\left\{1-\frac{2mn}{(m+n)^2}\right\}=\frac{H(m^2+n^2)}{(m+n)^2}$$

156 $\cos\theta+\cos 3\theta=\frac{1}{2}$, therefore $\cos\theta+4\cos^3\theta-3\cos\theta=\frac{1}{2}$,

therefore $4\cos^3\theta-2\cos\theta-\frac{1}{2}=0$, therefore $4\left(\cos^3\theta+\frac{1}{8}\right)-2\left(\cos\theta+\frac{1}{2}\right)=0$,

therefore $\quad 2\left(\cos\theta+\frac{1}{2}\right)\left(\cos^2\theta-\frac{1}{2}\cos\theta+\frac{1}{4}\right)-\left(\cos\theta+\frac{1}{2}\right)=0,$

therefore $\quad \left(\cos\theta+\frac{1}{2}\right)\left(2\cos^2\theta-\cos\theta-\frac{1}{2}\right)=0$

Thus either $\quad \cos\theta+\frac{1}{2}=0$ or $2\cos^2\theta-\cos\theta-\frac{1}{2}=0,$

the former gives $\cos\theta=-\frac{1}{2}$, the latter gives $\cos\theta=\frac{1\pm\sqrt{5}}{4}$

157 $\quad \cos\beta\cos\gamma\sin(\gamma-\beta)=\frac{1}{2}\{\cos(\beta-\gamma)+\cos(\beta+\gamma)\}\sin(\gamma-\beta)$

$$=\frac{1}{4}\{\sin(2\gamma-2\beta)+\sin 2\gamma-\sin 2\beta\}$$

Transform the other two terms in the same way, and thus we obtain finally as the sum

$$\frac{1}{4}\{\sin(2\gamma-2\beta)+\sin(2\alpha-2\gamma)+\sin(2\beta-2\alpha)\}$$

Again, $\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)$

$$=\frac{1}{2}\{\cos(\alpha+\gamma-2\beta)-\cos(\alpha-\gamma)\}\sin(\gamma-\alpha)$$

$$=\frac{1}{4}\{\sin(2\gamma-2\beta)+\sin(2\beta-2\alpha)+\sin(2\alpha-2\gamma)\}$$

Thus the proposed expressions are equal.

Or thus from Example VIII 12 we see that

$$\sin\beta\sin\gamma\sin(\gamma-\beta)+\sin\gamma\sin\alpha\sin(\alpha-\gamma)+\sin\alpha\sin\beta\sin(\beta-\alpha)$$
$$=\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)$$

In this formula change α, β, γ into $\frac{\pi}{2}+\alpha$, $\frac{\pi}{2}+\beta$, $\frac{\pi}{2}+\gamma$ respectively; and thus we obtain the required result

158 $\quad \sin A\sin(A-B)\sin(A-C)=\frac{1}{2}\sin A\{\cos(C-B)-\cos(2A-B-C)\}$

$$=\frac{1}{4}\{\sin(A+C-B)+\sin(A+B-C)-\sin(3A-B-C)-\sin(B+C-A)\}$$

$$=\frac{1}{4}\{\sin 2B+\sin 2C+\sin 4A-\sin 2A\}.$$

In this way we see that the expression on the left-hand side in the proposed formula

$$=\frac{1}{4}\{\sin 2A+\sin 2B+\sin 2C+\sin 4A+\sin 4B+\sin 4C\}$$

Then by Example VIII 33 we have

$$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C,$$

$$\sin 4A+\sin 4B+\sin 4C=-4\sin 2A\sin 2B\sin 2C$$

$$=-32\sin A\sin B\sin C\cos A\cos B\cos C.$$

Thus we obtain the required result.

159 Let A denote the bottom of the pole, B the point on the pole to which the man climbs, F the top of the window, E the bottom. Let AF and BE intersect at D, which is therefore the top of the wall Draw DC perpendicular to the ground, and produce FE to meet the ground at H Draw from B a horizontal straight line meeting FH at G

Then from the triangle BAF we get $BF=\frac{c\cos\alpha}{\sin(\alpha-\beta)}$,

$$BG=BF\cos\beta=\frac{c\cos\alpha\cos\beta}{\sin(\alpha-\beta)}=\frac{c}{\tan\alpha-\tan\beta},$$

$$EG=BG\tan\gamma=\frac{c\tan\gamma}{\tan\alpha-\tan\beta},$$

$$EH=c+EG=\frac{c(\tan\alpha-\tan\beta+\tan\gamma)}{\tan\alpha-\tan\beta}.$$

160. From the triangle CEB we have $\frac{CE}{a}=\frac{\sin\frac{1}{2}B}{\sin\left(C+\frac{1}{2}B\right)}$;

and from the triangle CDA we have $\frac{CD}{b}=\frac{\sin\frac{1}{2}A}{\sin\left(C+\frac{1}{2}A\right)}$.

Thus the area of the triangle $CED=\frac{1}{2}CE\ CD\sin C$

$$=\frac{ab\sin C\sin\frac{1}{2}A\sin\frac{1}{2}B}{2\sin\left(C+\frac{1}{2}B\right)\sin\left(C+\frac{1}{2}A\right)}=\frac{S\sin\frac{1}{2}A\sin\frac{1}{2}B}{\cos\frac{C-A}{2}\cos\frac{C-B}{2}}.$$

161. We have $p=2\cos A+\cos^3 A\,(-5+4\cos^2 A)$

$$=2\cos A+\frac{1}{4}(\cos 3A+3\cos A)(-5+4\cos^2 A)$$

$$=2\cos A+\frac{1}{4}(\cos 3A+3\cos A)(-3+2\cos 2A)$$

$$=2\cos A-\frac{3}{4}\cos 3A-\frac{9}{4}\cos A+\frac{1}{2}\cos 3A\cos 2A+\frac{3}{2}\cos A\cos 2A$$

$$=-\frac{3}{4}\cos 3A-\frac{1}{4}\cos A+\frac{1}{4}(\cos 5A+\cos A)+\frac{3}{4}(\cos 3A+\cos A)$$

$$=\frac{1}{4}(\cos 5A+3\cos A)$$

In the same way we find that

$$q=\frac{1}{4}(\sin 5A+3\sin A)$$

Therefore

$$p\cos 3A+q\sin 3A=\frac{1}{4}(\cos 5A+3\cos A)\cos 3A+\frac{1}{4}(\sin 5A+3\sin A)\sin 3A$$

$$=\frac{1}{4}(\cos 5A\cos 3A+\sin 5A\sin 3A)+\frac{3}{4}(\cos 3A\cos A+\sin 3A\sin A)$$

$$=\frac{1}{4}\cos(5A-3A)+\frac{3}{4}\cos(3A-A)=\cos 2A.$$

And

$$p\sin 3A-q\cos 3A=\frac{1}{4}(\cos 5A+3\cos A)\sin 3A-\frac{1}{4}(\sin 5A+3\sin A)\cos 3A$$

$$=\frac{1}{4}(\cos 5A\sin 3A-\sin 5A\cos 3A)+\frac{3}{4}(\sin 3A\cos A-\cos 3A\sin A)$$

$$=-\frac{1}{4}\sin(5A-3A)+\frac{3}{4}\sin(3A-A)=\frac{1}{2}\sin 2A$$

162 Let $u=\left(\cos\frac{a}{n}\right)^{\cot^2\frac{\beta}{n}}$, therefore

$$\log u=\cot^2\frac{\beta}{n}\log\cos\frac{a}{n}=\cos^2\frac{\beta}{n}\times\operatorname{cosec}^2\frac{\beta}{n}\log\cos\frac{a}{n}$$

Now as in the solution of Example XII 33 we can shew that

$$\operatorname{cosec}^2\frac{\beta}{n}\log\cos\frac{a}{n}=-\frac{a^2}{2\beta^2}\text{ when } n \text{ is infinite}$$

And $\cos^2\frac{\beta}{n}=1$ when n is infinite

Thus $\log u=-\frac{a^2}{2\beta^2}$, and therefore $u=e^{-\frac{a^2}{2\beta^2}}$.

163 If n be a positive integer, we have $1+2+2^2+\ldots+2^n=2^{n+1}-1$. Hence the infinite series

$$=2-1+\frac{2^2-1}{\lfloor 2}+\frac{2^3-1}{\lfloor 3}+\frac{2^4-1}{\lfloor 4}+\ldots$$

$$=2+\frac{2^2}{\lfloor 2}+\frac{2^3}{\lfloor 3}+\frac{2^4}{\lfloor 4}+\ldots-\left\{1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\ldots\right\}$$

$$=e^2-1-\{e-1\}=e^2-e$$

164 Here $$\sec\alpha\cos(x+y)=\frac{\cos(x-y)}{\cos x\cos y},$$

and $$\sec\beta\cos(x-y)=\frac{\cos(x+y)}{\cos x\cos y},$$

therefore, by division, $\dfrac{\cos\beta}{\cos\alpha}\quad\dfrac{\cos(x+y)}{\cos(x-y)}=\dfrac{\cos(x-y)}{\cos(x+y)}$,

so that $$\frac{\cos(x-y)}{\cos(x+y)}=\sqrt{\frac{\cos\beta}{\cos\alpha}} \qquad (1).$$

And $\cos(x-y)+\cos(x+y)=2\cos x\cos y=2\cos\alpha\dfrac{\cos(x-y)}{\cos(x+y)}$

$$=2\sqrt{\cos\alpha\cos\beta} \qquad (2).$$

From (1) and (2) we have

$$\cos(x+y)\left\{\sqrt{\frac{\cos\beta}{\cos\alpha}}+1\right\}=2\sqrt{\cos\alpha\cos\beta};$$

therefore $$\cos(x+y)=\frac{2\cos\alpha\sqrt{\cos\beta}}{\sqrt{\cos\alpha}+\sqrt{\cos\beta}}.$$

Then by (1) we have $\cos(x-y)=\dfrac{2\cos\beta\sqrt{\cos\alpha}}{\sqrt{\cos\alpha}+\sqrt{\cos\beta}}$

165. It may be shewn as in the solution of Example xx. 4 that

$$\frac{a\cos\frac{1}{2}(B-C)}{bc\cos\frac{1}{2}(B+C)}=\frac{b+c}{bc}=\frac{a(b+c)}{abc},$$

similarly $\dfrac{b\cos\frac{1}{2}(C-A)}{ca\cos\frac{1}{2}(C+A)}=\dfrac{b(c+a)}{abc}$, and $\dfrac{c\cos\frac{1}{2}(A-B)}{ab\cos\frac{1}{2}(A+B)}=\dfrac{c(a+b)}{abc}$.

Hence by addition we obtain the required result

166 $$AP = OA\cos OAP,\quad AS = OA\cos OAS,$$

therefore the area of the triangle $APS = \frac{1}{2}OA^2\cos OAP\cos OAS\sin A.$

In the same way the area of the triangle OPS

$$=\frac{1}{2}OA^2\sin OAP\sin OAS\sin POS = \frac{1}{2}OA^2\sin OAP\sin OAS\sin(180^\circ - A)$$

$$=\frac{1}{2}OA^2\sin OAP\sin OAS\sin A$$

Hence triangle APS − triangle OPS

$$=\frac{1}{2}OA^2\sin A\{\cos OAP\cos OAS - \sin OAP\sin OAS\}$$

$$=\frac{1}{2}OA^2\sin A\cos(OAP + OAS) = \frac{1}{2}OA^2\sin A\cos A = \frac{1}{4}OA^2\sin 2A$$

In the same way we obtain

$$\text{triangle } BQP - \text{triangle } OQP = \frac{1}{4}OB^2\sin 2B,$$

$$\text{triangle } CRQ - \text{triangle } ORQ = \frac{1}{4}OC^2\sin 2C,$$

and $$\text{triangle } DSR - \text{triangle } OSR = \frac{1}{4}OD^2\sin 2D$$

Hence by addition we have

$$\text{triangle } APS + \text{triangle } BQP + \text{triangle } CRQ + \text{triangle } DSR - \text{quadrilateral } PQRS$$

$$=\frac{1}{4}\{OA^2\sin 2A + OB^2\sin 2B + OC^2\sin 2C + OD^2\sin 2D\}$$

But the sum of the four triangles and the quadrilateral

$$=\text{the quadrilateral } ABCD$$

Hence by subtraction we have

$$\text{twice the quadrilateral } PQRS = \text{the quadrilateral } ABCD$$

$$-\frac{1}{4}\{OA^2\sin 2A + OB^2\sin 2B + OC^2\sin 2C + OD^2\sin 2D\}$$

167 We have $a = 2R\sin A,\ b = 2R\sin B,\ c = 2R\sin C,$
thus the proposed expression

$$=4R\left\{\sin\frac{B-C}{2}\cos\frac{A}{2} + \sin\frac{C-A}{2}\cos\frac{B}{2} + \sin\frac{A-B}{2}\cos\frac{C}{2}\right\}$$

$$=4R\left\{\sin\frac{B-C}{2}\sin\frac{B+C}{2}+\sin\frac{C-A}{2}\sin\frac{C+A}{2}+\sin\frac{A-B}{2}\sin\frac{A+B}{2}\right\}$$

$$=4R\left\{\sin^2\frac{B}{2}-\sin^2\frac{C}{2}+\sin^2\frac{C}{2}-\sin^2\frac{A}{2}+\sin^2\frac{A}{2}-\sin^2\frac{B}{2}\right\}$$

$$=0$$

168 Assume $x=\tan A$, $y=\tan B$, $z=\tan C$, then since $x+y+z=xyz$ it will follow in the manner of Art 114 that $\tan(A+B+C)$ is zero, therefore $A+B+C=n\pi$ where n is zero or some integer Therefore $3A+3B+3C=3n\pi$, and therefore in the manner of Art. 114 we have

$$\tan 3A+\tan 3B+\tan 3C=\tan 3A\tan 3B\tan 3C.$$

But $$\tan 3A=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}=\frac{3x-x^3}{1-3x^2},$$

similarly $$\tan 3B=\frac{3y-y^3}{1-3y^2},\quad \tan 3C=\frac{3z-z^3}{1-3z^2}:$$

thus the required result follows

169 We have $l=R\cos A$, $m=R\cos B$, $n=R\cos C$, thus we have to shew that

$$\frac{4a}{R\cos A}+\frac{4b}{R\cos B}+\frac{4c}{R\cos C}=\frac{abc}{R^3\cos A\cos B\cos C}$$

Now $a=2R\sin A$, $b=2R\sin B$, $c=2R\sin C$, thus the proposed identity becomes

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C,$$

and this is true by Art. 114

170 $$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=\frac{3}{2}-\frac{1}{2}(\cos A+\cos B+\cos C)$$

$$=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},\text{ by Art 114}$$

Now we have seen in Example XIII 40 that $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ cannot be greater than $\frac{1}{8}$, hence $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}$ cannot be less than $1-\frac{1}{4}$, that is than $\frac{3}{4}$

171 $\dfrac{\sin a\theta}{\sin b\theta}=\dfrac{a}{b}\dfrac{\sin a\theta}{a\theta}\cdot\dfrac{b\theta}{\sin b\theta}$, and when θ is indefinitely diminished the limit of $\dfrac{\sin a\theta}{a\theta}$ is unity, and so also is the limit of $\dfrac{b\theta}{\sin b\theta}$ thus the limit of $\dfrac{\sin a\theta}{\sin b\theta}$ is $\dfrac{a}{b}$

Also $$\frac{\text{vers}\, a\theta}{\text{vers}\, b\theta}=\frac{1-\cos a\theta}{1-\cos b\theta}=\frac{\sin^2\frac{a\theta}{2}}{\sin^2\frac{b\theta}{2}}=\left\{\frac{\sin\frac{a\theta}{2}}{\sin\frac{b\theta}{2}}\right\}^2.$$

Now the limit of $\dfrac{\sin\frac{a\theta}{2}}{\sin\frac{b\theta}{2}}$ is $\dfrac{a}{b}$ in the manner just shewn, therefore the limit of $\dfrac{\text{vers}\, a\theta}{\text{vers}\, b\theta}$ is $\dfrac{a^2}{b^2}$

172 $$\frac{1}{4}\left\{\frac{1}{1.1}+\frac{1}{2.3}+\frac{1}{3.5}+\frac{1}{4.7}+\dots\right\}$$
$$=\frac{1}{2}\left\{\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\frac{1}{7.8}+\dots\right\}$$
$$=\frac{1}{2}\log 2,\ \text{by Art } 146,\ =\log\sqrt{2}$$

173. $$\tan\frac{A}{2}+\cos\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}+\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}$$
$$=\frac{\cos^2\frac{A}{2}+\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}.$$

The numerator of this fraction $=1-\sin^2\frac{A}{2}+\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$$=1+\sin\frac{A}{2}\left\{\cos\frac{B}{2}\cos\frac{C}{2}-\cos\frac{B+C}{2}\right\}=1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Thus the fraction $=\sec\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}+\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$

Similarly the other two proposed expressions may be reduced to this symmetrical form, and thus the three expressions are equal.

174 $$\sin(\pi\cot\theta)=\cos(\pi\tan\theta),$$

therefore $$\cos(\pi\tan\theta)=\cos\left(\frac{\pi}{2}-\pi\cot\theta\right),$$

therefore all the solutions are comprised in

$$\pi\tan\theta=2n\pi\pm\left(\frac{\pi}{2}-\pi\cot\theta\right),$$

where n is zero, or some integer, positive or negative

Take the upper sign; thus $2n+\frac{1}{2}=\tan\theta+\cot\theta=\frac{1}{\sin\theta\cos\theta}$, so that $n+\frac{1}{4}=\frac{1}{\sin 2\theta}$.

Take the lower sign; thus $\frac{1}{2}-2n=\cot\theta-\tan\theta=2\cot 2\theta$, so that $\cot 2\theta=\frac{1}{4}-n$

Thus either cosec 2θ or $\cot 2\theta$ takes the prescribed form.

175 We have $a=2R\sin A$, $b=2R\sin B$, $c=2R\sin C$

Thus the left-hand member $=\left(\frac{1}{2R}\right)^2$.

And the right-hand member $=\frac{\sin 2A+\sin 2B+\sin 2C}{16R^2\sin A\sin B\sin C}=\frac{1}{4R^2}$, by Art. 114

176 Let θ be the angle of the sector, then we see from a diagram that $\frac{r}{a-r}=\sin\frac{\theta}{2}$ But $2c=2a\sin\frac{\theta}{2}$ Therefore $\frac{r}{a-r}=\frac{c}{a}$, therefore $\frac{a-r}{r}=\frac{a}{c}$, therefore $\frac{1}{r}=\frac{1}{a}+\frac{1}{c}$

177 $$\frac{\sin x}{\cos x}+\frac{\sin 4x}{\cos 4x}+\frac{\sin 2x}{\cos 2x}+\frac{\sin 3x}{\cos 3x}=0,$$

therefore $$\frac{\sin x\cos 4x+\cos x\sin 4x}{\cos x\cos 4x}+\frac{\sin 2x\cos 3x+\cos 2x\sin 3x}{\cos 2x\cos 3x}=0;$$

therefore $$\frac{\sin 5x}{\cos x\cos 4x}+\frac{\sin 5x}{\cos 2x\cos 3x}=0,$$

therefore either $\sin 5x=0$ or $\frac{1}{\cos x\cos 4x}+\frac{1}{\cos 2x\cos 3x}=0$

If we take the former, then $5x=n\pi$

If we take the latter, then $\cos 2x\cos 3x+\cos x\cos 4x=0$,

therefore $$\cos 2x(4\cos^3 x-3\cos x)+\cos x\cos 4x=0,$$

therefore either $\cos x=0$ or $\cos 2x(4\cos^2 x-3)+\cos 4x=0$

If we take the former, then $x=(2m+1)\frac{\pi}{2}$

If we take the latter, then $\cos 2x(2+2\cos 2x-3)+2\cos^2 2x-1=0$;

therefore $$4\cos^2 2x-\cos 2x-1=0,$$

and by solving this quadratic we obtain $\cos 2x=\frac{1\pm\sqrt{17}}{8}$.

178 We may proceed as in the solution of Example 147, and seek for the values of x, y, and z, which make

$$x\sin(\theta-\beta)\sin(\theta-\gamma)+y\sin(\theta-\gamma)\sin(\theta-\alpha)+z\sin(\theta-\alpha)\sin(\theta-\beta)$$

always equal to 1 Then we shall find that $x=\dfrac{1}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}$, and so on

Or we may verify the formula by direct work. For reduce the three fractions to the common denominator $\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)$ Then the numerator will become $L\sin^2\theta+M\sin\theta\cos\theta+N\cos^2\theta$, where

$$L=\cos\beta\cos\gamma\sin(\gamma-\beta)+\cos\gamma\cos\alpha\sin(\alpha-\gamma)+\cos\alpha\cos\beta\sin(\beta-\alpha),$$

$$M=-\sin(\gamma+\beta)\sin(\gamma-\beta)-\sin(\alpha+\gamma)\sin(\alpha-\gamma)-\sin(\beta+\alpha)\sin(\beta-\alpha),$$

$$N=\sin\beta\sin\gamma\sin(\gamma-\beta)+\sin\gamma\sin\alpha\sin(\alpha-\gamma)+\sin\alpha\sin\beta\sin(\beta-\alpha)$$

It is obvious by Art 83 that $M=0$, and we have seen in the solution of Example 157 that L and N are each equal to the common denominator, so that $L\sin^2\theta+N\cos^2\theta$ is also equal to this denominator, and the expression is equal to unity.

179. By Euclid VI 2 we find that $BD=\dfrac{ac}{b+c}$, and $CD=\dfrac{ab}{b+c}$

Similar expressions hold for the segments of the other sides of ABC

Therefore the area of the triangle DCE

$$=\frac{1}{2}\frac{ab}{b+c}\;\frac{ab}{a+c}\sin C=\frac{Sab}{(a+c)(b+c)}.$$

Similar expressions hold for the areas of EFA and FDB.

Therefore the area of DEF

$$=S\left\{1-\frac{ab}{(a+c)(b+c)}-\frac{bc}{(b+a)(c+a)}-\frac{ca}{(c+b)(a+b)}\right\}$$

$$=\frac{S}{(a+b)(b+c)(c+a)}\{(a+b)(b+c)(c+a)-ab(a+b)-bc(b+c)-ca(c+a)\}$$

$$=\frac{2abcS}{(a+b)(b+c)(c+a)}=2S\cdot\frac{a}{b+c}\;\frac{b}{c+a}\;\frac{c}{a+b}$$

Now $$\frac{a}{b+c}=\frac{\sin A}{\sin B+\sin C}=\frac{\sin\frac{A}{2}\cos\frac{A}{2}}{\sin\frac{B+C}{2}\cos\frac{B-C}{2}}=\frac{\sin\frac{A}{2}}{\cos\frac{B-C}{2}}.$$

Similarly $$\frac{b}{c+a}=\frac{\sin\frac{B}{2}}{\cos\frac{C-A}{2}},\text{ and }\frac{c}{a+b}=\frac{\sin\frac{C}{2}}{\cos\frac{A-B}{2}}$$

Thus the required result is obtained.

180 Let x denote the height of the mountain, then the distances of the two stations from the point in the horizontal plane which is vertically under the top of the mountain are $x\cot\alpha$ and $x\cot\beta$ respectively.

Thus $\quad c^2=x^2\cot^2\alpha+x^2\cot^2\beta-2x^2\cot\alpha\cot\beta\cos\gamma$, (Art 215)

therefore
$$x^2=\frac{c^2}{\cot^2\alpha+\cot^2\beta-2\cot\alpha\cot\beta\cos\gamma}$$
$$=\frac{c^2\sin^2\alpha\sin^2\beta}{\sin^2\beta\cos^2\alpha+\sin^2\alpha\cos^2\beta-2\sin\alpha\cos\alpha\sin\beta\cos\beta\cos\gamma}.$$

The denominator of this fraction may be put in the form
$$(\sin\beta\cos\alpha+\cos\beta\sin\alpha)^2-\sin 2\alpha\sin 2\beta\cos^2\frac{\gamma}{2},$$
so that with the specified value of ϕ it becomes $\sin^2(\alpha+\beta)\cos^2\phi$,

and therefore
$$x=\frac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)\cos\phi}.$$

181 Let θ denote the angle, then $\tan\frac{\theta}{2}=\frac{1}{3\times 450}$, therefore approximately $\frac{\theta}{2}=\frac{1}{1350}$, therefore $\theta=\frac{1}{675}$ Hence the number of degrees in the angle is $\frac{180}{\pi}\times\frac{1}{675}$, and the number of minutes is $\frac{180}{\pi}\times\frac{60}{675}$, that is $\frac{4}{45}\times\frac{180}{\pi}$, that is $\frac{4}{45}\times 57\ 29$, that is about 5.

182 The general term of the series is $\frac{n^2}{\lfloor n+1}$; for we obtain all the terms by putting successively 1, 2, 3, for n in this expression.

Now
$$\frac{n^2}{\lfloor n+1}=\frac{n(n+1)-(n+1)+1}{\lfloor n+1}=\frac{1}{\lfloor n-1}-\frac{1}{\lfloor n}+\frac{1}{\lfloor n+1}.$$

If then we split up each term into three in this manner, beginning with the second term, we obtain
$$\frac{1}{\lfloor 2}+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+$$
$$-\left\{\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\frac{1}{\lfloor 5}+\qquad\right\}$$
$$+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\frac{1}{\lfloor 5}+\dots\quad,$$
that is $\frac{1}{2}+e-1-(e-2)+e-2-\frac{1}{2}$, that is $e-1$.

183 Here

$$\frac{2}{\cos\theta}=\frac{1}{\cos(\theta+2\alpha)}+\frac{1}{\cos(\theta-2\alpha)}=\frac{2\cos\theta\cos 2\alpha}{\cos(\theta+2\alpha)\cos(\theta-2\alpha)}=\frac{2\cos\theta\cos 2\alpha}{\cos^2\theta-\sin^2 2\alpha},$$

therefore $$\cos^2\theta-\sin^2 2\alpha=\cos^2\theta\cos 2\alpha,$$

therefore $$\cos^2\theta(1-\cos 2\alpha)=\sin^2 2\alpha=4\sin^2\alpha\cos^2\alpha,$$

therefore $$\cos^2\theta=2\cos^2\alpha$$

184 Here $4\sin(\theta+\phi)\cos(\theta-\phi)=1$, and $2\sin(\theta+\phi)=1$,

therefore $$\sin(\theta+\phi)=\frac{1}{2},\quad\text{and }\cos(\theta-\phi)=\frac{1}{2};$$

therefore $$\theta+\phi=n\pi+(-1)^n\frac{\pi}{6},\quad\text{and }\theta-\phi=2m\pi\pm\frac{\pi}{3}$$

185 $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$, by Example VIII. 16.

And $$\tan\frac{A}{2}+\tan\frac{B}{2}=\frac{\sin\frac{A+B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}=\frac{\cos\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}},$$

therefore $$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{\cos\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}$$

$$=\frac{\cos^2\frac{C}{2}+\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}},$$

the numerator $=1-\sin^2\frac{C}{2}+\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}$

$$=1+\sin\frac{C}{2}\left\{\cos\frac{A}{2}\cos\frac{B}{2}-\cos\frac{A+B}{2}\right\}=1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

and thus the fraction $=\dfrac{1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}.$

Hence by multiplication we obtain the required result.

186 Proceed as in the solution of Example 166 Then we obtain the following expression for the excess of the sum of all the triangles at the corners above the second polygon

$$\frac{r^2}{4}\{\sin 2A+\sin 2B+\sin 2C+\sin 2D+\ \ \},$$

where r is the radius of the circle

Hence this vanishes if $\sin 2A+\sin 2B+\ \ =0$, and then the sum of the triangles at the corners is equal to the second polygon, and therefore the first polygon is double the second

187 We have $a=2R\sin A$, $b=2R\sin B$, $c=2R\sin C$, hence the proposed expression

$$=4R\left\{\sin\frac{B-C}{2}\sin\frac{A}{2}+\sin\frac{C-A}{2}\sin\frac{B}{2}+\sin\frac{A-B}{2}\sin\frac{C}{2}\right\}$$

$$=2R\left\{\cos\frac{A+C-B}{2}-\cos\frac{A+B-C}{2}+\cos\frac{B+A-C}{2}-\cos\frac{B+C-A}{2}\right.$$

$$\left.+\cos\frac{B+C-A}{2}-\cos\frac{A+C-B}{2}\right\}=0$$

188 We have $a=2R\sin A$, $b=2R\sin B$, $c=2R\sin C$, hence the proposed expression

$$=4R^2\left\{\frac{\sin^2 A\sin(B-C)}{\sin B+\sin C}+\frac{\sin^2 B\sin(C-A)}{\sin C+\sin A}+\frac{\sin^2 C\sin(A-B)}{\sin A+\sin B}\right\}$$

Now $$\frac{\sin^2 A\sin(B-C)}{\sin B+\sin C}=\frac{\sin A\sin(B+C)\sin(B-C)}{\sin B+\sin C}$$

$$=\frac{\sin A(\sin^2 B-\sin^2 C)}{\sin B+\sin C}=\sin A(\sin B-\sin C).$$

In this way the proposed expression

$$=4R^2\{\sin A(\sin B-\sin C)+\sin B(\sin C-\sin A)+\sin C(\sin A-\sin B)\}=0.$$

189 If n be the number of sides in the first polygon we have

$$a=2r\sin\frac{\pi}{n},\quad b=2r\sin\frac{\pi}{2n}.$$

By Art 100, since $\frac{\pi}{n}$ lies between 0 and $\frac{\pi}{2}$, we have

$$2\sin\frac{\pi}{2n}=\sqrt{\left(1+\sin\frac{\pi}{n}\right)}-\sqrt{\left(1-\sin\frac{\pi}{n}\right)};$$

therefore $$\frac{b}{r}=\sqrt{\left(1+\frac{a}{2r}\right)}-\sqrt{\left(1-\frac{a}{2r}\right)}.$$

Multiply by r, and we obtain the required result.

190 $\left(1-\sin\frac{B}{2}\right)\left(1-\sin\frac{C}{2}\right)\cos\frac{A}{2}$

$$=\cos\frac{A}{2}-\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\cos\frac{A}{2}+\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}$$

Develope each of the other two terms in the same way, the aggregate

$$=\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}-\sin\frac{A+B}{2}-\sin\frac{B+C}{2}-\sin\frac{C+A}{2}$$

$$+\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}+\sin\frac{C}{2}\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.$$

But $\cos\frac{A}{2}=\sin\frac{B+C}{2},\ \cos\frac{B}{2}=\sin\frac{C+A}{2},\ \cos\frac{C}{2}=\sin\frac{A+B}{2},$

thus the expression

$$=\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}+\sin\frac{C}{2}\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$$

$$=\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}+\sin\frac{A}{2}\left(\sin\frac{C}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}\right)$$

$$=\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}+\sin\frac{A}{2}\sin\frac{B+C}{2}$$

$$=\cos\frac{A}{2}\left\{\sin\frac{A}{2}+\sin\frac{B}{2}\sin\frac{C}{2}\right\}=\cos\frac{A}{2}\left\{\cos\frac{B+C}{2}+\sin\frac{B}{2}\sin\frac{C}{2}\right\}$$

$$=\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Or instead of the last four lines we may use Art 118, observing that here $\cos\left(\frac{A}{2}+\frac{B}{2}+\frac{C}{2}\right)=0$

191 Let D denote the top of the object

From the triangle ABD we have $\frac{AB}{BD}=1$, for the angles BAD and BDA are equal From the triangle BDC we have $\frac{BD}{BC}=\frac{\sin 3\alpha}{\sin\alpha}$

Therefore $$\frac{AB}{BC}=\frac{\sin 3\alpha}{\sin\alpha}=3-4\sin^2\alpha$$

Since the object is very distant α is very small; therefore $AB=3BC$ nearly.

192 $$\frac{1}{x}+\frac{1}{\log_e(1-x)}=\frac{1}{x}-\frac{1}{x+\frac{1}{2}x^2+\frac{1}{3}x^3+}=\frac{\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+}{x^2+\frac{1}{2}x^3+\frac{1}{3}x^4+}.$$

Here every term in the numerator is less than the corresponding term in the denominator, and thus the fraction is less than unity

193 Here $$\frac{6\sin B}{\cos(A+B)} = \frac{6\sin B\cos B}{\cos(A+2B)};$$

thus either $\sin B = 0$ or $\cos(A+2B) = \cos(A+B)\cos B$,

the latter gives $\cos(A+B)\cos B - \sin(A+B)\sin B = \cos(A+B)\cos B$,

so that either $\sin B = 0$ or $\sin(A+B) = 0$

Suppose that $\sin(A+B)=0$, then since

$$\frac{3\sin 2B}{\cos(A+B+B)} = \frac{2\sin 3B}{\cos(A+B+2B)},$$

we have $$\frac{3\sin 2B}{\cos(A+B)\cos B} = \frac{2\sin 3B}{\cos(A+B)\cos 2B},$$

so that $$\frac{3\sin B}{\cos(A+B)} = \frac{\sin 3B}{\cos(A+B)\cos 2B},$$

therefore $$3\sin B\cos 2B = \sin 3B,$$

therefore $$3\sin B(1-2\sin^2 B) = 3\sin B - 4\sin^3 B,$$

therefore $$6\sin^3 B = 4\sin^3 B,$$

therefore $$\sin B = 0$$

194 Here $\tan\theta = \frac{x}{y}$, therefore $\sin^2\theta = \frac{x^2}{x^2+y^2}$, and $\cos^2\theta = \frac{y^2}{x^2+y^2}$. Substitute in the second given equation; thus

$$\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)\frac{1}{x^2+y^2} = \frac{6}{x^2+y^2}, \text{ therefore } \frac{x^2}{y^2}+\frac{y^2}{x^2} = 6$$

From this quadratic in $\frac{x^2}{y^2}$ we find $\frac{x^2}{y^2} = 3\pm 2\sqrt{2} = (\sqrt{2}\pm 1)^2$, therefore $\tan\theta = \pm(\sqrt{2}+1)$ or $\pm(\sqrt{2}-1)$ The former gives $\theta = n\pi \pm \frac{3\pi}{8}$, and the latter gives $\theta = n\pi \pm \frac{\pi}{8}$

195 Since the sines of the angles are in Harmonical Progression, so are the sides of the opposite angles Thus a, b, c are in Harmonical Progression, and we have to shew that $\frac{(s-b)(s-c)}{bc}$, $\frac{(s-c)(s-a)}{ca}$, $\frac{(s-a)(s-b)}{ab}$ are so also Multiply each term by $\frac{abc}{(s-a)(s-b)(s-c)}$, thus we see it is sufficient to shew that $\frac{a}{s-a}$, $\frac{b}{s-b}$, $\frac{c}{s-c}$ are in Harmonical Progression, or that $\frac{s-a}{a}$, $\frac{s-b}{b}$, $\frac{s-c}{c}$ are in Arithmetical Progression, or that $\frac{s}{a}$, $\frac{s}{b}$, $\frac{s}{c}$ are in Arithmetical Progression; and this is the case since a, b, c are in Harmonical Progression

196 We have $$a=2R\sin\frac{\pi}{5},$$

therefore $$\frac{R}{a}=\frac{1}{2\sin\frac{\pi}{5}}=\frac{2}{\sqrt{10-2\sqrt{5}}}=\frac{2\sqrt{10+2\sqrt{5}}}{\sqrt{80}}=\frac{2\sqrt{200+40\sqrt{5}}}{\sqrt{80\times 20}}$$

$$=\frac{\sqrt{200+40\sqrt{5}}}{20}=\frac{\sqrt{289\ \ 44}}{20}=\frac{17}{20}\text{ nearly}$$

197 Here $$\frac{\sin(B+C)}{\sin B}=\frac{m}{n},\quad \frac{\cos(B+C)}{\cos B}=-\frac{p}{q},$$

therefore $$\cos C+\cot B\sin C=\frac{m}{n},\quad \tan B\sin C-\cos C=\frac{p}{q},$$

therefore $$\sin^2 C=\left(\frac{m}{n}-\cos C\right)\left(\frac{p}{q}+\cos C\right),$$

therefore $$1=\frac{mp}{nq}+\cos C\left(\frac{m}{n}-\frac{p}{q}\right),$$

therefore $$\cos C=\frac{mp-nq}{np-mq}$$

198 We have $$OA=\frac{r}{\sin\frac{A}{2}},\quad OB=\frac{r}{\sin\frac{B}{2}},\quad OC=\frac{r}{\sin\frac{C}{2}},$$

$$AF=r\cot\frac{A}{2},\quad BD=r\cot\frac{B}{2},\quad CE=r\cot\frac{C}{2}$$

Hence we have to show that

$$\frac{r^4}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\left\{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right\}=4Rr^3\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

By Example VIII 15 the left-hand member

$$=\frac{r^4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}},$$

thus we have to show that

$$4R\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C=r$$

The left-hand member

$$=4R\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}=\frac{4RS^2}{sabc}=\frac{S}{s}=r.$$

199. The radius of the circle inscribed in the triangle OBC

$$=\frac{\text{area of the triangle}}{\text{semiperimeter}}=\frac{\frac{1}{2}\rho^2\sin 2\alpha}{\rho(1+\sin\alpha)}=\frac{\rho\sin 2\alpha}{2(1+\sin\alpha)}.$$

Let P denote the position of the centre, then

$$OP=\frac{\rho\sin 2\alpha}{2(1+\sin\alpha)}\times\frac{1}{\sin\alpha}=\frac{\rho\cos\alpha}{1+\sin\alpha}$$

Again, let Q denote the position of the centre of the circle inscribed in the triangle OAB, then, as 2α is now to be changed to $\pi-2\alpha$, we have

$$OQ=\frac{\rho\cos\left(\frac{\pi}{2}-\alpha\right)}{1+\sin\left(\frac{\pi}{2}-\alpha\right)}=\frac{\rho\sin\alpha}{1+\cos\alpha}.$$

And since POQ is a right angle, $PQ^2=OP^2+OQ^2$

$$=\rho^2\left\{\frac{\cos^2\alpha}{(1+\sin\alpha)^2}+\frac{\sin^2\alpha}{(1+\cos\alpha)^2}\right\}$$

$$=\rho^2\left\{\frac{1-\sin\alpha}{1+\sin\alpha}+\frac{1-\cos\alpha}{1+\cos\alpha}\right\}=\frac{\rho^2(2-\sin 2\alpha)}{(1+\sin\alpha)(1+\cos\alpha)},$$

therefore
$$PQ=\frac{\rho\sqrt{2-\sin 2\alpha}}{\sqrt{(1+\sin\alpha)(1+\cos\alpha)}}.$$

200 Let A and B be the two objects Suppose a circle to pass through A and B, and to *touch* the straight line at P, then P is the point at which the greatest angle is subtended see *Appendix to Euclid*, page 308 Produce AB to meet the straight line at Q Let the angle $BPQ=\alpha$, and let β be the angle between AP and the straight line Then also $PAB=\alpha$, and $PBA=\beta$, by Euclid III 32 Let $PQ=c$

Then
$$\frac{BP}{PQ}=\frac{\sin(\beta-\alpha)}{\sin\beta},\quad \frac{AB}{BP}=\frac{\sin(\beta+\alpha)}{\sin\alpha},$$

therefore
$$\frac{AB}{c}=\frac{\sin(\beta+\alpha)\sin(\beta-\alpha)}{\sin\alpha\sin\beta}$$

201 We have

$$r_1+r_2+r_3-r=S\left\{\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}\right\}$$

$$=S\left\{\frac{2s-a-b}{(s-a)(s-b)}+\frac{c}{s(s-c)}\right\}=cS\left\{\frac{1}{(s-a)(s-b)}+\frac{1}{s(s-c)}\right\}$$

$$=\frac{cS}{S^2}\{s(s-c)+(s-a)(s-b)\}=\frac{c}{S}\{2s^2-s(a+b+c)+ab\}$$

$$=\frac{abc}{S}=4R$$

202 $$\sin^{-1}\frac{1}{\sqrt{5}}=\tan^{-1}\frac{1}{2}, \quad \cot^{-1}3=\tan^{-1}\frac{1}{3};$$

$$\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}=\tan^{-1}1=\frac{\pi}{4}$$

203. The angle of the second triangle which is opposite to the angle C of the first triangle will be found to be $\frac{\pi}{2}-\frac{C}{2}$, similarly the corresponding angle of the third triangle will be $\frac{\pi}{2}-\frac{1}{2}\left(\frac{\pi}{2}-\frac{C}{2}\right)$, that is $\frac{\pi}{2}-\frac{\pi}{4}+\frac{C}{4}$. Proceeding in this way we find that the corresponding angle of the n^{th} triangle is

$$\pi\left\{\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\dots-\frac{(-1)^{n-1}}{2^{n-1}}\right\}+\frac{(-1)^{n-1}C}{2^{n-1}},$$

that is $$\frac{\pi}{2}\,\frac{1-\left(-\frac{1}{2}\right)^{n-1}}{1+\frac{1}{2}}+\frac{(-1)^{n-1}C}{2^{n-1}},$$

that is $$\frac{\pi}{3}\left\{1-\frac{(-1)^{n-1}}{2^{n-1}}\right\}+\frac{(-1)^{n-1}C}{2^{n-1}}$$

Similar expressions hold for the other angles

204. Suppose $\theta=\tan^{-1}a$, then we require $\cos 4\theta$.

Now $$\tan\theta=a, \quad \cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{1-a^2}{1+a^2},$$

$$\cos 4\theta=2\cos^2 2\theta-1=\frac{2(1-a^2)^2}{(1+a^2)^2}-1=\frac{1-6a^2+a^4}{(1+a^2)^2}$$

205. We have $e^{ax}\cos(bx+c)=e^{ax}(\cos bx\cos c-\sin bx\sin c)$, then by Art. 323 the required general term is

$$\frac{(a^2+b^2)^{\frac{n}{2}}}{\underline{|n}}(\cos n\theta\cos c-\sin n\theta\sin c),$$

that is $$\frac{(a^2+b^2)^{\frac{n}{2}}}{\underline{|n}}\cos(n\theta+c),$$

where θ is such that $\tan\theta=\frac{b}{a}$

206 We have $AD^2 = AB^2 + BD^2 - 2AB\ BD\cos B$

$$= c^2 + (s-c)^2 - 2c(s-c)\cos B, \text{ by Art 250;}$$

therefore $$s^2 - AD^2 = s^2 - c^2 - (s-c)^2 + 2c(s-c)\cos B$$

$$= (s-c)\{s+c-(s-c)+2c\cos B\}$$

$$= 2c(s-c)(1+\cos B) = 4c(s-c)\cos^2\frac{B}{2},$$

therefore $$a(s^2 - AD^2) = 4ac(s-c)\cos^2\frac{B}{2} = 4s(s-b)(s-c)$$

207 We have

$$\alpha + \beta\sqrt{-1} = \cos(\theta + \phi\sqrt{-1}) = \cos\theta\cos\phi\sqrt{-1} - \sin\theta\sin\phi\sqrt{-1}$$

$$= \cos\theta\frac{\epsilon^{-\phi}+\epsilon^{\phi}}{2} - \sin\theta\frac{\epsilon^{-\phi}-\epsilon^{\phi}}{2\sqrt{-1}}$$

$$= \cos\theta\frac{\epsilon^{\phi}+\epsilon^{-\phi}}{2} - \sin\theta\frac{\epsilon^{\phi}-\epsilon^{-\phi}}{2}\sqrt{-1}$$

Hence by equating the possible and the impossible parts we have

$$\alpha = \cos\theta\frac{\epsilon^{\phi}+\epsilon^{-\phi}}{2}, \quad \beta = -\sin\theta\frac{\epsilon^{\phi}-\epsilon^{-\phi}}{2}$$

Therefore $$\frac{\alpha^2}{\cos^2\theta} - \frac{\beta^2}{\sin^2\theta} = \left(\frac{\epsilon^{\phi}+\epsilon^{-\phi}}{2}\right)^2 - \left(\frac{\epsilon^{\phi}-\epsilon^{-\phi}}{2}\right)^2 = 1;$$

and $$\frac{\alpha^2}{(\epsilon^{\phi}+\epsilon^{-\phi})^2} + \frac{\beta^2}{(\epsilon^{\phi}-\epsilon^{-\phi})^2} = \frac{\cos^2\theta+\sin^2\theta}{4} = \frac{1}{4}.$$

208. $$\log\sec\theta = \frac{1}{2}\log\frac{1}{\cos^2\theta} = \frac{1}{2}\log\frac{2}{1+\cos 2\theta}$$

$$= \frac{1}{2}\log\frac{4}{2+\epsilon^{2\iota\theta}+\epsilon^{-2\iota\theta}} = \frac{1}{2}\log\frac{4}{(1+\epsilon^{2\iota\theta})(1+\epsilon^{-2\iota\theta})}$$

$$= \frac{1}{2}\{2\log 2 - \log(1+\epsilon^{2\iota\theta}) - \log(1+\epsilon^{-2\iota\theta})\},$$

therefore $$2\log\sec\theta = 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \quad\right)$$

$$- \left(\epsilon^{2\iota\theta} - \frac{1}{2}\epsilon^{4\iota\theta} + \frac{1}{3}\epsilon^{6\iota\theta} - \frac{1}{4}\epsilon^{8\iota\theta} + \quad\right)$$

$$- \left(\epsilon^{-2\iota\theta} - \frac{1}{2}\epsilon^{-4\iota\theta} + \frac{1}{3}\epsilon^{-6\iota\theta} - \frac{1}{4}\epsilon^{-8\iota\theta} + \quad\right).$$

Now $$2-e^{2i\theta}-e^{-2i\theta}=-(e^{i\theta}-e^{-i\theta})^2=4\sin^2\theta;$$

$$\frac{1}{2}(-2+e^{4i\theta}+e^{-4i\theta})=\frac{1}{2}(e^{2i\theta}-e^{-2i\theta})^2=-\frac{4}{2}\sin^2 2\theta,$$

$$\frac{1}{3}(2-e^{6i\theta}-e^{-6i\theta})=-\frac{1}{3}(e^{3i\theta}-e^{-3i\theta})^2=\frac{4}{3}\sin^2 3\theta,$$

and so on, thus

$$2\log\sec\theta=4\left\{\sin^2\theta-\frac{1}{2}\sin^2 2\theta+\frac{1}{3}\sin^2 3\theta-\ \ \right\},$$

therefore $$\log\sec\theta=2\left\{\sin^2\theta-\frac{1}{2}\sin^2 2\theta+\frac{1}{3}\sin^2 3\theta-\ \ \right\}$$

209 $$\sec\alpha\sec(\alpha+\beta)=\frac{1}{\sin\beta}\{\tan(\alpha+\beta)-\tan\alpha\},$$

$$\sec(\alpha+\beta)\sec(\alpha+2\beta)=\frac{1}{\sin\beta}\{\tan(\alpha+2\beta)-\tan(\alpha+\beta)\},$$

and so on

Then by addition we obtain the required result

210 The regular hexagon may be divided into six equilateral triangles; and thus the area of the first hexagon $=\frac{6a^2\sqrt{3}}{4}$

By Art. 257 the radius of the first circle $=\frac{a}{2}\cot 30^\circ=\frac{a\sqrt{3}}{2}$, and the side of the second hexagon is equal to this, so that the area of the second hexagon $=\frac{6a^2\sqrt{3}}{4}\left(\frac{\sqrt{3}}{2}\right)^2$. In this way we see that the areas of the hexagons form a geometrical progression of which the ratio is $\frac{3}{4}$; and the sum of

the areas $$=\frac{6a^2\sqrt{3}}{4}\,\frac{1-\left(\frac{3}{4}\right)^n}{1-\frac{3}{4}}=6a^2\sqrt{3}\left\{1-\left(\frac{3}{4}\right)^n\right\}$$

211 We have $a=2R\sin A,\quad b=2R\sin B,\quad c=2R\sin C;$

thus the proposed expression

$$\begin{aligned}&=2R(\sin A\cos A+\sin B\cos B+\sin C\cos C)\\&=R(\sin 2A+\sin 2B+\sin 2C)\\&=4R\sin A\sin B\sin C,\ \text{by Art 114},\\&=2a\sin B\sin C\end{aligned}$$

The expression is now adapted to logarithms

212 Let θ denote the angle APB, ϕ the angle BPC, and ψ the angle ABP

We have
$$\frac{AB}{PB}=\frac{\sin\theta}{\sin(\psi+\theta)}, \quad \frac{BC}{PB}=\frac{\sin\phi}{\sin(\psi-\phi)},$$
but AB is supposed equal to BC, and thus
$$\frac{\sin(\psi+\theta)}{\sin\theta}=\frac{\sin(\psi-\phi)}{\sin\phi},$$
therefore
$$\sin\psi\cot\theta+\cos\psi=\sin\psi\cot\phi-\cos\psi,$$
therefore
$$2\cot\psi=\cot\phi-\cot\theta,$$
therefore
$$\frac{2}{T}=\frac{1}{t'}-\frac{1}{t}$$

213 It is shown in Art 319 that $\frac{\pi}{4}=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}$;
hence we have only to shew that
$$\tan^{-1}\frac{1}{239}=2\tan^{-1}\frac{1}{408}-\tan^{-1}\frac{1}{1393},$$
or that
$$\tan^{-1}\frac{1}{239}+\tan^{-1}\frac{1}{1393}=2\tan^{-1}\frac{1}{408}$$

Now
$$\tan^{-1}\frac{1}{239}+\tan^{-1}\frac{1}{1393}=\tan^{-1}\frac{\frac{1}{239}+\frac{1}{1393}}{1-\frac{1}{239\times1393}}$$
$$=\tan^{-1}\frac{1393+239}{239\times1393-1}=\tan^{-1}\frac{1632}{332926}=\tan^{-1}\frac{816}{166463},$$
and
$$2\tan^{-1}\frac{1}{408}=\tan^{-1}\frac{\frac{2}{408}}{1-\left(\frac{1}{408}\right)^2}=\tan^{-1}\frac{2\times408}{166463},$$

Thus the required result is established

214 By the diagram of Art. 253 we see that
$$\frac{PA}{AB}=\frac{\sin PBA}{\sin APB}=\frac{\sin\left(\frac{\pi}{2}-A\right)}{\sin(A+B)}=\frac{\cos A}{\sin C};$$
therefore
$$PA=\frac{c\cos A}{\sin C}=\frac{a\cos A}{\sin A},$$
therefore
$$PA^2=\frac{a^2(1-\sin^2A)}{\sin^2A}=\frac{a^2}{\sin^2A}-a^2=4R^2-a^2.$$

215 $\dfrac{(\cos a+\sqrt{-1}\sin a)(\cos 2a+\sqrt{-1}\sin 2a)}{\cos 3a-\sqrt{-1}\sin 3a}=\dfrac{\cos 3a+\sqrt{-1}\sin 3a}{\cos 3a-\sqrt{-1}\sin 3a}$,

multiply both numerator and denominator by $\cos 3a+\sqrt{-1}\sin 3a$, thus we obtain unity in the denominator, and $\cos 6a+\sqrt{-1}\sin 6a$ in the numerator. and this numerator $=\sqrt{-1}$ since $a=15^0$

216 The new triangle will have for its angular points the centres of the escribed circles of the original triangle Now from Art 250 we have

$$OC=CE\sec OCE=(s-b)\operatorname{cosec}\frac{C}{2},$$

and in the same manner the distance from C of the centre of the circle which touches BC and BA produced $=(s-a)\operatorname{cosec}\dfrac{C}{2}$ Hence the sum of these two $=(2s-b-a)\operatorname{cosec}\dfrac{C}{2}=c\operatorname{cosec}\dfrac{C}{2}=2R\sin C\operatorname{cosec}\dfrac{C}{2}=4R\cos\dfrac{C}{2}$.

This is the length of the side of the second triangle which passes through the point C, similar expressions hold for the other two sides

217. By the preceding Example the sum of the squares

$$=16R^2\left\{\cos^2\frac{1}{2}A+\cos^2\frac{1}{2}B+\cos^2\frac{1}{2}C\right\}$$

$$=8R^2\{3+\cos A+\cos B+\cos C\}$$

$$=8R^2\left\{4+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right\}, \text{ by Art 114,}$$

$$=32R^2+\frac{32R^2S^2}{sabc}=32R^2+8Rr$$

218 The numerator can be expressed in powers of $\cos\theta$, and it will be found to reduce to $2^5\cos^6\theta$, in like manner the denominator will be found to reduce to $2^4\cos^5\theta$ see Art 292 Hence the expression reduces to $2\cos\theta$

219 $\cos\theta$ is less than $1-\dfrac{\theta^2}{2}+\dfrac{\theta^4}{16}$, that is less than $\left(1-\dfrac{\theta^2}{4}\right)^2$, therefore $\sqrt{\cos\theta}$ is less than $1-\dfrac{\theta^2}{4}$, this holds if θ lies between 0 and $\dfrac{\pi}{2}$ see Art 328 Again, $\cos\dfrac{\theta}{\sqrt{2}}$ is greater than $1-\dfrac{1}{2}\left(\dfrac{\theta}{\sqrt{2}}\right)^2$, that is greater than $1-\dfrac{\theta^2}{4}$, this holds as long as $\cos\dfrac{\theta}{\sqrt{2}}$ and $1-\dfrac{\theta^2}{4}$ remain both positive, and this certainly holds if θ lies between 0 and $\dfrac{\pi}{2}$. Hence $\sqrt{\cos\theta}$ is less than $\cos\dfrac{\theta}{\sqrt{2}}$

220 Suppose the polygon has n sides Let O be the centre of the circle inscribed in the polygon, and S the assumed point. Let $OS=c$, and suppose OS to be inclined at an angle α to the first perpendicular which is drawn, put β for $\frac{2\pi}{n}$, and r for the radius of the inscribed circle Then the length of the first perpendicular will be $r+c\cos\alpha$, that of the second $r+c\cos(\alpha+\beta)$, that of the third $r+c\cos(\alpha+2\beta)$, and so on Hence the sum of one set of perpendiculars

$$=\frac{nr}{2}+c\left\{\cos\alpha+\cos(\alpha+2\beta)+\cos(\alpha+4\beta)+\quad\text{to } \frac{n}{2} \text{ terms}\right\}$$

By Art 327 the sum of the series of cosines contains the factor $\sin\frac{n}{2}\beta$, that is $\sin\pi$, that is zero

Hence the sum of the set of perpendiculars $=\frac{nr}{2}$

Similarly the sum of the other set of perpendiculars has the same value

221 $$r=\frac{S}{s},\quad r_1=\frac{S}{s-a},\quad r_2=\frac{S}{s-b},\quad r_3=\frac{S}{s-c},$$

therefore $$rr_1r_2r_3=\frac{S^4}{s(s-a)(s-b)(s-c)}=\frac{S^4}{S^2}=S^2,$$

therefore $$\sqrt{rr_1r_2r_3}=S$$

222 We have $$2\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{\frac{2}{7}}{1-\frac{1}{49}}=\tan^{-1}\frac{7}{24};$$

thus $$4\tan^{-1}\frac{1}{7}=2\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{2\times\frac{7}{24}}{1-\left(\frac{7}{24}\right)^2}=\tan^{-1}\frac{2\times7\times24}{(24+7)(24-7)}$$

$$=\tan^{-1}\frac{336}{527}$$

Then $$5\tan^{-1}\frac{1}{7}=4\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{336}{527}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{\frac{336}{527}+\frac{1}{7}}{1-\frac{336}{7\times527}}=\tan^{-1}\frac{2879}{3353}.$$

Again $$2\tan^{-1}\frac{3}{79}=\tan^{-1}\frac{2\times\frac{3}{79}}{1-\left(\frac{3}{79}\right)^2}=\tan^{-1}\frac{2\times3\times79}{(79+3)(79-3)}$$

$$=\tan^{-1}\frac{237}{3116}$$

Finally $$\tan\left(\frac{\pi}{4}-\tan^{-1}\frac{2879}{3353}\right)=\frac{1-\frac{2879}{3353}}{1+\frac{2879}{3353}}=\frac{474}{6232}=\frac{237}{3116};$$

so that $$\frac{\pi}{4}-\tan^{-1}\frac{2879}{3353}=\tan^{-1}\frac{237}{3116}$$

223 In the expression for $\tan n\theta$ put $\frac{\pi}{4}$ for θ, then $\tan\theta=1$.

If n is an odd number we have $\tan n\theta=(-1)^{\frac{n-1}{2}}$, so that the numerator of the expression is numerically equal to the denominator

If n is an even number, $\tan n\theta$ is either zero or infinite, so that in the former case the numerator of the expression must vanish, and in the latter case the denominator must vanish.

224 We have $\sin^4\theta\cos^5\theta=(1-\cos^2\theta)^2\cos^5\theta=\cos^9\theta-2\cos^7\theta+\cos^5\theta$

Now $$\cos^9\theta=\frac{1}{2^8}\left\{\cos 9\theta+9\cos 7\theta+36\cos 5\theta+84\cos 3\theta+126\cos\theta\right\},$$

$$\cos^7\theta=\frac{1}{2^6}\left\{\cos 7\theta+7\cos 5\theta+21\cos 3\theta+35\cos\theta\right\},$$

$$\cos^5\theta=\frac{1}{2^4}\left\{\cos 5\theta+5\cos 3\theta+10\cos\theta\right\}.$$

Hence

$$\cos^9\theta-2\cos^7\theta+\cos^5\theta=\frac{1}{256}(\cos 9\theta+\cos 7\theta)-\frac{1}{64}(\cos 5\theta+\cos 3\theta)+\frac{3}{128}\cos\theta$$

Or we may proceed thus

$$\sin^4\theta\cos^5\theta=\sin^4\theta\cos^4\theta\sin\theta=\frac{1}{16}(\sin 2\theta)^4\cos\theta$$

$$=\frac{1}{16}\left\{\frac{1}{8}\cos 8\theta-\frac{1}{2}\cos 4\theta+\frac{3}{8}\right\}\cos\theta$$

$$=\frac{1}{256}(\cos 9\theta+\cos 7\theta)-\frac{1}{64}(\cos 5\theta+\cos 3\theta)+\frac{3}{128}\cos\theta$$

225. We have

$$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$$

$$= \frac{3}{4}\left\{x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \quad + \frac{(-1)^n x^{2n+1}}{\lfloor 2n+1} + \quad\right\}$$

$$-\frac{1}{4}\left\{3x - \frac{(3x)^3}{\lfloor 3} + \frac{(3x)^5}{\lfloor 5} - \frac{(3x)^7}{\lfloor 7} + \quad + \frac{(-1)^n (3x)^{2n+1}}{\lfloor 2n+1} + \quad\right\},$$

then by arranging according to powers of x we obtain the required result

226 Put s for $\cos\phi + \cos 3\phi + \cos 9\phi$, and t for $\cos 5\phi + \cos 7\phi + \cos 11\phi$

Then $\qquad s + t = \cos\phi + \cos 3\phi + \cos 5\phi + \cos 7\phi + \cos 9\phi + \cos 11\phi$

$$= \frac{\cos(\phi + 5\phi)\sin 6\phi}{\sin\phi} \text{ (Art 327) } = \frac{\sin 12\phi}{2\sin\phi} = \frac{\sin\phi}{2\sin\phi} = \frac{1}{2}$$

And $\qquad st = (\cos\phi + \cos 3\phi + \cos 9\phi)(\cos 5\phi + \cos 7\phi + \cos 11\phi)$

$$= \cos\phi(\cos 5\phi + \cos 7\phi + \cos 11\phi) +$$

Resolve each product into the sum of two cosines by Art 84, thus we get

$$2st = \cos 6\phi + \cos 4\phi + \cos 8\phi + \cos 6\phi + \cos 12\phi + \cos 10\phi$$
$$+ \cos 8\phi + \cos 2\phi + \cos 10\phi + \cos 4\phi + \cos 14\phi + \cos 8\phi$$
$$+ \cos 14\phi + \cos 4\phi + \cos 16\phi + \cos 2\phi + \cos 20\phi + \cos 2\phi$$
$$= 3\cos 2\phi + 3\cos 4\phi + 2\cos 6\phi + 3\cos 8\phi + 2\cos 10\phi$$
$$+ \cos 12\phi + 2\cos 14\phi + \cos 16\phi + \cos 20\phi$$

Now since $\phi = \frac{\pi}{13}$ we have $\cos 14\phi = \cos 12\phi$, $\cos 20\phi = -\cos 7\phi = \cos 6\phi$, $\cos 16\phi = -\cos 3\phi = \cos 10\phi$ Thus

$$2st = 3\{\cos 2\phi + \cos 4\phi + \cos 6\phi + \cos 8\phi + \cos 10\phi + \cos 12\phi\}$$

$$= \frac{3\cos(2\phi + 5\phi)\sin 6\phi}{\sin\phi} = -\frac{3\cos 6\phi\sin 6\phi}{\sin\phi} = -\frac{3\sin 12\phi}{2\sin\phi} = -\frac{3}{2};$$

therefore

$$st = -\frac{3}{4}$$

Then, since $s + t = \frac{1}{2}$, and $st = -\frac{3}{4}$, we find by Algebra that

$$s = \frac{1 \pm \sqrt{13}}{4} \text{ and } t = \frac{1 \mp \sqrt{13}}{4},$$

and it is obvious that the upper sign must be taken, because s is positive, for $\cos\phi$ and $\cos 3\phi$, which are positive, are both numerically greater than $\cos 9\phi$, which is negative

227. Suppose the polygon has n sides Let O be the centre of the circle inscribed in the polygon, and S the assumed point. Let $OS=c$, and suppose OS to be inclined at an angle a to the first perpendicular which is drawn, put β for $\frac{2\pi}{n}$, and r for the radius of the inscribed circle Then the length of the first perpendicular will be $r+c\cos a$, that of the second $r+c\cos(a+\beta)$, that of the third $r+c\cos(a+2\beta)$, and so on

Then for the squares on the sides of the new polygon we obtain the expressions

$$\{r+c\cos a\}^2+\{r+c\cos(a+\beta)\}^2-2\{r+c\cos a\}\{r+c\cos(a+\beta)\}\cos\beta,$$

$$\{r+c\cos(a+\beta)\}^2+\{r+c\cos(a+2\beta)\}^2-2\{r+c\cos(a+\beta)\}\{r+c\cos(a+2\beta)\}\cos\beta,$$

and so on

Thus for the square on the m^{th} side of the new polygon we shall obtain

$$2r^2(1-\cos\beta)+2rc\{\cos(a+m\beta-\beta)+\cos(a+m\beta)\}(1-\cos\beta)$$
$$+c^2\{\cos^2(a+m\beta-\beta)+\cos^2(a+m\beta)-2\cos\beta\cos(a+m\beta-\beta)\cos(a+m\beta)\},$$

that is

$$2r^2(1-\cos\beta)+2rc\{\cos(a+m\beta-\beta)+\cos(a+m\beta)\}(1-\cos\beta)$$
$$+\frac{c^2}{2}\{1+\cos(2a+2m\beta-2\beta)+1+\cos(2a+2m\beta)$$
$$-2\cos\beta[\cos\beta+\cos(2a+2m\beta-\beta)]\}$$

We have to obtain the sum formed from this expression by giving to m all integral values from 1 to n, both inclusive, the result, by Art. 328,

$$=2nr^2(1-\cos\beta)+\frac{c^2}{2}\{2n-2n\cos^2\beta\}$$

$$=4nr^2\sin^2\frac{\beta}{2}+nc^2\sin^2\beta$$

228 We have $\cos 5\theta+\sin 5\theta=\sqrt{2}\cos\left(5\theta-\frac{\pi}{4}\right)$

$$=\sqrt{2}\cos 5\left(\theta-\frac{\pi}{20}\right)=\sqrt{2}\cos 5(\theta-\beta)$$

And by Art. 342 we have $\cos 5(\theta-\beta)\sin\frac{5\pi}{2}$

$$=2^4\cos(\theta-\beta)\cos(\theta-\beta+2a)\cos(\theta-\beta+4a)\cos(\theta-\beta+6a)\cos(\theta-\beta+8a),$$

where $a=\frac{\pi}{10}=2\beta$

Thus $\cos 5(\theta-\beta)$

$$=2^4\cos(\theta-\beta)\cos(\theta+3\beta)\cos(\theta+7\beta)\cos(\theta+11\beta)\cos(\theta+15\beta).$$

Also
$$\cos(\theta+7\beta)=\sin(3\beta-\theta)=-\sin(\theta-3\beta),$$
$$\cos(\theta+11\beta)=\cos\left(\theta+\beta+\frac{\pi}{2}\right)=-\sin(\theta+\beta),$$
$$\cos(\theta+15\beta)=\cos\left(\theta+\frac{3\pi}{4}\right)=-\frac{1}{\sqrt{2}}(\cos\theta+\sin\theta)$$

Hence $\sqrt{2}\cos 5(\theta-\beta)$
$$=-2^4\cos(\theta-\beta)\cos(\theta+3\beta)\sin(\theta-3\beta)\sin(\theta+\beta)(\sin\theta+\cos\theta),$$
hence also $\cos 5\theta+\sin 5\theta$ is equal to the last expression, which had to be shewn.

229 We have
$$\sin\theta\sqrt{(\cos^2\alpha\cos^2\beta+\sin^2\alpha\sin^2\beta)}+\cos\theta\sqrt{(\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta)}=\sin(\alpha+\beta)$$

Assume $\qquad r\cos\phi=\sqrt{(\cos^2\alpha\cos^2\beta+\sin^2\alpha\sin^2\beta)},$

and $\qquad r\sin\phi=\sqrt{(\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta)},$

so that
$$r^2=2\cos^2\alpha\cos^2\beta \tag{1}$$
and
$$\tan^2\phi=\frac{\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta}{\cos^2\alpha\cos^2\beta+\sin^2\alpha\sin^2\beta} \tag{2}$$
Thus
$$r\sin(\theta+\phi)=\sin(\alpha+\beta) \tag{3}$$

Now it is obvious that r may be found from (1) by logarithms. Also ϕ may be determined by logarithms, for we have from (2)
$$\frac{1-\tan^2\phi}{1+\tan^2\phi}=\frac{\sin^2\alpha\sin^2\beta}{\cos^2\alpha\cos^2\beta},$$
that is
$$\cos 2\phi=\tan^2\alpha\tan^2\beta,$$
which is adapted to logarithms

Thus θ can be found from (3) by logarithms

230 If A, B, and C are angles of a triangle, we have by Art 114, and Example VIII 16,

$$\sin A+\sin B+\sin C-(\sin 2A+\sin 2B+\sin 2C)$$
$$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}-4\sin A\sin B\sin C$$
$$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\left\{1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right\},$$
and by Example XIII 40 this expression can never be negative.

231. Let $A, B, C, \ldots M, N$ denote the angular points of the polygon taken in order, and let $\alpha=\frac{\pi}{n}$. Suppose P the point in the circumference from which chords are drawn, so that P is between N and A.

Then $\frac{1}{2}c_1c_2 \sin\alpha =$ the area of the triangle PAB,

$\frac{1}{2}c_2c_3 \sin\alpha =$ the area of the triangle PBC,

$\frac{1}{2}c_{n-1}c_n \sin\alpha =$ the area of the triangle PMN.

Therefore $\frac{1}{2}(c_1c_2+c_2c_3+\ldots+c_{n-1}c_n)\sin\alpha$

$=$ the area of the triangles PAB, PBC, $\ldots$ PMN

Also $\frac{1}{2}c_nc_1 \sin\alpha =$ the area of the triangle PNA

Thus $\frac{1}{2}(c_1c_2+c_2c_3+\ldots+c_{n-1}c_n-c_nc_1)\sin\alpha$

$=$ the area of the regular polygon,

so that $c_1c_2+c_2c_3+\ldots+c_{n-1}c_n-c_nc_1$

$$=\frac{2}{\sin\alpha}\times \text{the area of the regular polygon}$$

This result is the same for all positions of P on the circumference of the circle

232 Let θ be the angle of one sector, and 2θ the angle of the other Let a and b be the corresponding radii. Then, since the areas are equal, $a^2\frac{\theta}{2}=b^2\frac{2\theta}{2}$, and since there is a common chord, $2a\sin\frac{\theta}{2}=2b\sin\frac{2\theta}{2}$

Thus $a\sin\frac{\theta}{2}=b\sin\theta=2b\sin\frac{\theta}{2}\cos\frac{\theta}{2}$, therefore $\cos\frac{\theta}{2}=\frac{a}{2b}$,

therefore $\cos^2\frac{\theta}{2}=\frac{a^2}{4b^2}=\frac{2b^2}{4b^2}=\frac{1}{2}$, therefore $\frac{\theta}{2}=\frac{\pi}{4}$.

Therefore $\theta=\frac{\pi}{2}$ and $2\theta=\pi$.

233 We have
$$\tan(\phi-\theta)=\frac{\tan\phi-\tan\theta}{1+\tan\phi\tan\theta}=\frac{\frac{x\sqrt{3}}{2k-x}-\frac{2x-k}{k\sqrt{3}}}{1+\frac{x(2x-k)\sqrt{3}}{(2k-x)k\sqrt{3}}}$$
$$=\frac{1}{\sqrt{3}}\cdot\frac{3kx-(2k-x)(2x-k)}{(2k-x)k+x(2x-k)}=\frac{1}{\sqrt{3}}\,\frac{2x^2-2kx+2k^2}{2x^2-2kx+2k^2}=\frac{1}{\sqrt{3}}$$

Therefore one value of $\phi-\theta$ is $\frac{\pi}{6}$

234. We have here $\frac{\sin\theta}{\theta}$ very nearly equal to unity, so we may infer that θ is small hence $\sin\theta=\theta-\frac{\theta^3}{6}$ nearly Therefore $1-\frac{\theta^2}{6}=\frac{863}{864}$ nearly, therefore $\frac{\theta^2}{6}=\frac{1}{864}$ nearly, therefore $\theta^2=\frac{1}{144}$ nearly, therefore $\theta=\frac{1}{12}$ nearly Hence the number of degrees in the angle is nearly $\frac{1}{12}\ \frac{180}{\pi}$, that is about 5

235 Let ABC be any triangle, let D, E, F be the centres of the escribed circles opposite to A, B, C respectively

Then AD bisects the angle of the triangle at A, and EF bisects the *exterior* angle at A Therefore AD is perpendicular to EF

Similarly EB is perpendicular to FD, and FC is perpendicular to DE

Therefore the circle described round ABC is the nine points circle of DEF

236 As in Art. 295 we have

$2^{2n}(-1)^n\sin^{2n+1}\theta$

$$=\sin(2n+1)\theta-(2n+1)\sin(2n-1)\theta+\frac{(2n+1)2n}{\lfloor 2}\sin(2n-3)\theta- \quad .$$

Now suppose each side were to be expanded in powers of θ, on the left-hand side we should have $2^{2n}(-1)^n\left\{\theta-\frac{\theta^3}{\lfloor 3}+\quad\right\}^{2n+1}$, by Art 286

On the right-hand side each sine gives rise to a series Since the lowest power of θ on the left-hand side is θ^{2n+1} it follows that the whole coefficient of every lower power of θ on the right-hand side must be zero. The whole coefficient of θ is

$$2n+1-(2n+1)(2n-1)+\frac{(2n+1)2n}{\lfloor 2}(2n-3)- \quad \text{to } n+1 \text{ terms},$$

hence this is zero, and dividing by $2n+1$ we obtain the required result.

Similarly, supposing n to be greater than unity, we can obtain another result by equating to zero the whole coefficient of θ^3 on the right-hand side And so on.

237 We have

$$\cos\alpha+\sqrt{-1}\sin\alpha=\cos(\theta+\phi\sqrt{-1})=\cos\theta\cos\phi\sqrt{-1}-\sin\theta\sin\phi\sqrt{-1}$$

$$=\cos\theta\frac{e^{-\phi}+e^{\phi}}{2}-\sin\theta\frac{e^{-\phi}-e^{\phi}}{2\sqrt{-1}}=\cos\theta\frac{e^{-\phi}+e^{\phi}}{2}+\sin\theta\frac{e^{-\phi}-e^{\phi}}{2}\sqrt{-1}.$$

Hence, by equating the possible and the impossible parts, we have

$$\cos\theta\frac{e^{-\phi}+e^{\phi}}{2}=\cos\alpha,\quad \sin\theta\frac{e^{-\phi}-e^{\phi}}{2}=\sin\alpha,$$

so that
$$\frac{e^{-\phi}+e^{\phi}}{2}=\frac{\cos\alpha}{\cos\theta},\quad \frac{e^{-\phi}-e^{\phi}}{2}=\frac{\sin\alpha}{\sin\theta}.$$

Square and subtract; thus

$$1=\frac{\cos^2\alpha}{\cos^2\theta}-\frac{\sin^2\alpha}{\sin^2\theta},$$

therefore
$$\sin^2\theta\cos^2\alpha-\cos^2\theta\sin^2\alpha=\sin^2\theta\cos^2\theta;$$

therefore
$$\sin^2\theta(1-\cos^2\theta)=\sin^2\alpha,$$

therefore
$$\sin^4\theta=\sin^2\alpha,$$

therefore
$$\sin^2\theta=\pm\sin\alpha$$

238 On the left hand side the numerator

$$=\sin x+\sin(3x+\pi)+\sin(5x+2\pi)+\ \text{to } n \text{ terms},$$

$$=\frac{\sin\left\{x+\frac{n-1}{2}(2x+\pi)\right\}\sin\frac{n}{2}(2x+\pi)}{\sin\frac{1}{2}(2x+\pi)};$$

in like manner the denominator

$$=\frac{\cos\left\{x+\frac{n-1}{2}(2x+\pi)\right\}\sin\frac{n}{2}(2x+\pi)}{\sin\frac{1}{2}(2x+\pi)}.$$

Divide the former by the latter and we obtain

$$\tan\left\{x+\frac{n-1}{2}(2x+\pi)\right\},\ \text{that is}\ \tan\left(nx+\frac{n-1}{2}\pi\right).$$

239 Let O denote the centre of the circles, r the radius of the circle $ABCP$, and R the radius of the circle $DEFQ$

Suppose the angle QOA is equal to θ, then the angle QOB will be $\theta+\frac{2\pi}{3}$, and the angle QOC will be $\theta+\frac{4\pi}{3}$, or at least the angles may be

so denoted by suitably choosing A, B, and C. Then

$$QA^2 = QO^2 + OA^2 - 2QO \cdot OA \cos\theta$$
$$= R^2 + r^2 - 2Rr\cos\theta,$$

similarly $$QB^2 = R^2 + r^2 - 2Rr\cos\left(\theta + \frac{2\pi}{3}\right),$$

and $$QC^2 = R^2 + r^2 - 2Rr\cos\left(\theta + \frac{4\pi}{3}\right).$$

Hence by addition, and Art. 328, we have

$$QA^2 + QB^2 + QC^2 = 3(R^2 + r^2).$$

In the same way we find that

$$PD^2 + PE^2 + PF^2 = 3(R^2 + r^2).$$

240. Put for each cosine its exponential value, then the proposed series

$$= \frac{1}{2}(1 - ae^{\iota cx})^n + \frac{1}{2}(1 - ae^{-\iota cx})^n$$

$$= \frac{1}{2}(1 - a\cos cx - \iota a\sin cx)^n + \frac{1}{2}(1 - a\cos cx + \iota a\sin cx)^n.$$

Now assume $1 - a\cos cx = r\cos\theta$ and $a\sin cx = r\sin\theta$,

then the sum $$= \frac{1}{2}(r\cos\theta - \iota r\sin\theta)^n + \frac{1}{2}(r\cos\theta + \iota r\sin\theta)^n$$

$$= \frac{r^n}{2}(\cos n\theta - \iota\sin n\theta) + \frac{r^n}{2}(\cos n\theta + \iota\sin n\theta)$$

$$= r^n\cos n\theta.$$

241. By addition $3 - p\sin\theta - q\cos\theta = 0$.

By subtraction $\cos^2\theta - \sin^2\theta = -p\sin\theta + q\cos\theta$.

Therefore $3(\cos^2\theta - \sin^2\theta) = q^2\cos^2\theta - p^2\sin^2\theta$,

therefore $3(2\cos^2\theta - 1) = q^2\cos^2\theta - p^2 + p^2\cos^2\theta$,

therefore $$\cos^2\theta = \frac{p^2 - 3}{p^2 + q^2 - 6},$$

therefore $$\sin^2\theta = \frac{q^2 - 3}{p^2 + q^2 - 6}.$$

Substitute in the equation $3 = p\sin\theta + q\cos\theta$, thus

$$3\sqrt{(p^2 + q^2 - 6)} = p\sqrt{(q^2 - 3)} + q\sqrt{(p^2 - 3)}.$$

This is the result of the elimination, the radicals are not necessarily positive. By squaring, transposing, and squaring again, we obtain finally

$$\{p^2q^2 - 6(p^2 + q^2) + 27\}^2 = p^2q^2(p^2 - 3)(q^2 - 3).$$

242 $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$

$$=\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}+\sqrt{\frac{s(s-b)}{(s-a)(s-c)}}+\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$=\frac{\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}}\{s-a+s-b+s-c\}=\frac{s^2}{S}$$

Hence the proposed expression $=s^2-\frac{s^2}{S}=S$

243 Here $$2\tan^{-1}ax+\sec^{-1}bx=\frac{\pi}{2},$$

therefore $$\sin^{-1}\frac{2ax}{1+a^2x^2}=\frac{\pi}{2}-\cos^{-1}\frac{1}{bx}=\sin^{-1}\frac{1}{bx},$$

therefore $$\frac{2ax}{1+a^2x^2}=\frac{1}{bx},\quad \text{therefore } 2abx^2=1+a^2x^2,$$

therefore $$x^2=\frac{1}{2ab-a^2}.$$

244 With the diagram of Art. 253 we have $OA=R$, also the angle $OAB=\frac{\pi}{2}-C$, and the angle $BAP=\frac{\pi}{2}-B$, so that the angle $OAP=C-B$

Hence $$OP^2=R^2+AP^2-2R\ AP\cos(B-C)$$

Now, as in Example 214, we have $AP=\frac{a\cos A}{\sin A}=2R\cos A$,

so that $$OP^2=R^2+4R^2\cos^2 A-4R^2\cos A\cos(B-C)$$
$$=R^2+2R^2(1+\cos 2A)+4R^2\cos(B+C)\cos(B-C)$$
$$=3R^2+2R^2\cos 2A+2R^2(\cos 2B+\cos 2C)$$
$$=3R^2+2R^2(\cos 2A+\cos 2B+\cos 2C)$$

245 The values of x, y, z are given in Example 216; and the values of α, β, γ in Example XVI 31. Hence

$$\frac{\beta z+\gamma y}{x}=\frac{4R\cos\frac{1}{2}C\,b\sec\frac{1}{2}B+4R\cos\frac{1}{2}B\,c\sec\frac{1}{2}C}{4R\cos\frac{1}{2}A}$$

$$=\frac{4R\left(\cos\frac{1}{2}C\sin\frac{1}{2}B+\cos\frac{1}{2}B\sin\frac{1}{2}C\right)}{\cos\frac{1}{2}A}=\frac{4R\sin\frac{1}{2}(B+C)}{\cos\frac{1}{2}A}$$

$$=4R$$

Similarly the other expressions are also equal to $4R$.

246. We have $c=b\cos A+a\cos B=b\cos A-a\cos(\pi-B)$

$$=b\sqrt{(1-\sin^2 A)}-a\cos\theta=b\sqrt{\left(1-\frac{a^2}{b^2}\sin^2\theta\right)}-a\cos\theta$$

We wish to expand this in powers of θ, as far as terms involving θ^4

Now $$\sqrt{\left(1-\frac{a^2}{b^2}\sin^2\theta\right)}=1-\frac{a^2}{2b^2}\sin^2\theta-\frac{a^4}{8b^4}\sin^4\theta-$$

Put for $\sin\theta$ its value $\theta-\frac{\theta^3}{6}+\quad$, thus we obtain

$$1-\frac{a^2}{2b^2}\left(\theta-\frac{\theta^3}{6}+\quad\right)^2-\frac{a^4}{8b^4}\left(\theta-\frac{\theta^3}{6}+\quad\right)^4,$$

that is $$1-\frac{a^2}{2b^2}\left(\theta^2-\frac{\theta^4}{3}\right)-\frac{a^4}{8b^4}\theta^4+$$

And $$\cos\theta=1-\frac{\theta^2}{\lfloor 2}+\frac{\theta^4}{\lfloor 4}-$$

Hence approximately

$$c=b-\frac{a^2}{2b}\left(\theta^2-\frac{\theta^4}{3}\right)-\frac{a^4}{8b^3}\theta^4-a\left(1-\frac{\theta^2}{\lfloor 2}+\frac{\theta^4}{\lfloor 4}\right)$$

$$=b-a+\left(a-\frac{a^2}{b}\right)\frac{\theta^2}{2}+\left(\frac{a^2}{6b}-\frac{a^4}{8b^3}-\frac{a}{24}\right)\theta^4$$

$$=b-a+\frac{(b-a)a\theta^2}{2b}+\frac{\theta^4}{\lfloor 4}\left(\frac{4a^2}{b}-\frac{3a^4}{b^3}-a\right),$$

and $$\frac{4a^2}{b}-\frac{3a^4}{b^3}-a=\frac{a^2}{b}-a+3\left(\frac{a^2}{b}-\frac{a^4}{b^3}\right)$$

$$=\frac{a(a-b)}{b}+\frac{3a^2(b^2-a^2)}{b^3}=(a-b)\left\{\frac{a}{b}-\frac{3a^2}{b^3}(a+b)\right\}$$

Thus we obtain the required result

247. $$\sin^5\theta\cos^6\theta=\cos\theta(\sin\theta\cos\theta)^5=\frac{\cos\theta}{2^5}(\sin 2\theta)^5$$

$$=\frac{\cos\theta}{2^5}\times\frac{1}{2^4}\{\sin 10\theta-5\sin 6\theta+10\sin 2\theta\}$$

$$=\frac{1}{2^{10}}\{\sin 11\theta+\sin 9\theta-5(\sin 7\theta+\sin 5\theta)+10(\sin 3\theta+\sin\theta)\}$$

248 We have $\cos\alpha+\sqrt{-1}\sin\alpha=\dfrac{\sin(\theta+\sqrt{-1}\phi)}{\cos(\theta+\sqrt{-1}\phi)}$

$$=\frac{\sin\theta\cos\sqrt{-1}\phi+\cos\theta\sin\sqrt{-1}\phi}{\cos\theta\cos\sqrt{-1}\phi-\sin\theta\sin\sqrt{-1}\phi}=\frac{\sin\theta(e^{\phi}+e^{-\phi})-\sqrt{-1}\cos\theta(e^{-\phi}-e^{\phi})}{\cos\theta(e^{\phi}+e^{-\phi})+\sqrt{-1}\sin\theta(e^{-\phi}-e^{\phi})}$$

$$=\frac{\sin\theta+\sqrt{-1}\lambda\cos\theta}{\cos\theta-\sqrt{-1}\lambda\sin\theta},\quad\text{where } \lambda=\frac{e^{\phi}-e^{-\phi}}{e^{\phi}+e^{-\phi}}.$$

Hence $\sin\theta+\sqrt{-1}\lambda\cos\theta=(\cos\alpha+\sqrt{-1}\sin\alpha)(\cos\theta-\sqrt{-1}\lambda\sin\theta)$

$$=\cos\alpha\cos\theta+\lambda\sin\alpha\sin\theta+\sqrt{-1}(\sin\alpha\cos\theta-\lambda\sin\theta\cos\alpha),$$

therefore $\qquad \sin\theta=\cos\alpha\cos\theta+\lambda\sin\alpha\sin\theta,$

and $\qquad \lambda\cos\theta=\sin\alpha\cos\theta-\lambda\sin\theta\cos\alpha$

therefore $\qquad \dfrac{\sin\theta-\cos\alpha\cos\theta}{\sin\alpha\sin\theta}=\dfrac{\sin\alpha\cos\theta}{\cos\theta+\sin\theta\cos\alpha}$

Multiply up, thus we get $\cos\alpha(\sin^2\theta-\cos^2\theta)=0$,

therefore $\qquad \tan^2\theta=1$, and therefore $\theta=n\pi\pm\dfrac{\pi}{4}$.

249 By Art. 332 we have

$$\frac{1}{x}-2\cot 2x=\tan x+\frac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+\ldots,$$

and, since $2\cot 2x+\tan x=\cot x$, we have

$$\frac{1}{x}-\cot x=\frac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+\ldots,$$

then put $\dfrac{\pi}{2}$ for x, and divide by 2, thus $\dfrac{1}{\pi}=\dfrac{1}{4}\tan\dfrac{\pi}{4}+\dfrac{1}{8}\tan\dfrac{\pi}{8}+\ldots$.

250 Put $-\dfrac{\theta^2}{\pi^2}$ for x, then we require the coefficient of $\left(-\dfrac{\theta^2}{\pi^2}\right)^n$, that is of $\dfrac{(-1)^n\theta^{2n}}{\pi^{2n}}$ in the development of

$$\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots$$

Thus we require the coefficient of $\dfrac{(-1)^n\theta^{2n+1}}{\pi^{2n}}$ in the development of

$$\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots,$$

that is in $\sin\theta$. See Art. 311.

But the general term in the expansion of $\sin\theta$ is $\dfrac{(-1)^n\theta^{2n+1}}{\lfloor 2n+1}$.

Hence $\dfrac{1}{\pi^{2n}}\times$ the required coefficient $=\dfrac{1}{\lfloor 2n+1}$, so that the required coefficient is $\dfrac{\pi^{2n}}{\lfloor 2n+1}$

251 Proceed as in Example 241 We have

$$1+m+n=p\sin\theta+q\cos\theta,\quad \cos^2\theta-\sin^2\theta+n-m=q\cos\theta-p\sin\theta,$$

therefore $$(1+m+n)(2\cos^2\theta-1+n-m)=q^2\cos^2\theta-p^2\sin^2\theta,$$

therefore $$\cos^2\theta=\frac{p^2+n^2-(1+m)^2}{p^2+q^2-2(1+m+n)},$$

and $$\sin^2\theta=\frac{q^2+m^2-(1+n)^2}{p^2+q^2-2(1+m+n)}.$$

Substitute in $1+m+n=p\sin\theta+q\cos\theta$, and the elimination will be effected.

252 Let D, E, F be the points at which the bisectors of the angles A, B, C respectively meet the circumference Then the angle $DAC=\frac{1}{2}A$, and the angle CAE=the angle $CBE=\frac{1}{2}B$, therefore $DAE=\frac{1}{2}(A+B)$, and therefore DE subtends at the centre of the circle an angle equal to $A+B$. thus $DE=2R\sin\frac{1}{2}(A+B)=2R\cos\frac{1}{2}C$ Similarly $EF=2R\cos\frac{1}{2}A$, and the angle $DEF=\frac{1}{2}(A+C)$, thus the area of the triangle DEF

$$=\frac{1}{2}\ 4R^2\cos\frac{1}{2}A\cos\frac{1}{2}C\sin\frac{1}{2}(A+C)=2R^2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$$

$$=\frac{2R^2sS}{abc}=\frac{Rs}{2}$$

253. Here $$\sin^{-1}\frac{x}{a}+\sin^{-1}\frac{y}{b}=\sin^{-1}\frac{c^2}{ab}$$

Take the cosines of both sides, thus

$$\sqrt{\left(1-\frac{x^2}{a^2}\right)}\sqrt{\left(1-\frac{y^2}{b^2}\right)}-\frac{xy}{ab}=\sqrt{\left(1-\frac{c^4}{a^2b^2}\right)},$$

therefore $$\sqrt{\left(1-\frac{x^2}{a^2}\right)}\sqrt{\left(1-\frac{y^2}{b^2}\right)}=\frac{xy}{ab}+\sqrt{\left(1-\frac{c^4}{a^2b^2}\right)},$$

square both sides, thus

$$1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}=\frac{x^2y^2}{a^2b^2}+2\frac{xy}{ab}\sqrt{\left(1-\frac{c^4}{a^2b^2}\right)}+1-\frac{c^4}{a^2b^2},$$

therefore $$b^2x^2+a^2y^2+2xy\sqrt{(a^2b^2-c^4)}=c^4.$$

251. Let $PCA=\theta$; then $PCB=\frac{\pi}{2}-\theta$,

$$\frac{PC}{a}=\frac{\sin(\theta+\gamma)}{\sin\gamma},\quad \frac{PC}{b}=\frac{\sin\left(\frac{\pi}{2}-\theta+\gamma\right)}{\sin\gamma},$$

thus $$a\sin(\theta+\gamma)=b\sin\left(\frac{\pi}{2}-\theta+\gamma\right)=b\cos(\theta-\gamma),$$

therefore $$a(\sin\theta\cos\gamma+\cos\theta\sin\gamma)=b(\cos\theta\cos\gamma+\sin\theta\sin\gamma),$$

therefore $$\tan\theta=\frac{b\cos\gamma-a\sin\gamma}{a\cos\gamma-b\sin\gamma}$$

Hence $$\sin\theta=\frac{b\cos\gamma-a\sin\gamma}{\sqrt{\{(a\cos\gamma-b\sin\gamma)^2+(b\cos\gamma-a\sin\gamma)^2\}}}$$
$$=\frac{b\cos\gamma-a\sin\gamma}{\sqrt{(a^2+b^2-2ab\sin2\gamma)}};$$

and $$\cos\theta=\frac{a\cos\gamma-b\sin\gamma}{\sqrt{(a^2+b^2-2ab\sin2\gamma)}}.$$

Then $$PC=\frac{a(\sin\theta\cos\gamma+\cos\theta\sin\gamma)}{\sin\gamma}=\frac{ab\cos2\gamma}{\sin\gamma\sqrt{(a^2+b^2-2ab\sin2\gamma)}}.$$

255. $x^4-x^3+x^2-x+1=\frac{x^5+1}{x+1}$. Hence we must find the roots of $x^5+1=0$, and omit the root -1.

Now if $x^5=-1$ we may put $x^5=\cos n\pi\pm\sqrt{-1}\sin n\pi$, where n is any odd integer. Hence $x=(\cos n\pi\pm\sqrt{-1}\sin n\pi)^{\frac{1}{5}}=\cos\frac{n\pi}{5}\pm\sqrt{-1}\sin\frac{n\pi}{5}$.

Put in succession 1 and 3 for n, thus we obtain the assigned values. If we put 5 for n we obtain the root -1, which we had to omit.

256. Let θ denote the angle opposite to the side 1, then

$$\frac{\sin\theta}{\sin\frac{\pi}{6}}=\frac{1}{250},\quad\text{therefore }\sin\theta=\frac{1}{500}.$$

As θ is very small we may put θ for $\sin\theta$, thus $\theta=\frac{1}{500}$ approximately. Therefore the number of degrees in the angle $=\frac{1}{500}\times\frac{180}{\pi}$, and therefore the number of minutes $=\frac{60}{500}\times\frac{180}{\pi}=\frac{3}{25}\times\frac{180}{\pi}=\frac{3}{25}\times57\cdot3=7$ nearly.

257. First take the inscribed circle; see Art. 218.

$$FE=2r\sin FOA=2r\cos\frac{A}{2},\quad\text{similarly }FD=2r\cos\frac{B}{2}.$$

The angle $EFA=\frac{1}{2}(\pi-A)$, the angle $DFB=\frac{1}{2}(\pi-B)$, therefore the angle $EFD=\frac{1}{2}(A+B)$.

Hence the area of the triangle DFE

$$=\frac{1}{2}\ 4r^2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{A+B}{2}=2r^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{2r^2Ss}{abc}=\frac{rS}{2R}$$

Now take one of the escribed circles, as for instance that opposite to the angle A: see Art 250

$$DF=2r_1\sin DOB=2r_1\sin\frac{B}{2},\quad \text{similarly } DE=2r_1\sin\frac{C}{2}.$$

The angle FDE = the angle FDO + the angle EDO

$$=\frac{1}{2}(\pi-B)+\frac{1}{2}(\pi-C)=\pi-\frac{1}{2}(B+C)$$

Hence the area of the triangle DFE

$$=\frac{1}{2}\ 4r_1^2\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{B+C}{2}$$

$$=2r_1^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}=\frac{2r_1^2(s-a)S}{abc}=\frac{2r_1S^2}{abc}=\frac{r_1S}{2R}$$

258 Proceed as in Example 248, thus we obtain $\sin^2\theta=\cos^2\theta$.

If we take $\sin\theta=\cos\theta$ we get $1=\cos\alpha+k\sin\alpha$, thus

$$k=\frac{1-\cos\alpha}{\sin\alpha}=\tan\frac{\alpha}{2},\quad \text{that is}\quad \frac{e^\phi-e^{-\phi}}{e^\phi+e^{-\phi}}=\tan\frac{\alpha}{2};$$

therefore $$e^{2\phi}=\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}=\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

If we take $\sin\theta=-\cos\theta$ we get $1=-\cos\alpha+k\sin\alpha$; thus

$$k=\frac{1+\cos\alpha}{\sin\alpha}=\cot\frac{\alpha}{2},\quad \text{that is}\quad \frac{e^\phi-e^{-\phi}}{e^\phi+e^{-\phi}}=\cot\frac{\alpha}{2},$$

therefore $$e^{2\phi}=\frac{1+\cot\frac{\alpha}{2}}{1-\cot\frac{\alpha}{2}}=\frac{\tan\frac{\alpha}{2}+1}{\tan\frac{\alpha}{2}-1}=-\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right).$$

259 Put the exponential values for the cosines in the series denoted by s: thus

$$s=1+\frac{1}{2}z\,(e^{\iota\theta}+e^{-\iota\theta})+\frac{z^2}{2\,\lfloor 2}(e^{2\iota\theta}+e^{-2\iota\theta})+\frac{z^3}{2\,\lfloor 3}(e^{3\iota\theta}+e^{-3\iota\theta})+$$

$$=\frac{1}{2}(e^x+e^y),$$

where $$x=ze^{\iota\theta}\ \ =z\,(\cos\theta+\iota\sin\theta),$$

and $$y=ze^{-\iota\theta}=z\,(\cos\theta-\iota\sin\theta)$$

Thus $$s=\frac{1}{2}e^{z\cos\theta}\,(e^{\iota z\sin\theta}+e^{-\iota z\sin\theta})=e^{z\cos\theta}\cos(z\sin\theta).$$

Similarly we find that $\sigma=\frac{1}{2\iota}(e^x-e^y)=e^{z\cos\theta}\sin(z\sin\theta)$

Therefore $$\frac{\sigma}{s}=\frac{\sin(z\sin\theta)}{\cos(z\sin\theta)}=\tan(z\sin\theta),$$

so that $$z\sin\theta=\tan^{-1}\frac{\sigma}{s}.$$

And $$s^2+\sigma^2=e^{2z\cos\theta}\{\cos^2(z\sin\theta)+\sin^2(z\sin\theta)\}=e^{2z\cos\theta},$$

so that $$z\cos\theta=\frac{1}{2}\log(s^2+\sigma^2)$$

If $\theta=\frac{\pi}{2}$, we have $\frac{\sigma}{s}=\tan z$ and $s^2+\sigma^2=1$, so that $\sigma=\sin z$ and $s=\cos z$

260 Let c be the distance of the two given points, n the number of sides in the polygon; and put $\beta=\frac{2\pi}{n}$ Let α be the angle which the distance between the two given points makes with the first straight line which is drawn Then the numerical values of the successive perpendiculars are $c\sin\alpha$, $c\sin(\alpha+\beta)$, $c\sin(\alpha+2\beta)$, Hence the sum of the squares on the perpendiculars

$$=c^2\{\sin^2\alpha+\sin^2(\alpha+\beta)+\sin^2(\alpha+2\beta)+\quad\text{to } n \text{ terms}\}$$

$$=\frac{c^2}{2}\{1-\cos 2\alpha+1-\cos 2(\alpha+\beta)+1-\cos 2(\alpha+2\beta)+\qquad\}$$

$$=\frac{nc^2}{2}$$ See Art 328

261 We have

$$a\sin\theta+b=h\cos\theta,\ \text{and}\ \cos\theta\,(a+b\sin\theta)=k\sin\theta$$

Find $\cos\theta$ from the first equation, and substitute it in the second, thus we get

$$\sin^2\theta+\frac{a^2+b^2-hk}{ab}\sin\theta+1=0$$

Again, find $\sin\theta$ from the first equation, and substitute it in the second; thus we get

$$\cos^2\theta + \frac{a^2-b^2-hk}{bh}\cos\theta + \frac{k}{h} = 0$$

Then we may employ the process of Example 251

262 By Example XVI 50 we know that the sides of the new triangle are respectively $a\cos A$, $b\cos B$, and $c\cos C$ Thus the perimeter

$$= a\cos A + b\cos B + c\cos C = 4R\sin A\sin B\sin C, \text{ by Example XVI 22,}$$

$$= \frac{4R \quad 8S^3}{(abc)^2} = \frac{2S}{R}$$

263 As in Example 252 we shew that the area of the triangle thus formed is $2R^2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$, denote this by Σ

Also $$S = \frac{1}{2}ab\sin C = 2R^2\sin A\sin B\sin C.$$

Hence $$\frac{\Sigma}{S} = \frac{\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}{\sin A\sin B\sin C} = \frac{1}{8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

Now, as in Example XIII 40 we see that $8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ cannot be greater than unity, and therefore S cannot be greater than Σ.

264 $$r_1 = \frac{S}{s-a}, \quad r_2 = \frac{S}{s-b}, \quad r_3 = \frac{S}{s-c}.$$

Hence

$$(r_1+r_2)(r_2+r_3)(r_3+r_1) = S^3\left(\frac{1}{s-a}+\frac{1}{s-b}\right)\left(\frac{1}{s-b}+\frac{1}{s-c}\right)\left(\frac{1}{s-c}+\frac{1}{s-a}\right)$$

$$= \frac{S^3abc}{(s-a)^2(s-b)^2(s-c)^2} = \frac{s^2S^3abc}{S^4} = \frac{s^2abc}{S}.$$

And $$r_1r_2 + r_2r_3 + r_3r_1 = S^2\left\{\frac{1}{(s-a)(s-b)} + \frac{1}{(s-b)(s-c)} + \frac{1}{(s-c)(s-a)}\right\}$$

$$= \frac{S^2(3s-a-b-c)}{(s-a)(s-b)(s-c)} = \frac{S^2s^2}{S^2} = s^2$$

Divide the first result by the second, and thus we get $\frac{abc}{S}$.

265 The wall a feet high casts a shadow which extends $a\cot\alpha$ feet from the wall *measured in the direction of the meridian*, hence $a\cot\alpha\sin\beta$ is the breadth of the shadow *measured in the direction at right angles to the wall*

Thus $b = a \cot \alpha \sin \beta$ Similarly $b' = a' \cot \alpha \sin (\gamma - \beta)$.

From these two equations we have to find α and β.

We get $\dfrac{a \sin \beta}{b} = \dfrac{a' \sin (\gamma - \beta)}{b'}$, so that

$$\frac{a}{b} = \frac{a'}{b'}(\sin \gamma \cot \beta - \cos \gamma),$$

therefore
$$\cot \beta = \cot \gamma + \frac{ab'}{a'b} \operatorname{cosec} \gamma$$

Then
$$\cot^2 \alpha = \frac{b^2}{a^2 \sin^2 \beta} = \frac{b^2}{a^2}(1 + \cot^2 \beta) = \frac{b^2}{a^2}\left\{1 + \left(\cot \gamma + \frac{ab'}{a'b} \operatorname{cosec} \gamma\right)^2\right\}$$
$$= \frac{b^2}{a^2}\left\{1 + \cot^2 \gamma + \frac{a^2 b'^2}{a'^2 b^2} \operatorname{cosec}^2 \gamma + \frac{2ab'}{a'b} \cot \gamma \operatorname{cosec} \gamma\right\}$$
$$= \left(\frac{b^2}{a^2} + \frac{b'^2}{a'^2}\right) \operatorname{cosec}^2 \gamma + \frac{2bb'}{aa'} \cot \gamma \operatorname{cosec} \gamma$$

260 Assume $a = r \cos \theta$, and $b = r \sin \theta$, so that $r^2 = a^2 + b^2$, and $\tan \theta = \dfrac{b}{a}$ Also assume $\alpha = \rho \cos \phi$, $\beta = \rho \sin \phi$, so that $\rho^2 = \alpha^2 + \beta^2$, and $\tan \phi = \dfrac{\beta}{\alpha}$

Then the proposed expression

$$= (r \cos \theta + \iota r \sin \theta)^{\rho \cos \phi + \iota \rho \sin \phi} = (re^{\iota\theta})^{\rho \cos \phi + \iota \rho \sin \phi}$$

Denote this by u, then

$$\log u = (\rho \cos \phi + \iota \rho \sin \phi) \log (re^{\iota\theta})$$
$$= (\rho \cos \phi + \iota \rho \sin \phi)(\iota\theta + \log r)$$
$$= \rho(\cos \phi \log r - \theta \sin \phi) + \rho\iota(\sin \phi \log r + \theta \cos \phi)$$
$$= \sigma + \iota\tau \text{ say},$$

therefore
$$u = e^{\sigma + \iota\tau} = e^{\sigma} e^{\iota\tau} = e^{\sigma}(\cos \tau + \iota \sin \tau)$$

To make this wholly real the term involving ι must vanish, therefore $\sin \tau$ must vanish, therefore τ must be zero or a multiple of π, therefore $\rho(\sin \phi \log r + \theta \cos \phi)$ must be zero or an even multiple of $\dfrac{\pi}{2}$, but $\rho \sin \phi = \beta$, and $\rho \cos \phi = \alpha$, so that $\dfrac{\beta}{2} \log (a^2 + b^2) + \alpha \tan^{-1} \dfrac{b}{a}$ must be zero or an even multiple of $\dfrac{\pi}{2}$.

267 $$\frac{a^2}{a^2\cos^2\theta+b^2\sin^2\theta}=\frac{2a^2}{a^2(1+\cos 2\theta)+b^2(1-\cos 2\theta)}$$

$$=\frac{2a^2}{a^2+b^2+(a^2-b^2)\cos 2\theta}$$

$$=\frac{4a^2}{2(a^2+b^2)+(a^2-b^2)(e^{2\theta\iota}+e^{-2\theta\iota})}=\frac{4a^2}{(a+b)^2(1+ce^{2\theta\iota})(1+ce^{-2\theta\iota})}$$

$$=\frac{(1+c)^2}{(1+ce^{2\theta\iota})(1+ce^{-2\theta\iota})}.$$

Therefore

$$\log\frac{a^2}{a^2\cos^2\theta+b^2\sin^2\theta}=2\log(1+c)-\log(1+ce^{2\theta\iota})-\log(1+ce^{-2\theta\iota})$$

$$=2\left\{c-\frac{c^2}{2}+\frac{c^3}{3}-\frac{c^4}{4}+\quad\right\}$$

$$-\left\{ce^{2\theta\iota}-\frac{c^2}{2}e^{4\theta\iota}+\frac{c^3}{3}e^{6\theta\iota}-\frac{c^4}{4}e^{8\theta\iota}+\quad\right\}$$

$$-\left\{ce^{-2\theta\iota}-\frac{c^2}{2}e^{-4\theta\iota}+\frac{c^3}{3}e^{-6\theta\iota}-\frac{c^4}{4}e^{-8\theta\iota}+\quad\right\}$$

The term which involves c is $-c(e^{\theta\iota}-e^{-\theta\iota})^2$, that is $4c\sin^2\theta$.

The term which involves c^2 is $\frac{c^2}{2}(e^{2\theta\iota}-e^{-2\theta\iota})^2$, that is $-\frac{4c^2}{2}\sin^2 2\theta$

The term which involves c^3 is $-\frac{c^3}{3}(e^{3\theta\iota}-e^{-3\theta\iota})^2$, that is $\frac{4c^3}{3}\sin^2 3\theta$

And so on

Thus we obtain the required result

268 Let O denote the centre of the inscribed circle, D and E the centres of the escribed circles Then D, C, E are on a straight line which is at right angles to OC The area of the triangle ODE

$$=\frac{1}{2}OC\quad DE=\frac{1}{2}r\operatorname{cosec}\frac{C}{2}(r_1+r_2)\sec\frac{C}{2}$$

$$=\frac{r(r_1+r_2)}{\sin C}=\frac{S}{s\sin C}\left(\frac{S}{s-a}+\frac{S}{s-b}\right)=\frac{S^2c}{s(s-a)(s-b)\sin C}$$

$$=\frac{(s-c)c}{\sin C}=\frac{abc\cos^2\frac{1}{2}C}{s\sin C}=\frac{abc}{2s}\cot\frac{C}{2}.$$

269 The angle $OBC=\frac{1}{2}B$, and the angle $OCB=\frac{1}{2}C$ Hence, as on page 187, line 5 of the *Trigonometry*, we have

$$r_a\left(\cot\frac{B}{4}+\cot\frac{C}{4}\right)=a,$$

therefore
$$\frac{a}{r_a}=\cot\frac{B}{4}+\cot\frac{C}{4}$$

Similarly $\frac{b}{r_b}=\cot\frac{C}{4}+\cot\frac{A}{4}$, and $\frac{c}{r_c}=\cot\frac{A}{4}+\cot\frac{B}{4}$.

Hence by addition we get the required result

270 We easily see that $\tan^{-1}\frac{1}{2r^2}=\tan^{-1}(2r+1)-\tan^{-1}(2r-1)$

Resolve each of the given terms into two by this formula Then by addition we find that the sum $=\tan^{-1}(2n+1)-\tan^{-1}1=\tan^{-1}\frac{n}{n+1}$

271
$$\cot A=\frac{\cos A}{\sin A}=\frac{b^2+c^2-a^2}{2bc\sin A}=\frac{b^2+c^2-a^2}{4S},$$

similar expressions hold for $\cot B$ and $\cot C$ Thus

$$\cot A+\cot B+\cot C=\frac{a^2+b^2+c^2}{4S}$$

Hence if S be given the sum of the cotangents of the angles varies as the sum of the squares of the sides.

272 By Art 251 we have

$$OI^2=R^2-2Rr,$$

and
$$OD^2+OE^2+OF^2=3R^2+2R(r_1+r_2+r_3).$$

Thus by addition we obtain

$$OI^2+OD^2+OE^2+OF^2=4R^2+2R(r_1+r_2+r_3-r)$$
$$=4R^2+8R^2, \text{ by Example 201, } =12R^2$$

273 Let $\theta=\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$, then $\tan\theta=\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}$,

$$\cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{a+b-(a-b)\tan^2\frac{x}{2}}{a+b+(a-b)\tan^2\frac{x}{2}}$$

$$=\frac{b+a\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}{a+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}=\frac{b+a\cos x}{a+b\cos x}$$

274 Let O denote the centre of the circles Let $ABCD$ be a straight line cutting the outer circumference at A and D, and the inner circumference at B and C Let $OB=r$, and $OA=nr$ Let the angle $AOD=2\alpha$, and the angle $BOC=2\beta$, so that the angle $AOB=\alpha-\beta$

Then $$AB^2=n^2r^2+r^2-2nr^2\cos(\alpha-\beta)$$

Now $$\frac{AB}{BD}=\frac{AB^2}{AB\ BD}=\frac{AB^2}{AB\,.\,AC}.$$

But $AB\ AC$ = the square on the straight line drawn from A to touch the inner circumference $=(n^2-1)r^2$

Therefore $$\frac{AB}{BD}=\frac{n^2-2n\cos(\alpha-\beta)+1}{n^2-1}$$

275 Proceed as in the solution of Example 266 That the expression may be wholly imaginary we must have $\cos\tau=0$, and therefore τ must be an odd multiple of $\frac{\pi}{2}$, therefore $\rho(\sin\phi\log r+\theta\cos\phi)$ must be an odd multiple of $\frac{\pi}{2}$, but $\rho\sin\phi=\beta$, and $\rho\cos\phi=\alpha$, so that $\frac{\beta}{2}\log(a^2+b^2)+\alpha\tan^{-1}\frac{b}{a}$ must be an odd multiple of $\frac{\pi}{2}$

276 $$\frac{1}{a}\left(\tan\frac{C}{2}+\tan\frac{B}{2}\right)=\frac{1}{a}\left(\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}+\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}\right)=\frac{\cos\frac{A}{2}}{a\cos\frac{C}{2}\cos\frac{B}{2}}$$

$$=\frac{1}{r}\tan\frac{C}{2}\tan\frac{B}{2},\ \text{by Art 249}$$

In this way we find that the proposed expression

$$=\frac{abc}{2r}\left\{\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}\right\}$$

$$=\frac{abc}{2r},\ \text{by Example VIII 15},$$

and this is the area of the triangle by Example XVI 34

Or thus Let I denote the centre of the inscribed circle, O the centre of the escribed circle opposite to A, then the area of the quadrilateral $IBOC=\frac{a}{2}(r+r_1)=\frac{a}{2}(s-a+s)\tan\frac{A}{2}=\frac{a}{2}(b+c)\tan\frac{A}{2}$ see Arts 249 and 250

In this way we obtain for the whole required area the given expression

277. We have
$$1+2\cos\theta=\frac{\sin\frac{3\theta}{2}}{\sin\frac{\theta}{2}},$$
$$1+2\cos 3\theta=\frac{\sin\frac{3^2\theta}{2}}{\sin\frac{3\theta}{2}},$$

and so on Then, as the sum of the logarithms of any set of quantities is equal to the logarithm of the product of those quantities, we see that the required sum is the logarithm of

$$\frac{\sin\frac{3\theta}{2}}{\sin\frac{\theta}{2}}\cdot\frac{\sin\frac{3^2\theta}{2}}{\sin\frac{3\theta}{2}}\quad\ldots\quad\frac{\sin\frac{3^n\theta}{2}}{\sin\frac{3^{n-1}\theta}{2}},\text{ that is the logarithm of }\frac{\sin\frac{3^n\theta}{2}}{\sin\frac{\theta}{2}}$$

278 Put β for $\frac{\pi}{n}$ The path consists of a set of arcs of circles, each of which corresponds to the angle 2β, and the radii of which are the respective distances of any assumed point from all the other angular points. The radii thus are $2R\sin\beta$, $2R\sin 2\beta$, $2R\sin 3\beta$,

Hence the required sum

$$=2R\{\sin\beta+\sin 2\beta+\sin 3\beta+\quad\ldots\quad+\sin n\beta\}\,2\beta$$

The term $\sin n\beta$ is zero, and may be omitted if we please

By Art 303 this expression

$$=4R\beta\frac{\sin\left(\beta+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{1}{2}\beta}=4R\beta\frac{\sin\frac{n+1}{2n}\pi}{\sin\frac{\pi}{2n}}$$

$$=4R\beta\cot\frac{\pi}{2n}=\frac{4R\pi}{n}\cot\frac{\pi}{2n}$$

279 The sum of the areas of all the sectors will be

$$4R^2\{\sin^2\beta+\sin^2 2\beta+\sin^2 3\beta+\quad\ldots\quad+\sin^2 n\beta\}\,\beta$$
$$=2R^2\beta\{1-\cos 2\beta+1-\cos 4\beta+\quad\ldots\quad\}$$
$$=2R^2\beta\left\{n-\frac{\cos(2\beta+\overline{n-1}\beta)\sin n\beta}{\sin\beta}\right\}=2R^2n\beta=2R^2\pi$$

If we wish to have the whole area of the figure bounded by the straight line and by the arcs between two points where they cross the straight line, we must add to the above a set of triangles which make up the whole polygon, that is $\frac{n}{2}R^2\sin\frac{2\pi}{n}$

280 We have

$$\frac{\sin 2r\theta}{\sin(2r-1)\theta\sin(2r+1)\theta}=\frac{1}{2\cos\theta}\left\{\frac{1}{\sin(2r-1)\theta}+\frac{1}{\sin(2r+1)\theta}\right\}.$$

If we resolve each term of the proposed series into two by the aid of this formula we find that the sum of $2n$ terms $=\dfrac{1}{2\cos\theta}\left\{\dfrac{1}{\sin\theta}-\dfrac{1}{\sin(4n+1)\theta}\right\}$

281 As in Example 214 we have

$$\alpha=\frac{a\cos A}{\sin A},\qquad \beta=\frac{b\cos B}{\sin B},\qquad \gamma=\frac{c\cos C}{\sin C}.$$

Hence $\dfrac{1}{4}(a\alpha+b\beta+c\gamma)=\dfrac{1}{4}\left(\dfrac{a^2\cos A}{\sin A}+\dfrac{b^2\cos B}{\sin B}+\dfrac{c^2\cos C}{\sin C}\right)$

$$=R^2(\sin A\cos A+\sin B\cos B+\sin C\cos C)$$

$$=\frac{R^2}{2}(\sin 2A+\sin 2B+\sin 2C)=2R^2\sin A\sin B\sin C,\text{ by Art 114,}$$

$$=\frac{1}{2}ab\sin C,\text{ by Art 252, } =S$$

Also, $a^2\alpha\operatorname{cosec}A+b^2\beta\operatorname{cosec}B+c^2\gamma\operatorname{cosec}C=\dfrac{a^3\cos A}{\sin^2 A}+\dfrac{b^3\cos B}{\sin^2 B}+\dfrac{c^3\cos C}{\sin^2 C}$

$$=8R^3(\sin A\cos A+\sin B\cos B+\sin C\cos C)$$

$$=4R^3(\sin 2A+\sin 2B+\sin 2C)=16R^3\sin A\sin B\sin C$$

$$=8RS,\text{ by the former part of the Example, } =2abc$$

282 We obtain immediately from a diagram

$$2r'=R(1-\cos A),\quad 2r''=R(1-\cos B),\quad 2r'''=R(1-\cos C),$$

hence $$8r'r''r'''=8R^3\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}=\frac{8R^3S^4}{a^2b^2c^2s^2}.$$

Therefore $$64Rr'r''r'''=\frac{64R^4S^4}{a^2b^2c^2s^2}=\frac{a^2b^2c^2}{4s^2}=\left(\frac{abc}{a+b+c}\right)^2$$

283 Let $\theta=\sin^{-1}\dfrac{\sqrt{(x^2-c^2)}}{\sqrt{(a^2-c^2)}}$ and $\phi=\sin^{-1}\dfrac{c\sqrt{(a^2-x^2)}}{x\sqrt{(a^2-c^2)}}$,

then $$\cos\theta=\frac{\sqrt{(a^2-x^2)}}{\sqrt{(a^2-c^2)}}\quad\text{and}\quad\cos\phi=\frac{a\sqrt{(x^2-c^2)}}{x\sqrt{(a^2-c^2)}},$$

therefore $$\sin(\theta-\phi)=\frac{\sqrt{(x^2-c^2)}}{\sqrt{(a^2-c^2)}}\cdot\frac{a\sqrt{(x^2-c^2)}}{x\sqrt{(a^2-c^2)}}-\frac{\sqrt{(a^2-x^2)}}{\sqrt{(a^2-c^2)}}\cdot\frac{c\sqrt{(a^2-x^2)}}{x\sqrt{(a^2-c^2)}}$$

$$=\frac{a(x^2-c^2)}{x(a^2-c^2)}-\frac{c(a^2-x^2)}{x(a^2-c^2)}=\frac{x^2(a+c)-ac(a+c)}{x(a^2-c^2)}=\frac{x^2-ac}{x(a-c)}.$$

284. Suppose D the middle point of BC. Then

$$AB^2 = AD^2 + BD^2 - 2AD \cdot BD \cos ADB,$$
$$AC^2 = AD^2 + CD^2 - 2AD \cdot CD \cos ADC,$$

therefore by addition $b^2 + c^2 = 2h^2 + \frac{a^2}{2}$, so that $h^2 = \frac{1}{2}(b^2 + c^2) - \frac{a^2}{4}$;

similarly $k^2 = \frac{1}{2}(c^2 + a^2) - \frac{b^2}{4}$, and $l^2 = \frac{1}{2}(a^2 + b^2) - \frac{c^2}{4}$.

Therefore by addition $4(h^2 + k^2 + l^2) = 3(a^2 + b^2 + c^2)$.

Also $(4h^2)^2 + (4k^2)^2 + (4l^2)^2$

$$= (2b^2 + 2c^2 - a^2)^2 + (2c^2 + 2a^2 - b^2)^2 + (2a^2 + 2b^2 - c^2)^2$$
$$= 9(a^4 + b^4 + c^4), \text{ by development.}$$

Again, from what has been already shewn,

$$16(h^2 + k^2 + l^2)^2 = 9(a^2 + b^2 + c^2)^2,$$

and $$16(h^4 + k^4 + l^4) = 9(a^4 + b^4 + c^4),$$

subtract and divide by 2, thus

$$16(h^2k^2 + k^2l^2 + l^2h^2) = 9(a^2b^2 + b^2c^2 + c^2a^2).$$

285. The area of the triangle which can be formed with the straight lines h, k, l, by Arts. 247 and 218,

$$= \frac{1}{4}\sqrt{(2h^2k^2 + 2k^2l^2 + 2l^2h^2 - h^4 - k^4 - l^4)}$$
$$= \frac{1}{4}\sqrt{\frac{9}{16}(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4)}$$
$$= \frac{3}{16}\sqrt{(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4)} = \frac{3}{4}S.$$

286. $a\cos\theta - b\cos(\theta - \alpha) = \cos\theta(a - b\cos\alpha) - b\sin\alpha\sin\theta$,

assume $a - b\cos\alpha = k\cos\beta$, and $b\sin\alpha = k\sin\beta$,

then $a\cos\theta - b\cos(\theta - \alpha) = k(\cos\theta\cos\beta - \sin\theta\sin\beta) = k\cos(\theta + \beta)$.

Similarly

$$a\sin\theta - b\sin(\theta - \alpha) = \sin\theta(a - b\cos\alpha) + b\sin\alpha\cos\theta = k\sin(\beta + \theta).$$

Thus the proposed expression

$$= \{k\cos(\theta + \beta) + k\sqrt{-1}\sin(\theta + \beta)\}^{\frac{1}{n}}$$
$$= k^{\frac{1}{n}}\left\{\cos\frac{\theta + \beta}{n} + \sqrt{-1}\sin\frac{\theta + \beta}{n}\right\}.$$

287 Denote the point by O, and let OD, OE, OF be the perpendiculars on BC, CA, AB respectively Then OA is the diameter of the circle which would go round $OEAF$, so that $OA=\dfrac{EF}{\sin A}$, by Art. 252 Therefore

$$OA^2\ a \sin A = a\ OA\ EF = a\,(OE \cdot FA + OF\ AE),\text{ by Euclid VI D,}$$
$$= a\ \beta\ AF + a\ \gamma\ AE$$

Transform the other two terms in like manner, thus we obtain

$$\alpha\beta\,(AF+BF)+\beta\gamma\,(BD+CD)+\gamma\alpha\,(AE+EC)=\alpha\beta c+\beta\gamma a+\gamma a b$$

288. We have $\quad \alpha\beta c=\dfrac{c}{\sin C}\alpha\beta \sin C=2R\times 2$ area of OED

Transform the other two terms similarly, thus we obtain

$$4R\,(\text{area of } OED + \text{area of } ODF + \text{area of } OFE)$$

289 We have

$$\frac{1}{\cos^2 B}-\frac{1}{\cos^2 A}=\frac{\cos^2 A-\cos^2 B}{\cos^2 A\cos^2 B}=\frac{\sin^2 B-\sin^2 A}{\cos^2 B\cos^2 A}=\frac{\sin(B-A)\sin(B+A)}{\cos^2 B\cos^2 A},$$

so that $$\frac{\sin(B+A)}{\cos^2 B\cos^2 A}=\frac{1}{\sin(B-A)}\left\{\frac{1}{\cos^2 B}-\frac{1}{\cos^2 A}\right\}$$

Apply this transformation to every term of the proposed series, then we find that the sum

$$=\frac{1}{\sin\theta}\left\{\frac{1}{\cos^2 n\theta}-\frac{1}{\cos^2 0}\right\}=\frac{1}{\sin\theta}\left\{\frac{1}{\cos^2 n\theta}-1\right\}=\operatorname{cosec}\theta\tan^2 n\theta$$

290 By De Moivre's Theorem the equation becomes

$$\cos(\theta+2\theta+\quad+n\theta)+\sqrt{-1}\sin(\theta+2\theta+\quad+n\theta)=1,$$

that is $$\cos\frac{n(n+1)}{2}\theta+\sqrt{-1}\sin\frac{n(n+1)}{2}\theta=1$$

Hence we must have $\cos\dfrac{n(n+1)}{2}\theta=1$, and $\sin\dfrac{n(n+1)}{2}\theta=0$, so that $\dfrac{n(n+1)}{2}\theta=2m\pi$, where m is zero or any integer

291 We have $r'=\dfrac{BC}{2\sin BDC}=\dfrac{a}{2\sin\dfrac{B+C}{2}}=\dfrac{a}{2\cos\dfrac{A}{2}}$, and similar expressions hold for r'' and r''' Thus

$$r'r''r'''=\frac{abc}{8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{a^2b^2c^2}{8sS}=\frac{16R^2S^2}{8sS}=\frac{2R^2S}{s}=2R^2r$$

292 We have $R=\frac{a}{2\sin A}$, so that $R\sin A=\frac{a}{2}$ (1)

Suppose that in consequence of the error γ in C there is an error α in A, and an error ρ in R Thus

$$(R+\rho)\sin(A+\alpha)=\frac{a}{2},$$

therefore approximately by Art 181

$$(R+\rho)(\sin A+\alpha\cos A)=\frac{a}{2} \qquad (2)$$

From (1) and (2) by subtraction, neglecting the product $\alpha\rho$,

$$\alpha R\cos A+\rho\sin A=0,$$

so that
$$\alpha=-\frac{\rho}{R}\tan A$$

Similarly if β be the error in B arising from the error γ in C, we have

$$\beta=-\frac{\rho}{R}\tan B$$

But $\alpha+\beta+\gamma=0$, since the sum of the three angles of a triangle is equal to a fixed quantity, namely two right angles

Thus
$$\gamma-\frac{\rho}{R}(\tan A+\tan B)=0,$$

therefore
$$\rho=\frac{R\gamma}{\tan A+\tan B}=\frac{R\gamma\cos A\cos B}{\sin(A+B)}=\frac{c\gamma\cos A\cos B}{2\sin^2 C}$$

And since $\sin C=\frac{c\sin A}{a}$ and $=\frac{c\sin B}{b}$, we have $\rho=\frac{ab\gamma\cot A\cot B}{2c}$

293 Let C denote the right angle, CA and CB the equal sides, produce CA to D and CB to E, then since DE is n times AB, it follows that CD and and CE are each n times CA or CB. Let AE and BD intersect at O. Then the angle DOA = the sum of the angles OBA and OAB, and these are equal; so that the angle DOA = twice the angle OAB But the angle OAB = the angle EAC − the angle BAC, so that

$$\tan OAB=\tan(EAC-BAC)=\frac{\tan EAC-\tan BAC}{1+\tan EAC\tan BAC}=\frac{n-1}{1+n}$$

294 As θ continually increases from 0 to $\frac{\pi}{2}$ the value of $\cos\theta$ continually decreases from 1 to 0, so that there must be one value of θ, and only one, in this range, which makes $\theta=\cos\theta$ Also as $\cos\theta$ is greater than θ when $\theta=0$, and is less than θ when $\theta=\frac{\pi}{4}$, this value is less than $\frac{\pi}{4}$

As θ changes from 0 to $-\frac{\pi}{2}$, the cosine is always positive, and so we cannot have $\cos\theta=\theta$

When θ is numerically greater than $\frac{\pi}{2}$ it is numerically greater than unity, and so cannot be equal to $\cos\theta$

Hence there must be one, and only one, solution of the equation $\theta=\cos\theta$

295 Suppose β the circular measure of an angle between 0 and $\frac{\pi}{2}$, which is greater than the solution of $\theta=\cos\theta$, so that $\beta-\cos\beta$ is positive Let $\beta-\alpha$ denote the solution, so that $\beta-\alpha=\cos(\beta-\alpha)=\cos\beta\cos\alpha+\sin\beta\sin\alpha$, therefore $\alpha=\dfrac{\beta-\cos\beta\cos\alpha}{1+\sin\beta\dfrac{\sin\alpha}{\alpha}}$. Now $\frac{\sin\alpha}{\alpha}$ is less than unity, and so is $\cos\alpha$, hence $\frac{\beta-\cos\beta}{1+\sin\beta}$ is less than the true value of α, and is a positive quantity Therefore $\beta-\frac{\beta-\cos\beta}{1+\sin\beta}$ is nearer than β to the solution of the equation, and is still too large

296 As in the solution of Example 248 we get

$$\tan\alpha+\sqrt{-1}\sec\alpha=\frac{\sin\theta+\sqrt{-1}k\cos\theta}{\cos\theta-\sqrt{-1}\lambda\sin\theta},$$

therefore $(\tan\alpha+\sqrt{-1}\sec\alpha)(\cos\theta-\sqrt{-1}\lambda\sin\theta)=\sin\theta+\sqrt{-1}k\cos\theta$,

therefore $$\sin\theta=\tan\alpha\cos\theta+k\sin\theta\sec\alpha,$$

and $$k\cos\theta=\sec\alpha\cos\theta-\lambda\sin\theta\tan\alpha,$$

therefore $(\sin\theta-\tan\alpha\cos\theta)(\cos\theta+\sin\theta\tan\alpha)=\sin\theta\cos\theta\sec^2\alpha$,

therefore $\sin\theta\cos\theta(1-\sec^2\alpha-\tan^2\alpha)=\tan\alpha(\cos^2\theta-\sin^2\theta)$,

therefore $$-\tan\alpha=\frac{\cos 2\theta}{\sin 2\theta}=\cot 2\theta$$

Hence $\cot 2\theta=\cot\left(\frac{\pi}{2}+\alpha\right)$; therefore $2\theta=n\pi+\frac{\pi}{2}+\alpha$

And $$\lambda=\frac{\sin\theta-\tan\alpha\cos\theta}{\sin\theta\sec\alpha}=\frac{\sin(\theta-\alpha)}{\sin\theta},$$

therefore $$\frac{1+\lambda}{1-\lambda}=\frac{\sin(\theta-\alpha)+\sin\theta}{\sin\theta-\sin(\theta-\alpha)}=\tan\left(\theta-\frac{\alpha}{2}\right)\cot\frac{\alpha}{2}.$$

Now $$\tan\left(\theta-\frac{\alpha}{2}\right)=\tan\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)=\pm 1,$$

thus $$\frac{1+\lambda}{1-\lambda}=\pm\cot\frac{\alpha}{2},\quad\text{that is } \epsilon^{2\phi}=\pm\cot\frac{\alpha}{2}$$

297. When the figure is constructed it will be found to have ten sides, five of which are respectively equal to the other five

The sum of five sides will be found to be

$$2r\{\sin 30^\circ+\sin 6^\circ+\sin 24^\circ+\sin 12^\circ+\sin 18^\circ\};$$

and by Art 326 this $=\dfrac{2r\sin(6^\circ+12^\circ)\sin 15^\circ}{\sin 3^\circ}=\dfrac{2r\sin 18^\circ\sin 15^\circ}{\sin 3^\circ}$.

298. The first term $=\dfrac{\cos\theta\,(1+\cos\theta)}{1-\cos 3\theta}=\dfrac{\cos\theta\,(1+\cos\theta)}{(1-\cos\theta)(1+2\cos\theta)^2}$

$$=\frac{\cos\theta\,(1+\cos\theta)+\frac{1}{4}-\frac{1}{4}}{(1-\cos\theta)(1+2\cos\theta)^2}=\frac{\frac{1}{4}}{1-\cos\theta}-\frac{\frac{1}{4}}{(1-\cos\theta)(1+2\cos\theta)^2}$$

$$=\frac{\frac{1}{4}}{1-\cos\theta}-\frac{\frac{1}{4}}{1-\cos 3\theta}.$$

Each term is to be resolved into two in this manner, so that the sum

$$=\frac{1}{4}\left\{\frac{1}{1-\cos\theta}-\frac{1}{1-\cos 3^n\theta}\right\}.$$

299 Put β for $\dfrac{\pi}{n}$. The first perpendicular $=r\sin\phi$, the second perpendicular $=r\sin(\phi+\beta)$, the third $=r\sin(\phi+2\beta)$, and so on Hence the product

$$=r^n\sin\phi\,\sin(\phi+\beta)\,\sin(\phi+2\beta)\quad\ldots\quad\sin(\phi+n\beta-\beta),$$

and this by Art 342 $=\dfrac{r^n}{2^{n-1}}\sin n\phi$

300 Let r denote the radius When all the stones are taken to the centre each stone is carried over a length r, so that the labour may be denoted by nr When all the stones are taken to the position of one stone the labour in like manner may be represented by the sum of the straight lines drawn from one corner of the polygon to all the other corners

Let $\beta=\dfrac{\pi}{n}$ then this sum

$$=2r\{\sin\beta+\sin 2\beta+\sin 3\beta+\ldots+\sin n\beta\}$$

$$=\frac{2r\sin\left(\beta+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}=\frac{2r\sin\frac{n+1}{2}\beta}{\sin\frac{\beta}{2}}=2r\cot\frac{\beta}{2}$$

Hence the required ratio $=\dfrac{nr}{2r\cot\frac{\beta}{2}}=\dfrac{n}{2}\tan\dfrac{\beta}{2}=\dfrac{n}{2}\tan\dfrac{\pi}{2n}$

To find the value of this when n is indefinitely increased we put it in the form $\dfrac{\pi}{4}\dfrac{\tan\frac{\pi}{2n}}{\frac{\pi}{2n}}$, then by Art 118 the limit is $\dfrac{\pi}{4}$

301 $$\sec^2\phi = 1+\tan^2\phi = \frac{(\cos\theta-\cos\alpha)^2+\sin^2\alpha\sin^2\theta}{(\cos\theta-\cos\alpha)^2}$$

$$=\frac{\cos^2\theta-2\cos\theta\cos\alpha+\cos^2\alpha+\sin^2\alpha\,(1-\cos^2\theta)}{(\cos\theta-\cos\alpha)^2}$$

$$=\frac{\cos^2\theta\cos^2\alpha-2\cos\theta\cos\alpha+1}{(\cos\theta-\cos\alpha)^2},$$

$$\pm\cos\phi=\frac{\cos\theta-\cos\alpha}{\cos\theta\cos\alpha-1}$$

But $$\tan\phi=\frac{\sin\alpha\sin\theta}{\cos\theta-\cos\alpha}.$$

$$\pm\sin\phi=\frac{\sin\alpha\sin\theta}{\cos\theta\cos\alpha-1}$$

$$\pm\cos\phi-\cos\alpha=\frac{\cos\theta-\cos\alpha}{\cos\theta\cos\alpha-1}-\cos\alpha$$

$$=\frac{\cos\theta-\cos\alpha-\cos\theta\cos^2\alpha+\cos\alpha}{\cos\theta\cos\alpha-1}$$

$$=\frac{\cos\theta\sin^2\alpha}{\cos\theta\cos\alpha-1},$$

and $$\pm\sin\phi\sin\alpha=\frac{\sin\theta\sin^2\alpha}{\cos\theta\cos\alpha-1}$$

Dividing these two results, we get

$$\tan\theta=\frac{\pm\sin\phi\sin\alpha}{\pm\cos\phi-\cos\alpha}$$

$$=\frac{\sin\phi\sin\alpha}{\cos\phi\mp\cos\alpha}$$

Or thus $$\tan\phi\cos\theta-\sin\alpha\sin\theta=\tan\phi\cos\alpha$$

$$\tan^2\phi\cos^2\theta-2\sin\alpha\tan\phi\sin\theta\cos\theta+\sin^2\alpha\sin^2\theta$$
$$=\tan^2\phi\cos^2\alpha\,(\sin^2\theta+\cos^2\theta)$$

This equation reduces to

$$\tan^2\theta\,(\cos^2\phi-\cos^2\alpha)-2\sin\alpha\sin\phi\cos\phi\tan\theta+\sin^2\phi\sin^2\alpha=0,$$

or, $$\{\tan\theta\,(\cos\phi+\cos\alpha)-\sin\phi\sin\alpha\}\,\{\tan\theta\,(\cos\phi-\cos\alpha)-\sin\phi\sin\alpha\}=0$$

302 $$\frac{b^2p}{a^2q}=\frac{\sin^2B}{\sin^2A}\,\frac{\tan A}{\tan B}=\frac{\sin B\cos B}{\sin A\cos A}=\frac{\sin 2B}{\sin 2A}.$$

$$\frac{b^2p-a^2q}{a^2q}=\frac{\sin 2B-\sin 2A}{\sin 2A}=\frac{2\cos(A+B)\sin(B-A)}{\sin 2A}$$

$$=\frac{2\cos C\sin(A-B)}{\sin 2A}$$

Also
$$\frac{p}{q}=\frac{\sin A\cos B}{\cos A\sin B}$$

$$\frac{p-q}{q}=\frac{\sin(A-B)}{\cos A\sin B}=\frac{a\sin(A-B)}{b\sin A\cos A}$$

By division,
$$\frac{b^2p-a^2q}{a^2(p-q)}=\frac{b}{a}\cos C$$

$$\cos C=\frac{b^2p-a^2q}{ab(p-q)}$$

303 By Art 185 we have

$$\cos(\alpha_1+\alpha_2+\quad+\alpha_n)=\cos\alpha_1\cos\alpha_2\quad\cos\alpha_n(1-s_2+s_4-\quad),$$
$$\sin(\alpha_1+\alpha_2+\quad+\alpha_n)=\cos\alpha_1\cos\alpha_2\quad\cos\alpha_n(s_1-s_3+s_5-\quad)$$

Square and add,

$$\therefore 1=\cos^2\alpha_1\cos^2\alpha_2\quad\cos^2\alpha_n\{(1-s_2+s_4-\quad)^2+(s_1-s_3+s_5-\quad)^2\}$$
$$(1-s_2+s_4-\quad)^2+(s_1-s_3+s_5-\quad)^2$$
$$=\sec^2\alpha_1\sec^2\alpha_2$$
$$=(1+\tan^2\alpha_1)(1+\tan^2\alpha_2)$$
$$=1+S_1+S_2+$$

304 Put $\cos\alpha=x$, $\cos\beta=y$, $\cos\gamma=z$ and transpose, we have to prove that

$$(1-x^2-y^2-z^2+2xyz)^2-2(yz-x)(xy-z)(xz-y)$$
$$+(1-x^2)(yz-x)^2+(1-y^2)(zx-y)^2+(1-z^2)(xy-z)^2$$
$$=(1-x^2)(1-y^2)(1-z^2)$$

If $x=1$ the left-hand side

$$=(y^2+z^2-2yz)^2-2(yz-1)(y-z)(z-y)+(1-y^2)(z-y)^2+(1-z^2)(y-z)^2$$
$$=(y-z)^2(y^2+z^2-2yz+2yz-2+1-y^2+1-z^2)$$
$$=0$$

$1-x$ is a factor of the left-hand side

If $x=-1$ the left-hand side

$$=(y+z)^2(y^2+z^2+2yz-2yz-2+1-y^2+1-z^2)$$
$$=0$$

Hence $1-x^2$, and similarly $1-y^2$, $1-z^2$ are factors

Now $(1-x^2)(1-y^2)(1-z^2)$ contains a term of the sixth degree, therefore there can be no more algebraic factors Also the term without x, y or z is unity; therefore the result follows.

305 Let $\kappa=p\sin^2\beta+q\sin^2\alpha=p\sin^2\gamma+r\sin^2\alpha=q\sin^2\gamma+r\sin^2\beta$

Solve these equations for p, q, r (Ch XIII Ex 39)

$$p=\frac{\kappa\cos\alpha}{\sin\beta\sin\gamma},\quad q=\frac{\kappa\cos\beta}{\sin\alpha\sin\gamma},\quad r=\frac{\kappa\cos\gamma}{\sin\alpha\sin\beta}$$

Also
$$\frac{\kappa}{pq}+\frac{\kappa}{pr}+\frac{\kappa}{qr}=2\left(\frac{\sin^2\alpha}{p}+\frac{\sin^2\beta}{q}+\frac{\sin^2\gamma}{r}\right),$$

or
$$\frac{\sin^2\alpha}{p}+\frac{\sin^2\beta}{q}+\frac{\sin^2\gamma}{r}=\frac{\kappa}{2pqr}(p+q+r)$$

$$(px^2+qy^2+rz^2)\left(\frac{\sin^2\alpha}{p}+\frac{\sin^2\beta}{q}+\frac{\sin^2\gamma}{r}\right)-(x\sin\alpha+y\sin\beta+z\sin\gamma)^2$$
$$=px^2\left(\frac{\sin^2\beta}{q}+\frac{\sin^2\gamma}{r}\right)+qy^2\left(\frac{\sin^2\alpha}{p}+\frac{\sin^2\gamma}{r}\right)+rz^2\left(\frac{\sin^2\alpha}{p}+\frac{\sin^2\beta}{q}\right)$$
$$-2yz\sin\beta\sin\gamma-2zx\sin\gamma\sin\alpha-2xy\sin\alpha\sin\beta$$
$$=\frac{\kappa px^2}{qr}+\frac{\kappa qy^2}{pr}+\frac{\kappa rz^2}{pq}-\frac{2yz}{p}\kappa\cos\alpha-\frac{2zx}{q}\kappa\cos\beta-\frac{2xy}{r}\kappa\cos\gamma$$
$$=\frac{\kappa}{pqr}(p^2x^2+q^2y^2+r^2z^2-2qryz\cos\alpha-2prxz\cos\beta-2pqxy\cos\gamma)$$

Hence we obtain
$$p^2x^2+q^2y^2+r^2z^2-2qryz\cos\alpha-2prxz\cos\beta-2pqxy\cos\gamma$$
$$=\frac{1}{2}(p+q+r)\left\{px^2+qy^2+rz^2-\frac{(x\sin\alpha+y\sin\beta+z\sin\gamma)^2}{\frac{1}{p}\sin^2\alpha+\frac{1}{q}\sin^2\beta+\frac{1}{r}\sin^2\gamma}\right\},$$

and from the given conditions this is independent of x, y, z

306 $(\cos\alpha+\cos\beta+\cos\gamma)^2+(\sin\alpha+\sin\beta+\sin\gamma)^2$
$$=3+2\cos\alpha\cos\beta+2\sin\alpha\sin\beta+\&c$$
$$=3+2\cos(\alpha-\beta)+2\cos(\beta-\gamma)+2\cos(\gamma-\alpha)$$
$$=3-3=0$$

Hence
$$\left.\begin{aligned}\cos\alpha+\cos\beta+\cos\gamma=0\\ \sin\alpha+\sin\beta+\sin\gamma=0\end{aligned}\right\}$$
and
$$(\cos\beta+\cos\gamma)^2+(\sin\beta+\sin\gamma)^2=\cos^2\alpha+\sin^2\alpha.$$
$$\therefore\ 2+2\cos(\beta-\gamma)=1$$
$$\beta-\gamma=\frac{2\pi}{3}$$

Similarly $$\gamma-\alpha=\frac{2\pi}{3}.$$

$$\alpha=\gamma-\frac{2\pi}{3},\ \beta=\gamma+\frac{2\pi}{3}.$$

$$\cos n\alpha+\cos n\beta+\cos n\gamma=\cos\left(n\gamma-\frac{2n\pi}{3}\right)+\cos\left(n\gamma+\frac{2n\pi}{3}\right)+\cos n\gamma$$

$$=\cos n\gamma\left(1+2\cos\frac{2n\pi}{3}\right)$$

If n be not a multiple of 3, $\cos\frac{2n\pi}{3}=-\frac{1}{2}$ and this expression vanishes

If n be a multiple of 3, $\cos\frac{2n\pi}{3}=1$ and

$$\cos n\alpha+\cos n\beta+\cos n\gamma=3\cos n\gamma=3\cos\frac{1}{3}n(\alpha+\beta+\gamma)$$

307 First suppose that $B'C'$ is within the triangle, and let $B'C'$ produced cut AB, AC in P and Q respectively

Let p_1=perpendicular from A on BC

Then $$C'Q=\frac{y}{\sin C}=\frac{aby}{ab\sin C}=\frac{by}{2\Delta}\cdot a,$$

Δ being the area of the triangle ABC

$$B'P=\frac{z}{\sin B}=\frac{acz}{ac\sin C}=\frac{cz}{2\Delta}\cdot a$$

$$\frac{PQ}{a}=\frac{p_1-x}{p_1}=1-\frac{ax}{ap_1}=1-\frac{ax}{2\Delta}$$

$$B'C'=PQ-C'Q-B'P=a\left(1-\frac{ax+by+cz}{2\Delta}\right).$$

Now area $A'B'C'$: area $ABC=B'C'^2 : a^2$,

$$\therefore \text{area } A'B'C'=\Delta\left(1-\frac{ax+by+cz}{2\Delta}\right)^2$$

If $B'C'$ be on the other side of BC we find $B'C'$ and the area of $A'B'C'$ by putting $-x$ for x in the above

Hence the different values of $B'C'$ are

$$a\left(1-\frac{ax+by+cz}{2\Delta}\right),\ a\left(1-\frac{ax-by+cz}{2\Delta}\right),$$

$$a\left(1-\frac{ax-by-cz}{2\Delta}\right),\ a\left(1-\frac{ax+by-cz}{2\Delta}\right),$$

and four more, found by putting $-x$ for x in each of the above

The sum of these is $8a$

Thus the average perimeter is $a+b+c$

The sum of the areas $-\Delta$ is

$$\left(1-\frac{ax+by+cz}{2\Delta}\right)^2+\left(1-\frac{ax+by-cz}{2\Delta}\right)^2$$
$$+\left(1-\frac{ax-by+cz}{2\Delta}\right)^2+\left(1-\frac{ax-by-cz}{2\Delta}\right)^2$$

+four more terms found by putting $-x$ for x in these

$$=2\left\{\left(1-\frac{ax+by}{2\Delta}\right)^2+\frac{c^2z^2}{4\Delta^2}\right\}+2\left\{\left(1-\frac{ax-by}{2\Delta}\right)^2+\frac{c^2z^2}{4\Delta^2}\right\}$$

+two similar terms with $-x$ for x

$$=4\left\{\left(1-\frac{ax}{2\Delta}\right)^2+\frac{b^2y^2}{4\Delta^2}+\frac{c^2z^2}{4\Delta^2}\right\}$$
$$+4\left\{\left(1+\frac{ax}{2\Delta}\right)^2+\frac{b^2y^2}{4\Delta^2}+\frac{c^2z^2}{4\Delta^2}\right\}$$
$$=8\left\{1+\frac{a^2x^2+b^2y^2+c^2z^2}{4\Delta^2}\right\}$$

$$\text{the average area}=\Delta+\frac{a^2x^2+b^2y^2+c^2z^2}{4\Delta}$$

308 The perpendicular to PC through its middle point passes through O_1 and O_2, since PC is a chord of each of the circles round PCB, PCA

Thus the sides of the triangle $O_1O_2O_3$ are perpendicular to PA, PB, PC.

If X, Y, Z be the middle points of PA, PB, PC, then since O_1YPZ lie on a circle, $\angle O_1=\pi-\theta$, hence $O_2O_3=2\rho\sin\theta$, since ρ is the circumradius of the triangle $O_1O_2O_3$

Now area $O_1O_2O_3$=sum of the areas PO_2O_3, PO_3O_1, PO_1O_2

$$=\frac{1}{2}(PX\,.\,O_2O_3+PY\,.\,O_3O_1+PZ\ \ O_1O_2)$$
$$=\frac{1}{4}(x\ \ O_2O_3+y\ \ O_3O_1+z\,.\,O_1O_2)$$
$$=\frac{1}{2}(x\sin\theta+y\sin\phi+z\sin\psi)\ \ \rho$$

Also area $O_1O_2O_3=\frac{1}{2}O_1O_2\,.\,O_1O_3\sin\theta=2\rho^2\sin\theta\sin\phi\sin\psi$.

$$4\rho\sin\theta\sin\phi\sin\psi=x\sin\theta+y\sin\phi+z\sin\psi$$

309 The plane through the tops of the hills cuts the horizontal plane in a straight line, in which A, B, C must lie Let A', B', C' be the feet of the perpendiculars from the tops of the hills on the horizontal plane Then $A'B'$, $B'C'$, $A'C'$ pass through C, A, B

Now $$CA'\ \ CB'=x\ \ y$$

Through B' draw $B'N$ parallel to CBA and cutting $A'C'$ in N.

Hence $$\frac{BC}{B'N}=\frac{CA'}{A'B'}=\frac{x}{x-y},$$

and $$\frac{B'N}{AB}=\frac{B'C'}{AC'}=\frac{y-z}{z}.$$

multiplying, $$\frac{BC}{AB}=\frac{x(y-z)}{z(x-y)},$$

or $$\frac{z(x-y)}{AB}=\frac{x(y-z)}{BC}$$

Each fraction $$=\frac{z(x-y)+x(y-z)}{AB+BC}$$

$$=\frac{y(x-z)}{AC}=\frac{y(z-x)}{CA}$$

310 Let pA, pB, pC be perpendiculars from p on the straight lines qs, qt, qr, and let qt lie between qr, qs Since the angles at A, B, C are right angles, the circle on pq as diameter passes through A, B, C Also $pA=P_{qs}$, $pB=-P_{qt}$, $pC=P_{qr}$ The chord AB subtends at the circumference an angle sqt Hence $AB=pq\sin sqt$, and similarly $BC=pq\sin tqr$, $AC=pq\sin sqr$ Hence it is required to prove that

$$pC\ AB+pA\ BC=pB\ AC,$$

$ABCp$ being a quadrilateral in a circle, this is true by Euc VI, prop D

Let r, s, t denote the angles of the triangle rst, and R the radius of the circumscribing circle

$$\angle prq=\angle qrs-\angle prs=\frac{r}{2}-\left(\frac{\pi}{2}-t\right)=\frac{r}{2}+t-\frac{\pi}{2}$$

$$P_{qr}=R\sin prq=-R\cos\left(\frac{r}{2}+t\right)$$

$$P_{qr}\cos\frac{1}{2}r+P_{qs}\cos\frac{1}{2}s+P_{qt}\cos\frac{1}{2}t$$

$$=-\frac{R}{2}\left\{2\cos\frac{1}{2}r\cos\left(\frac{r}{2}+t\right)+2\cos\frac{1}{2}s\cos\left(\frac{s}{2}+r\right)+2\cos\frac{1}{2}t\cos\left(\frac{t}{2}+s\right)\right\}$$

$$=-\frac{R}{2}\left\{\cos(r+t)+\cos t+\cos(s+r)+\cos r+\cos(s+t)+\cos s\right\}$$

$$=-\frac{R}{2}\left\{-\cos s+\cos t-\cos t+\cos r-\cos r+\cos s\right\}$$

$$=0$$

311 Let O be the centre of the square, N the middle point of BC, let $OP=x=OQ$, hence, if $2a$ be the side of the square,

$$\tan B=\frac{PN}{BN}=\frac{a-x}{a},\ \tan C=\frac{QN}{CN}=\frac{a+x}{a}$$

$$\tan B+\tan C=2,\ \tan B\tan C=1-\frac{x^2}{a^2},$$

and $$\tan B-\tan C=-\frac{2x}{a}$$

Now $$\tan A=-\tan(B+C)=-\frac{\tan B+\tan C}{1-\tan B\tan C}$$

$$=-\frac{2}{1-\left(1-\frac{x^2}{a^2}\right)}=-2\frac{a^2}{x^2}.$$

$$\tan A(\tan B-\tan C)^2=-\frac{2a^2}{x^2}\times\frac{4x^2}{a^2}=-8$$

312 Let X, Y, Z be the centres of the three circles Let the circles whose centres are Y, Z touch BC at M and N, draw ZH perpendicular to YM

Since $$YZ=y+z \text{ and } YH=y-z,$$

$$MN^2=HZ^2=(y+z)^2-(y-z)^2=4yz$$

But $$MN=a-BM-CN=a-y\cot\frac{B}{2}-z\cot\frac{C}{2},$$

and $$a=r\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)$$

$$2\sqrt{yz}=(r-y)\cot\frac{B}{2}+(r-z)\cot\frac{C}{2}.$$

Similarly $$2\sqrt{zx}=(r-z)\cot\frac{C}{2}+(r-x)\cot\frac{A}{2},$$

and $$2\sqrt{xy}=(r-x)\cot\frac{A}{2}+(r-y)\cot\frac{B}{2}$$

Subtract the first of these from the sum of the 2nd and 3rd,

$$(r-x)\cot\frac{A}{2}=-\sqrt{yz}+\sqrt{zx}+\sqrt{xy},$$

which gives the first result

Put $\sqrt{yz}=v+w$, $\sqrt{zx}=u+w$, $\sqrt{xy}=u+v$

$$\cot\frac{A}{2}=\frac{2u}{r-x}$$

Now since $$\frac{1}{2}(A+B+C)=\frac{\pi}{2},$$

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

$$2u(r-y)(r-z)+2v(r-z)(r-x)+2w(r-x)(r-y)-8uvw=0,$$

coefficient of $r^2=2u+2v+2w=\sqrt{yz}+\sqrt{zx}+\sqrt{xy},$

coefficient of $-r=2u(y+z)+2v(z+x)+2w(x+y)$

$$=2x(v+w)+2y(w+u)+2z(u+v)$$

$$=2\sqrt{xyz}(\sqrt{x}+\sqrt{y}+\sqrt{z}),$$

absolute term $=2uyz+2vzx+2wxy-8uvw$

$$=2\{u(v+w)^2+v(w+u)^2+w(u+v)^2-4uvw\}$$

$$=2\{u(v^2+w^2)+v(w^2+u^2)+w(u^2+v^2)+2uvw\}$$

$$=2(u+v)(v+w)(w+u)$$

$$=2xyz.$$

$$r^2(\sqrt{yz}+\sqrt{zx}+\sqrt{xy})-2r\sqrt{xyz}(\sqrt{x}+\sqrt{y}+\sqrt{z})+2xyz=0$$

313. F divides AB in the ratio of b to a, hence

$$AF=\frac{b}{a+b}\,c, \text{ and similarly, } AE=\frac{c}{a+c}\,b$$

Now $$a=2R\sin A,\ EF=2R_1\sin A$$

$$EF^2=\frac{a^2R_1^2}{R^2}$$

Also $$EF^2=AF^2+AE^2-2AF\cdot AE\cos A$$

$$=b^2c^2\left[\frac{1}{(a+b)^2}+\frac{1}{(a+c)^2}\right]-\frac{bc(b^2+c^2-a^2)}{(a+b)(a+c)}$$

$$=b^2c^2\left[\frac{1}{a+b}-\frac{1}{a+c}\right]^2+\frac{bc}{(a+b)(a+c)}(2bc-b^2-c^2+a^2)$$

$$=\frac{bc(b-c)^2}{(a+b)(a+c)}\left\{\frac{bc}{(a+b)(a+c)}-1\right\}+\frac{a^2bc}{(a+b)(a+c)}$$

$$=\frac{abc(a+b+c)(b-c)^2}{(a+b)^2(a+c)^2}+\frac{a^2bc}{(a+b)(a+c)}$$

$$\Sigma\frac{a^2R_1^2}{R^2(b^2-c^2)}=\Sigma\left\{\frac{abc(a+b+c)(b^2-c^2)}{(a+b)^2(a+c)^2(b+c)^2}+\frac{abc}{(a+b)(a+c)(b+c)}\cdot\frac{a}{b-c}\right\}$$

$$=\frac{abc}{(a+b)(b+c)(c+a)}\left[\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right]$$

314. Suppose for convenience of reference that the triangle is within the circle and that O is within the triangle, let $OA=u$, $OB=v$, $OC=w$, $\angle BOC=\alpha$, $\angle COA=\beta$, $\angle AOB=\gamma$, $\angle AOP=\theta$, $\angle BOP=\gamma-\theta$. Then we have

$$AP^2=r^2+u^2-2ru\cos\theta$$

$$BP^2=r^2+v^2-2rv\cos(\gamma-\theta)$$

$$CP^2=r^2+w^2-2rw\cos(\beta+\theta)$$

Multiply these by $\frac{1}{2}vw \sin \alpha$, $\frac{1}{2}uw \sin\beta$, $\frac{1}{2}uv \sin\gamma$ respectively and add, the sum of the terms which depend on the position of P

$$= -ruvw\,\{\cos\theta \sin\alpha + \cos(\gamma-\theta)\sin\beta + \cos(\beta+\theta)\sin\gamma\}$$
$$= -ruvw\,\{\sin\alpha + \sin\beta\cos\gamma + \cos\beta\sin\gamma\}\cos\theta$$
$$= -ruvw\,\{\sin\alpha + \sin(\beta+\gamma)\}\cos\theta$$
$$= 0 \text{ since } \alpha+\beta+\gamma = 2\pi$$

315 Suppose the circle to be the inscribed circle of the triangle ABC, let $AB'C'$ be one of the other triangles, so that the circle is the escribed circle of the triangle $AB'C'$ Let dashed letters apply to the triangle $AB'C'$

The areas of the triangles are proportional to the squares of the sides,

$$\sqrt{\frac{\Delta}{\Delta_1}} = \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{s}{s'} = \frac{s-a}{s'-a'},$$

the radius of the circle $= \frac{\Delta}{s}$, and also $= \frac{\Delta_1}{s'-a'}$

$$\frac{\Delta}{\Delta_1} = \frac{s}{s'-a'}$$

But
$$\sqrt{\frac{\Delta_1}{\Delta}} = \frac{s'-a'}{s-a}$$

$$\frac{s}{s-a} = \sqrt{\frac{\Delta}{\Delta_1}}$$

Thus
$$\frac{s-a}{s} = \sqrt{\frac{\Delta_1}{\Delta}},\quad \frac{s-b}{s} = \sqrt{\frac{\Delta_2}{\Delta}},\quad \frac{s-c}{s} = \sqrt{\frac{\Delta_3}{\Delta}}$$

by addition,
$$1 = \frac{\sqrt{\Delta_1}+\sqrt{\Delta_2}+\sqrt{\Delta_3}}{\sqrt{\Delta}},$$

by multiplication,
$$\frac{s(s-a)(s-b)(s-c)}{s^4} = \sqrt{\frac{\Delta_1\Delta_2\Delta_3}{\Delta^3}},$$

$$\text{i e } \frac{\Delta^4}{s^4} = \Delta^2\sqrt{\frac{\Delta_1\Delta_2\Delta_3}{\Delta^3}} = (\Delta\Delta_1\Delta_2\Delta_3)^{\frac{1}{2}},$$

$$r = (\Delta\Delta_1\Delta_2\Delta_3)^{\frac{1}{8}}$$

Also
$$\cot^2\frac{A}{2} = \frac{s\ (s-a)}{(s-b)(s-c)} = \sqrt{\frac{\Delta}{\Delta_2}}\sqrt{\frac{\Delta_1}{\Delta_3}};$$

$$A = 2\cot^{-1}\left(\frac{\Delta\Delta_1}{\Delta_2\Delta_3}\right)^{\frac{1}{4}}$$

316 If Q be the orthocentre, the pedal line of P bisects PQ, and since the pedal line passes through G (the centroid) and is at right angles to PQ, therefore $PG=GQ$, also $GQ=\frac{2}{3}OQ$ (Ch XVI Ex. 18)

$$PG^2=\frac{4}{9}OQ^2=\frac{4}{9}R^2(1-8\cos A\cos B\cos C) \qquad \text{(Ch XVI Ex 41)}$$

Let D be the middle point of BC, and F the middle point of GA, so that AD is trisected in G and F

Since D is the middle point of BC

$$PB^2+PC^2=2PD^2+2BD^2,$$

similarly $$PA^2+PG^2=2PF^2+2FA^2,$$

and $$2(PD^2+PF^2)=2(2PG^2+2DG^2),$$

adding $$PA^2+PB^2+PC^2=3PG^2+2BD^2+2FA^2+4DG^2$$

$$=3PG^2+\frac{1}{2}a^2+6DG^2$$

$$=3PG^2+\frac{1}{2}a^2+\frac{2}{3}AD^2$$

$$=3PG^2+\frac{1}{2}a^2+\frac{1}{3}(AB^2+BC^2-2BD^2)$$

$$=3PG^2+\frac{1}{3}(a^2+b^2+c^2)$$

$$=3PG^2+\frac{4}{3}R^2(\sin^2 A+\sin^2 B+\sin^2 C)$$

$$=\frac{4}{3}R^2(1-8\cos A\cos B\cos C)+\frac{4}{3}R^2(2+2\cos A\cos B\cos C)$$

$$=4R^2(1-2\cos A\cos B\cos C)$$

317 Let $A'B'C'$ be the triangle so formed Since the angles at E and F are right angles, and the angles OFA', OEA' are equal or supplementary, a circle passes round $OA'FAE$, OA is the diameter of this circle, and OA' a chord subtending an angle θ (i e OFA') at the circumference, therefore $OA'=OA\sin\theta$, similarly $OB'=OB\sin\theta$, $OC'=OC\sin\theta$ Also $\angle OA'B'=\angle OAB$, since they are angles in the same segment of the circle round $AEOF$, for a similar reason $\angle OB'A'=\angle OBA$, also $\angle A'OB'=\angle AOB$

$$\text{area } OA'B'=\frac{1}{2}OA' \cdot OB'\sin A'OB'=\frac{1}{2}OA \cdot OB\sin AOB \cdot \sin^2\theta$$

$$=\text{area } AOB \cdot \sin^2\theta$$

In the same way, area $OB'C'=$area $AOB\ \sin^2\theta$, and area $OC'A'=$area $COA\ \sin^2\theta$

$$\text{area } A'B'C'=\text{area } ABC \cdot \sin^2\theta$$

318. Put $$r_1+r_2+r_3=2x$$
from the given equation
$$(x-r_1)(x-r_2)(x-r_3)+r_1r_2r_3=0,$$
$$x^3-x^2(2x)+x(r_1r_2+r_1r_3+r_2r_3)=0,$$
$$x^2=r_1r_2+r_1r_3+r_2r_3$$
$$=\Delta^2\left\{\frac{1}{(s-a)(s-b)}+\frac{1}{(s-a)(s-c)}+\frac{1}{(s-b)(s-c)}\right\}$$
$$=\frac{s\Delta^2\{3s-(a+b+c)\}}{s(s-a)(s-b)(s-c)}$$
$$=s^2$$
$$2x=2s$$
But $$r_1+r_2+r_3=r+4R \qquad \text{(Ch xvi Ex 20)}$$
$$r+4R=\text{the perimeter of the triangle}$$

Again $$r_1+r_2+r_3=r+4R, \text{ and } r_1r_2+r_1r_3+r_2r_3=s^2,$$
$$r_1^3+r_2^3+r_3^3-3r_1r_2r_3=(r_1+r_2+r_3)\{(r_1+r_2+r_3)^2-3(r_1r_2+r_1r_3+r_2r_3)\}$$
$$=(4R+r)\{(4R+r)^2-3s^2\}$$
$$=(4R+r)(4R+r+s\sqrt{3})(4R+r-s\sqrt{3})$$

319 $$C^3=r_1r_2r_3=\frac{s\Delta^3}{s(s-a)(s-b)(s-c)}=\Delta s,$$
$$B^2=r_1r_2+r_1r_3+r_2r_3=\frac{\Delta^2}{(s-a)(s-b)(s-c)}(s-a+s-b+s-c)=s^2$$
$$B=s$$
$$A=r_1+r_2+r_3=\frac{\Delta}{s-a}+\frac{\Delta}{s-b}+\frac{\Delta}{s-c}$$
$$=\frac{s\Delta}{s(s-a)(s-b)(s-c)}[(s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b)]$$
$$=\frac{s}{\Delta}[3s^2-2(a+b+c)s+bc+ca+ab]$$
$$=\frac{s}{\Delta}(-s^2+bc+ca+ab),$$
also $$\frac{4abc}{4\Delta}=4R=r_1+r_2+r_3-r \qquad \text{[Ch. xvi. Ex 20]}$$
$$=A-\frac{\Delta}{s}$$

Hence $$a+b+c=2B,$$

$$bc+ca+ab=s^2+A\,\frac{\Delta}{s}$$

$$=B^2+A\,\frac{C^3}{B^2},$$

$$abc=A\Delta-\frac{\Delta^2}{s}$$

$$=\frac{AC^3}{B}-\frac{C^6}{B^3},$$

∴ the equation for the sides is

$$x^3-2Bx^2+\left(B^2+\frac{AC^3}{B^2}\right)x+\frac{C^3}{B^3}(C^3-AB^2)=0$$

820. Let P be the point, α the angle PAB, β the angle PAC

Then $\cos\alpha=\dfrac{a^2+x^2-y^2}{2ax}$, $\cos\beta=\dfrac{a^2+x^2-z^2}{2ax}$, a being the side of the equilateral triangle

Also $$\sin^2\alpha=\frac{1}{4a^2x^2}(2a^2x^2+2a^2y^2+2x^2y^2-a^4-x^4-y^4)$$

Now $$\cos\beta=\cos(60^\circ-\alpha)=\frac{1}{2}\cos\alpha+\frac{\sqrt{3}}{2}\sin\alpha$$

$$3\sin^2\alpha=(2\cos\beta-\cos\alpha)^2$$

$$=\left(\frac{a^2+x^2-z^2}{ax}-\frac{a^2+x^2-y^2}{2ax}\right)^2$$

$$3(2a^2x^2+2a^2y^2+2x^2y^2-a^4-x^4-y^4)$$

$$=(a^2+x^2+y^2-2z^2)^2$$

$$=a^4+x^4+y^4+4z^4+2a^2x^2+2a^2y^2+2x^2y^2-4z^2(a^2+x^2+y^2),$$

$$4[x^2y^2+x^2z^2+y^2z^2+a^2(x^2+y^2+z^2)]=4(a^4+x^4+y^4+z^4)$$

821 Let r be the radius of the circle touching the semicircles, and let X, Y, Z be the centres of the semicircles, draw ON perpendicular to XY

Then $OX=r+a$, $OY=r+b$, $OZ=a+b-r$, $XZ=b$, $YZ=a$

By Euc II 13 we have

$$(r+a)^2=b^2+(a+b-r)^2\pm 2b\,.\,ZN,$$

$$(r+b)^2=a^2+(a+b-r)^2\mp 2a\,.\,ZN$$

These equations reduce to

$$(4a+2b)\,r=2b\,(a+b)\pm 2bZN,$$

$$(4b+2a)\,r=2a\,(a+b)\mp 2aZN$$

Multiply the first equation by a, the second by b, and add,

$$4\,(a^2+ab+b^2)\,r=4ab\,(a+b),$$

$$r=\frac{ab\,(a+b)}{a^2+ab+b^2}.$$

If α, β, γ be the sides of the triangle OXY, and $2\sigma=\alpha+\beta+\gamma$, then

$$\sigma=r+a+b,\quad \sigma-\alpha=a,\quad \sigma-\beta=b,\quad \sigma-\gamma=r,$$

$$\begin{aligned}\text{area of } OXY&=\sqrt{abr\,(a+b+r)}\\ &=\sqrt{\left\{\frac{a^2b^2\,(a+b)}{a^2+ab+b^2}\cdot\frac{(a+b)^3}{a^2+ab+b^2}\right\}}\\ &=\frac{ab\,(a+b)^2}{a^2+ab+b^2}=r\,(a+b),\end{aligned}$$

but
$$\text{area } OXY=\frac{1}{2}\;ON\,(a+b),$$

$$ON=2r$$

322 The angle $ABB_1=\frac{\pi}{2}-B$, also the angle $BAB_1=\frac{\pi}{2}$, the angle $B_1=$ the angle B Thus the triangles ABC and $A_1B_1C_1$ are similar, and therefore the radii of the circumscribing circles (R and R_1) are in the ratio of c to c_1, the suffixes referring to the triangle $A_1B_1C_1$.

$$\begin{aligned}c_1&=AB_1+AA_1=c\cot B+b\operatorname{cosec} A\\ &=c\left(\frac{\cos B}{\sin B}+\frac{\sin B}{\sin A\sin C}\right),\end{aligned}$$

$$\begin{aligned}\frac{R_1}{R}&=\frac{\sin A\sin C\cos B+\sin^2 B}{\sin A\sin B\sin C}\\ &=\frac{(\cos B+\cos A\cos C)\cos B+\sin^2 B}{\sin A\sin B\sin C}\\ &=\frac{1+\cos A\cos B\cos C}{\sin A\sin B\sin C}\\ &=\frac{\sin^2 A+\sin^2 B+\sin^2 C}{2\sin A\sin B\sin C}\end{aligned}$$

(Ch VIII Ex 23)

If this expression $=K$, then $R_1=KR$, in the same way $R_2=KR_1=K^2R$, and generally $R_n=K^n\,R$,

$$R_n=R\left(\frac{\sin^2 A+\sin^2 B+\sin^2 C}{2\sin A\sin B\sin C}\right)^n$$

323 The plane through A, B, C must cut the horizontal plane in a straight line in which P and Q lie Clearly the perpendiculars from A, B, C on the line PQ are proportional to the heights of the mountains, let these perpendiculars be p_1, p_2, p_3, then

$$\angle PAQ=\pi-(\phi+\psi+2a)=\angle PBQ=\angle PCQ$$

a circle passes round $ABCQP$, let R be the radius of this circle, since PA, PB, PC subtend angles ϕ, $\phi+a$, $\phi+2a$ at the circumference

$$PA=2R\sin\phi,\quad PB=2R\sin(\phi+a),\quad PC=2R\sin(\phi+2a),$$

$$p_1=PA\sin(\psi+2a)=2R\sin\phi\sin(\psi+2a),$$

$$p_2=PB\sin(\psi+a)\ =2R\sin(\phi+a)\sin(\psi+a),$$

$$p_3=PC\sin\psi\qquad=2R\sin(\phi+2a)\sin\psi,$$

the heights are in the ratio of

$$\sin\phi\sin(\psi+2a)\quad \sin(\phi+a)\sin(\psi+a)\quad \sin(\phi+2a)\sin\psi$$

Expand and divide through by $\sin\phi\sin\psi\sin 2a$, the heights are as

$$\cot 2a+\cot\psi\quad \frac{1}{2}(\cot a+\cot\phi)(\cot a+\cot\psi)\tan a\quad \cot 2a+\cot\phi$$

Again, $\angle QAC=\angle QPC=\psi$, $\angle ACQ=\pi-\angle APQ=\pi-(\psi+2a)$, and QD bisects $\angle AQC$,

$$\frac{AD}{DC}=\frac{AQ}{CQ}=\frac{\sin ACQ}{\sin QAC}=\frac{\sin(\psi+2a)}{\sin\psi},$$

$$\frac{AC}{CD}=\frac{\sin\psi+\sin(\psi+2a)}{\sin\psi}=\frac{2\sin(\psi+a)\cos a}{\sin\psi}=\sin 2a(\cot\psi+\cot a)$$

324 Let A', B', C' be the centres of the three circles, A', B', C' must lie in AI, BI, CI, where I is the centre of the inscribed circle of the triangle ABC

Now $$IB=\frac{r}{\sin\frac{B}{2}},\quad B'B=\frac{r_2}{\sin\frac{B}{2}},\quad IB'=\frac{r-r_2}{\sin\frac{B}{2}},$$

similarly $$IC'=\frac{r-r_3}{\sin\frac{C}{2}},\text{ also angle } BIC=\pi-\frac{1}{2}(B+C),$$

$$\text{area } B'IC'=\frac{1}{2}IB'\ IC'\sin B'IC'$$

$$=\frac{1}{2}(r-r_2)(r-r_3)\frac{\sin\frac{B+C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$=\frac{a}{2r}(r-r_2)(r-r_3)\qquad\text{(Art 249)}$$

We get similar expressions for areas of $C'IA'$, $A'IB'$,

$$\text{area } A'B'C'=\frac{1}{2r}\{a(r-r_2)(r-r_3)+b(r-r_3)(r-r_1)+c(r-r_1)(r-r_2)\}$$

325 Denote the angles BQC, CQA, AQB by θ, ϕ, ψ, so that

$$\theta+\phi+\psi=2\pi,$$

BC is a chord of a circle of radius R_1 and subtends at the circumference of the circle an angle θ,

$$\frac{a}{R_1}=2\sin\theta,\ \text{also}\ \frac{b}{R_2}=2\sin\phi,\ \ \frac{c}{R_3}=2\sin\psi$$

Hence, we have to prove that

$$(\sin\theta+\sin\phi+\sin\psi)(-\sin\theta+\sin\phi+\sin\psi)(\sin\theta-\sin\phi+\sin\psi)(\sin\theta+\sin\phi-\sin\psi)$$
$$=4\sin^2\theta\sin^2\phi\sin^2\psi$$

Putting $\pi-\theta$, $\pi-\phi$, $\pi-\psi$ for A, B, C in Ex. 16 and 17, Ch VIII, we see that

$$\sin\theta+\sin\phi+\sin\psi=4\sin\frac{\theta}{2}\sin\frac{\phi}{2}\sin\frac{\psi}{2},$$

$$-\sin\theta+\sin\phi+\sin\psi=4\sin\frac{\theta}{2}\cos\frac{\phi}{2}\cos\frac{\psi}{2},$$

$$\sin\theta-\sin\phi+\sin\psi=4\cos\frac{\theta}{2}\sin\frac{\phi}{2}\cos\frac{\psi}{2},$$

$$\sin\theta+\sin\phi-\sin\psi=4\cos\frac{\theta}{2}\cos\frac{\phi}{2}\sin\frac{\psi}{2}$$

Multiplying these we get the required result

326 The circle on PA as diameter passes through M, N, since the chord MN subtends an angle A at the circumference, we have $MN=PA\sin A$

Thus $$l=PA\sin A+PB\sin B+PC\sin C$$

Now

$$(PA^2+PB^2+PC^2)(\sin^2A+\sin^2B+\sin^2C)-(PA\sin A+PB\sin B+PC\sin C)^2$$
$$=(PB\sin C-PC\sin B)^2+(PC\sin A-PA\sin C)^2+(PA\sin B-PB\sin A)^2$$

The right-hand side is least when $\dfrac{PA}{\sin A}=\dfrac{PB}{\sin B}=\dfrac{PC}{\sin C}$, and is equal to 0

the least value of $PA^2+PB^2+PC^2$ is

$$\frac{(PA\sin A+PB\sin B+PC\sin C)^2}{\sin^2A+\sin^2B+\sin^2C},$$

that is, $$\frac{l^2}{\sin^2A+\sin^2B+\sin^2C}$$

327 Let PQR be the triangle formed by the tangents, and let OD be the perpendicular from the centre O to the line ABC. Draw RN perpendicular to ABC Let $DA=x$, $DB=y$, $DC=z$

$$\frac{AN}{RN}=\cot RAN=\tan OAD=\frac{p}{DA},$$

$$\frac{BN}{RN}=\cot RBN=\tan OBD=\frac{p}{DB}$$

∴ subtracting, $$\frac{AB}{RN}=\frac{p}{DA}\frac{AB}{DB}, \text{ or } RN=\frac{xy}{p}$$

$$\text{area } ABR=\frac{1}{2}RN\ AB=\frac{xy\,(y-x)}{2p}$$

Similarly $$\text{area } BPC=\frac{yz\,(z-y)}{2p},\ \text{area } QAC=\frac{zx\,(z-x)}{2p};$$

$$\text{area } PQR=\text{area } ABR+\text{area } BPC-\text{area } QAC$$

$$=\frac{1}{2p}\{xy\,(y-x)+yz\,(z-y)+xz\,(x-z)\}$$

$$=\frac{1}{2p}(y-x)(z-y)(z-x)=\frac{BC\ AC\ AB}{2p}.$$

328 The denominator of the fraction

$$=\cos\theta\,(\sin 2B-\sin 2C+\sin 2D-\sin 2A)$$
$$+\sin\theta\,(\cos 2B-\cos 2C+\cos 2D-\cos 2A)$$
$$=2\sin\theta\,(\cos 2B-\cos 2C), \text{ since } \sin 2B=-\sin 2D,\ \cos 2B=\cos 2D, \text{ \&c },$$
$$=4\sin\theta\,(\sin^2 C-\sin^2 B)$$
$$=\frac{\sin\theta}{R^2}(BD^2-AC^2), \text{ since } 2R\sin C=BD,\ 2R\sin B=AC$$

The numerator $=\cos\theta\,(ab\sin B+cd\sin B-da\sin A-bc\sin C)$
$$+\sin\theta\,(ab\cos B+cd\cos D-da\cos A-bc\cos C)$$
$$=\cos\theta\,(2\text{ area } ABCD-2\text{ area } ABCD)$$
$$+\frac{1}{2}\sin\theta\,\{a^2+b^2-AC^2+b^2+d^2-AC^2-(d^2+a^2-BD^2+b^2+c^2-BD^2)\}$$
$$=\sin\theta\,(BD^2-AC^2)$$

Hence the fraction $$=\frac{\sin\theta\,(BD^2-AC^2)}{\frac{1}{R^2}\sin\theta\,(BD^2-AC^2)}=R^2$$

329 We have

$$a^2+b^2-2ab\cos B=AC^2=c^2+d^2-2cd\cos D,$$

$$a^2+b^2-c^2-d^2=2ab\cos B-2cd\cos D,$$

$$(a+b)^2-(c-d)^2=4ab\cos^2\frac{1}{2}B+4cd\sin^2\frac{1}{2}D,$$

and
$$-(a-b)^2+(c+d)^2=4ab\sin^2\frac{1}{2}B+4cd\cos^2\frac{1}{2}D,$$

$$(s-c)(s-d)=ab\cos^2\frac{1}{2}B+cd\sin^2\frac{1}{2}D,$$

and
$$(s-a)(s-b)=ab\sin^2\frac{1}{2}B+cd\cos^2\frac{1}{2}D$$

Divide these by $(s-c)(s-d)$, $(s-a)(s-b)$ respectively,

$$1=\sin^2\psi+\frac{cd}{(s-c)(s-d)}\quad\sin^2\frac{1}{2}D,$$

and
$$1=\sin^2\phi+\frac{cd}{(s-a)(s-b)}\cdot\cos^2\frac{1}{2}D$$

Transpose, multiply the two results together and take the square root,

$$\cos\phi\cos\psi=\frac{cd\sin\frac{1}{2}D\cos\frac{1}{2}D}{\sqrt{(s-a)(s-b)(s-c)(s-d)}}$$

But
$$\sin\phi\sin\psi=\frac{ab\sin\frac{1}{2}B\cos\frac{1}{2}B}{\sqrt{(s-a)(s-b)(s-c)(s-d)}},$$

$$\sqrt{(s-a)(s-b)(s-c)(s-d)}\cos(\phi-\psi)$$

$$=cd\sin\frac{1}{2}D\cos\frac{1}{2}D+ab\sin\frac{1}{2}B\cos\frac{1}{2}B$$

$$=\frac{1}{2}(cd\sin D+ab\sin B)$$

$$=\text{area } ADC+\text{area } ABC=\text{area } ABCD$$

330 We have $\qquad 2r_1=\dfrac{BD}{\sin C},\ 2r_3=\dfrac{BD}{\sin A}$

$$\frac{1}{4}\left\{\frac{1}{r_1^2}+\frac{1}{r_3^2}+\frac{2\cos(A+C)}{r_1r_3}\right\}=\frac{\sin^2 A+\sin^2 C+2\cos(A+C)\sin A\sin C}{BD^2}$$

$$=\frac{\sin^2 A+\sin^2 C+2\cos A\cos C\sin A\sin C-2\sin^2 A\sin^2 C}{BD^2}$$

$$=\frac{\sin^2 A\,(1-\sin^2 C)+\sin^2 C\,(1-\sin^2 A)+2\cos A\cos C\sin A\sin C}{BD^2}$$

$$=\frac{(\sin A\cos C+\cos A\sin C)^2}{BD^2}=\frac{\sin^2(A+C)}{BD^2}.$$

Similarly $\qquad \dfrac{1}{4}\left\{\dfrac{1}{r_2^2}+\dfrac{1}{r_4^2}+\dfrac{2\cos(B+D)}{r_2r_4}\right\}=\dfrac{\sin^2(B+D)}{AC^2}$

Hence, since $\sin^2(A+C)=\sin^2(B+D)$ the result follows

331. Let A, B, C, D be the four angular points, P the point on the circumference, suppose x to represent the side of the polygon, θ the angle subtended by a side at the circumference, y the diagonal AC or BD

Applying Euc VI prop D, to the quadrilaterals $PBCD$, $PABC$, we obtain

$$yc=x(b+d),\text{ and } yb=x(a+c),$$

$$c(a+c)=b(b+d),$$

$$c^2-b^2=bd-ac \qquad (1),$$

which is the required relation

Again $\qquad \dfrac{x}{2r}=\sin\theta=\sin BAC=\dfrac{\sqrt{\left(x^2-\frac{1}{4}y^2\right)}}{x}$

$$r^2=\frac{x^4}{4x^2-y^2} \qquad (2)$$

Now $\qquad \dfrac{x}{y}=\dfrac{c}{b+d}=\dfrac{b}{a+c},$

$$\frac{x}{y}=\frac{c+b}{a+b+c+d} \qquad (3),$$

and $\qquad \dfrac{x}{y}=\dfrac{c-b}{b+d-a-c} \qquad (4);$

also from (3) $\qquad \dfrac{x}{2x-y}=\dfrac{c+b}{b+c-a-d} \qquad (5),$

and from (4) $\qquad \dfrac{x}{2x+y}=\dfrac{c-b}{c+d-a-b} \qquad (6)$

Again $\qquad y^2=a^2+c^2-2ac\cos 2\theta,$

and $\qquad y^2=b^2+d^2-2bd\cos 2\theta$

Hence eliminating $\cos 2\theta$,

$$y^2(bd-ac)=bd(a^2+c^2)-ac(b^2+d^2),$$

i e

$$y^2=\frac{(ab-cd)(ad-bc)}{bd-ac} \qquad (7)$$

multiplying together (3), (4), (5), (6), (7) and using equations (1), (2) we obtain

$$r^2=\frac{(ab-cd)(ad-bc)(bd-ac)}{(a+b+c+d)(b+d-a-c)(b+c-a-d)(c+d-a-b)}$$

332 Let PQ be that diameter of the circumscribing circle (whose centre is O) which bisects the diagonal BD of the given quadrilateral $ABCD$ Then AQ, CP bisect the angles A, C of the quadrilateral, and therefore pass through the centre of the inscribed circle, say I.

Then $$CI\,.\,IP=AI\,.\,IQ=R^2-d^2$$

Now $$r=AI\sin\frac{A}{2}=CI\sin\frac{C}{2}=CI\cos\frac{A}{2},$$

since $$A+C=\pi$$

Hence $$IP^2+IQ^2=(R^2-d^2)^2\left(\frac{1}{AI^2}+\frac{1}{CI^2}\right)$$

$$=(R^2-d^2)^2\left(\frac{1}{r^2}\sin^2\frac{A}{2}+\frac{1}{r^2}\cos^2\frac{A}{2}\right)$$

$$=(R^2-d^2)^2\ \frac{1}{r^2}$$

But $$IP^2+IQ^2=2IO^2+2PO^2=2d^2+2R^2$$

Hence $$\frac{1}{r^2}=\frac{2R^2+2d^2}{(R-d)^2}=\frac{1}{(R+d)^2}+\frac{1}{(R-d)^2}$$

333 Let $AB=x$, $AC=y$, $AD=z$, $AE=u$, $\angle BAC=\alpha$, $\angle CAD=\beta$, $\angle DAE=\gamma$ Then we have

$\sin(\alpha+\beta+\gamma)\sin\beta+\sin\alpha\sin\gamma=\sin(\alpha+\beta)\sin(\beta+\gamma)$ [Ch VIII Ex 11]

Multiply by $\frac{1}{4}xyzu$, then observing that

$\frac{1}{2}xy\sin\alpha=\triangle BAC$, $\frac{1}{2}yz\sin\beta=\triangle CAD$, &c, we obtain

$$\triangle BAE\,.\,\triangle CAD+\triangle BAC\ \ \triangle DAE=\triangle BAD\ \ \triangle CAE$$

But $\triangle CAD=A-b-c$, $\triangle BAD=A-c-e$, and $\triangle CAE=A-b-d$

Hence $$a(A-b-c)+be=(A-c-e)(A-b-d)$$

$$=A^2-(b+c+d+e)A+bc+cd+de+be.$$

$$A^2-(a+b+c+d+e)A+ab+bc+cd+de+ea=0$$

334 Since $ABCD$ is a quadrilateral inscribed in a circle, we have by Euc. VI, Prop D,

$$pq = bs - ac \quad \ldots \ldots \quad (1)$$

Similarly, from the quadrilateral $ABDE$,

$$st = qe + ad;$$

$$\therefore\ st - ad = qe = \frac{e}{p}(bs - ac) \text{ from (1)}.$$

$$\text{i e. } p(st - ad) = e(bs - ac).$$

Again, since CE is a diagonal of the quadrilateral $ACDE$, we have, from Art 255,

$$r^2 = \frac{(pd + ce)(pe - de)}{pe - cd}.$$

335 Let QQ' be one of the chords, O the centre of the circle, P the vertex of the right-angled isosceles triangle QPQ', then OP bisects QQ' at right angles, at N say. Then $NP = NQ$, and the angle $QON = \frac{\alpha}{2n}$

Now $$OP = ON - NQ = \cos\frac{\alpha}{2n} - \sin\frac{\alpha}{2n}$$

If X be the product of the $n-1$ distances, each equal to OP, then

$$X = \left(\cos\frac{\alpha}{2n} - \sin\frac{\alpha}{2n}\right)^{n-1};$$

$$\therefore\ \log X = (n-1)\log\left(1 - \frac{\alpha}{2n} - \frac{\alpha^2}{8n^2} - \ldots\right)$$

$$= (n-1)\frac{\alpha}{2n} + \text{terms which vanish when } n = \infty$$

$$= \frac{\alpha}{2} \text{ in the limit.}$$

$$X = e^{\frac{\alpha}{2}} \text{ in the limit.}$$

336 Subtract $\tan^{-1}(\tan\alpha)$ from the r^{th} term; using the formula

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1-xy},$$

we obtain

$$\tan^{-1}\frac{\dfrac{a_{r+1}\sin\alpha -}{a_r + \quad} - \dfrac{\sin\alpha}{\cos\alpha}}{1 - \dfrac{a_{r-1}\sin\alpha -}{a_r -}\cdot\dfrac{\sin\alpha}{\cos\alpha}} \quad \ldots \ldots \quad (1).$$

Clearing of fractions, the numerator of this fraction becomes

$$(a_{r+1}\sin\alpha + a_{r+2}\sin 2\alpha + \quad + a_{r-1}\sin\overline{n-1}\,\alpha)\cos\alpha$$

$$-(a_r + a_{r+1}\cos\alpha + a_{r+2}\cos 2\alpha + \quad + a_{r-1}\cos\overline{n-1}\,\alpha)\sin\alpha$$

$$= a_{r+2}\sin\alpha + a_{r+3}\sin 2\alpha + \quad + a_{r-1}\sin\overline{n-2}\,\alpha + a_r\sin\overline{n-1}\,\alpha,$$

since $\sin\overline{n-1}\,\alpha = -\sin\alpha$ when $n\alpha = 2\pi$

The denominator becomes

$$(a_r + a_{r+1}\cos\alpha + \quad + a_{r-1}\cos\overline{n-1}\,\alpha)\cos\alpha$$

$$+(a_{r+1}\sin\alpha + a_{r+2}\sin 2\alpha + \quad + a_{r-1}\sin\overline{n-1}\,\alpha)\sin\alpha$$

$$= a_{r+1} + a_{r+2}\cos\alpha + a_{r+3}\cos 2\alpha + \quad + a_r\cos\overline{n-1}\,\alpha,$$

since $\cos\overline{n-1}\,\alpha = +\cos\alpha$

Hence the expression (1) is the $(r+1)^{\text{th}}$ term of the series Therefore since $\tan^{-1}(\tan\alpha) = \alpha$, the terms form an arithmetical progression whose common difference is α

337 We shall require the following identities

$$\cos\alpha\sin(\beta-\gamma) + \cos\beta\sin(\gamma-\alpha) + \cos\gamma\sin(\alpha-\beta) = 0 \qquad (1),$$

$$\sin\alpha\sin(\beta-\gamma) + \sin\beta\sin(\gamma-\alpha) + \sin\gamma\sin(\alpha-\beta) = 0 \qquad (2),$$

$$\cos 2\alpha\sin(\beta-\gamma) + \cos 2\beta\sin(\gamma-\alpha) + \sin 2\gamma\sin(\alpha-\beta)$$

$$= -(\sin\overline{\beta-\gamma} + \sin\overline{\gamma-\alpha} + \sin\overline{\alpha-\beta})(\cos\overline{\beta+\gamma} + \cos\overline{\gamma+\alpha} + \cos\overline{\alpha+\beta})$$

$$= \quad 4\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\beta}{2}(\cos\overline{\beta+\gamma} + \cos\overline{\gamma+\alpha} + \cos\overline{\alpha+\beta}) \quad (3)$$

(2) is proved in Ch VIII Ex 8, and (1) follows from (2) by putting

$$\frac{\pi}{2}-\alpha,\ \frac{\pi}{2}-\beta,\ \frac{\pi}{2}-\gamma \text{ for } \alpha,\ \beta,\ \gamma$$

Multiply (1) by $\cos\alpha + \cos\beta + \cos\gamma$, (2) by $\sin\alpha + \sin\beta + \sin\gamma$, and subtract, therefore

$$\sin(\beta-\gamma)\{\cos 2\alpha + \cos(\alpha+\beta) + \cos(\alpha+\gamma)\} + \text{two similar terms} = 0$$

Now $\qquad \sin(\beta-\gamma)\cos(\beta+\gamma) + \text{two similar terms} = 0$

Therefore, by addition,

$$\sin(\beta-\gamma)\{\cos 2\alpha + \cos(\alpha+\beta) + \cos(\alpha+\gamma) + \cos(\beta+\gamma)\}$$

$$+ \text{two similar terms} = 0,$$

hence we have the identity (3).

Now put $$x\cos\phi+y\sin\phi+z+\cos 2\phi=u \tag{4}$$
Multiply the three given equations by $\sin(\beta-\gamma)$, $\sin(\gamma-\alpha)$, $\sin(\alpha-\beta)$ and add, using the above identities, we obtain

$$-4\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\beta}{2}\{z-\cos(\beta+\gamma)-\cos(\gamma+\alpha)-\cos(\alpha+\beta)\}=0 \tag{5}$$

Similarly, multiply the first two of the given equations by $\sin(\beta-\phi)$, $\sin(\phi-\alpha)$ and (4) by $\sin(\alpha-\beta)$ and add,

$$u\sin(\alpha-\beta)$$
$$=-4\sin\frac{\beta-\phi}{2}\sin\frac{\phi-\alpha}{2}\sin\frac{\alpha-\beta}{2}\{z-\cos(\beta+\phi)-\cos(\phi+\alpha)-\cos(\alpha+\beta)\}$$

Substitute for z from equation (5), the expression within the brackets becomes

$$\cos(\beta+\gamma)-\cos(\alpha+\phi)+\cos(\alpha+\gamma)-\cos(\beta+\phi)$$
$$=2\sin\frac{1}{2}(\alpha+\beta+\gamma+\phi)\left\{\sin\frac{1}{2}(\phi+\alpha-\beta-\gamma)+\sin\frac{1}{2}(\phi+\beta-\alpha-\gamma)\right\}$$
$$=4\sin\frac{1}{2}(\alpha+\beta+\gamma+\phi)\sin\frac{1}{2}(\phi-\gamma)\cos\frac{\alpha-\beta}{2}$$
$$u=8\sin\frac{1}{2}(\phi-\alpha)\sin\frac{1}{2}(\phi-\beta)\sin\frac{1}{2}(\phi-\gamma)\sin\frac{1}{2}(\phi+\alpha+\beta+\gamma)$$

Another solution

Put $$\cos\theta+\sqrt{-1}\sin\theta=\lambda=e^{i\theta},$$

$e^{i\alpha}$, $e^{i\beta}$, $e^{i\gamma}$ are three roots of the equation

$$x\left(\lambda+\frac{1}{\lambda}\right)-iy\left(\lambda-\frac{1}{\lambda}\right)+2z+\lambda^2+\frac{1}{\lambda^2}=0,$$

or $$\lambda^4+(x-iy)\lambda^3+2z\lambda^2+(x+iy)\lambda+1=0$$

The product of the four roots is unity, the fourth root is

$$e^{-i(\alpha+\beta+\gamma)}, \text{ or } e^{i\delta} \text{ say}$$

Thus $$\lambda^4+(x-iy)\lambda^3+\&c\equiv(\lambda-e^{i\alpha})(\lambda-e^{i\beta})(\lambda-e^{i\gamma})(\lambda-e^{i\delta})$$

Divide by λ^2,

$$\therefore x\left(\lambda+\frac{1}{\lambda}\right)-iy\left(\lambda-\frac{1}{\lambda}\right)+2z+\lambda^2+\frac{1}{\lambda^2}$$
$$\equiv e^{\frac{1}{2}i\alpha}\cdot e^{\frac{1}{2}i\beta}\,e^{\frac{1}{2}i\gamma}\,e^{\frac{1}{2}i\delta}\left(\sqrt{\frac{\lambda}{e^{i\alpha}}}-\sqrt{\frac{e^{i\alpha}}{\lambda}}\right)\left(\sqrt{\frac{\lambda}{e^{i\beta}}}-\sqrt{\frac{e^{i\beta}}{\lambda}}\right)$$
$$\left(\sqrt{\frac{\lambda}{e^{i\gamma}}}-\sqrt{\frac{e^{i\gamma}}{\lambda}}\right)\left(\sqrt{\frac{\lambda}{e^{i\delta}}}-\sqrt{\frac{e^{i\delta}}{\lambda}}\right)$$

For λ write $\cos\phi+\sqrt{-1}\sin\phi$, or $e^{\iota\phi}$, the above equation becomes

$$x\cos\phi+y\sin\phi+z+\cos 2\phi$$

$$=\frac{1}{2}\Pi\left(e^{\frac{1}{2}\overline{\phi-\alpha}\,\iota}-e^{-\frac{1}{2}\overline{\phi-\alpha}\,\iota}\right)=\frac{1}{2}\Pi\left(2\iota\sin\frac{1}{2}\overline{\phi-\alpha}\right)$$

$$=8\sin\frac{1}{2}(\phi-\alpha)\sin\frac{1}{2}(\phi-\beta)\sin\frac{1}{2}(\phi-\gamma)\sin\frac{1}{2}(\phi-\delta),$$

which is the required result, since $-\delta=\alpha+\beta+\gamma$

338 Multiply the first equation by $\sqrt{-1}$ and add;

$$x^2e^{2\iota\alpha}+y^2e^{2\iota\beta}+z^2e^{2\iota\gamma}-2yze^{\iota(\beta+\gamma)}-2zxe^{\iota(\gamma+\alpha)}-2xye^{\iota(\alpha+\beta)}=0,$$

$$(xe^{\iota\alpha}-ye^{\iota\beta}+ze^{\iota\gamma})^2=4xze^{\iota(\alpha+\gamma)},$$

$$xe^{\iota\alpha}\pm2\sqrt{xz}\,e^{\frac{1}{2}\iota(\alpha+\gamma)}+ze^{\iota\gamma}=ye^{\iota\beta},$$

$$\therefore\ \sqrt{(xe^{\iota\alpha})}\pm\sqrt{(ye^{\iota\beta})}\pm\sqrt{(ze^{\iota\gamma})}=0$$

Hence

$$\sqrt{x}\left(\cos\frac{\alpha}{2}+\iota\sin\frac{\alpha}{2}\right)\pm\sqrt{y}\left(\cos\frac{\beta}{2}+\iota\sin\frac{\beta}{2}\right)\pm\sqrt{z}\left(\cos\frac{\gamma}{2}+\iota\sin\frac{\gamma}{2}\right)=0,$$

or

$$\sqrt{x}\left(\cos\frac{\pi+\alpha}{2}+\iota\sin\frac{\pi+\alpha}{2}\right)\pm\sqrt{y}\left(\cos\frac{\beta}{2}+\iota\sin\frac{\beta}{2}\right)\pm\sqrt{z}\left(\cos\frac{\gamma}{2}+\iota\sin\frac{\gamma}{2}\right)=0,$$

or equations formed from these by putting $\pi+\beta$ or $\pi+\gamma$ for β and γ respectively

Equating real and imaginary parts we get equations such as

$$\left.\begin{aligned}\sqrt{x}\cos\frac{\alpha}{2}+\sqrt{y}\cos\frac{\beta}{2}+\sqrt{z}\cos\frac{\gamma}{2}=0,\\ \sqrt{x}\sin\frac{\alpha}{2}+\sqrt{y}\sin\frac{\beta}{2}+\sqrt{z}\sin\frac{\gamma}{2}=0,\end{aligned}\right\}$$

or

$$\left.\begin{aligned}-\sqrt{x}\sin\frac{\alpha}{2}+\sqrt{y}\cos\frac{\beta}{2}+\sqrt{z}\cos\frac{\gamma}{2}=0,\\ \sqrt{x}\cos\frac{\alpha}{2}+\sqrt{y}\sin\frac{\beta}{2}+\sqrt{z}\sin\frac{\gamma}{2}=0\end{aligned}\right\}$$

Solving the first pair for $\sqrt{x}$, $\sqrt{y}$, $\sqrt{z}$ in the usual way we obtain the first set of ratios, solving the second pair we obtain one of the other sets

339 Put $x = \cos\theta + \sqrt{-1}\sin\theta$, therefore $2\cos 2\theta = x^2 - \frac{1}{x^2}$, &c, the given equation becomes

$$a\left(x^2+\frac{1}{x^2}\right) - b\sqrt{-1}\left(x^2-\frac{1}{x^2}\right) + c\left(x+\frac{1}{x}\right) - d\sqrt{-1}\left(x-\frac{1}{x}\right) + 2e = 0,$$

or, $x^4(a - b\sqrt{-1}) + (c - d\sqrt{-1})x^3 + 2ex^2 + (c + d\sqrt{-1})x + a - b\sqrt{-1} = 0$

Let x_1, x_2, x_3, x_4 be the roots of this equation, then

$$\frac{a-b\sqrt{-1}}{1} = \frac{-c+d\sqrt{-1}}{x_1+x_2+x_3+x_4} = \frac{2e}{x_1x_2+x_1x_3+} = \frac{-c-d\sqrt{-1}}{x_1x_2x_3+} = \frac{a+b\sqrt{-1}}{x_1x_2x_3x_4}$$

$$\frac{a}{x_1x_2x_3x_4+1} = \frac{b\sqrt{-1}}{x_1x_2x_3x_4-1} = \frac{-c}{x_1+x_2+ \quad +x_1x_2x_3+}$$

$$= \frac{-d\sqrt{-1}}{x_1x_2x_3+ \quad -x_1-x_2-} = \frac{e}{x_1x_2+x_1x_3+},$$

or, $$\frac{a}{\sqrt{x_1x_2x_3x_4}+\frac{1}{\sqrt{x_1x_2x_3x_4}}} = \frac{b\sqrt{-1}}{\sqrt{x_1x_2x_3x_4}-\frac{1}{\sqrt{x_1x_2x_3x_4}}}$$

$$= \frac{-c}{\sqrt{\frac{x_1}{x_2x_3x_4}}+\sqrt{\frac{x_2x_3x_4}{x_1}}+} = \frac{+d\sqrt{-1}}{\sqrt{\frac{x_1}{x_2x_3x_4}}-\sqrt{\frac{x_2x_3x_4}{x_1}}+}$$

$$= \frac{e}{\sqrt{\frac{x_1x_2}{x_3x_4}}+\sqrt{\frac{x_3x_4}{x_1x_2}}+}$$

As in Ex 11, Ch XIX., $\sqrt{x_1x_2x_3x_4}+\frac{1}{\sqrt{x_1x_2x_3x_4}} = \cos s$,

$$\sqrt{\frac{x_1}{x_2x_3x_4}} - \sqrt{\frac{x_2x_3x_4}{x_1}} = 2\sqrt{-1}\sin\frac{1}{2}(\alpha-\beta-\gamma-\delta) = -2\sqrt{-1}\sin(s-\alpha),$$

and so on,

$$\frac{a}{\cos s} = \frac{b}{\sin s} = \frac{-c}{\Sigma\cos(s-\alpha)} = \frac{-d}{\Sigma\sin(s-\alpha)} = \frac{e}{\cos\frac{1}{2}(\beta+\gamma-\alpha-\delta)+}$$

340 Divide through by $(-b)^n$ and put $\frac{a}{-b} = \cos\theta$ Thus

$$\frac{S}{(-b)^n} = (2\cos\theta)^n - n(2\cos\theta)^{n-2} + \quad -2$$

$$= 2\cos n\theta - 2 \qquad \text{(Art 298)}$$

The roots of the equation $\cos n\theta = 1$ are given by $n\theta = 2\kappa\pi$,

$$(2\cos\theta)^n - n(2\cos\theta)^{n-2} + \ldots$$

$$= (2\cos\theta - 2\cos 0)\left(2\cos\theta - 2\cos\frac{2\pi}{n}\right)\left(2\cos\theta - 2\cos\frac{4\pi}{n}\right)\ldots$$

$$\ldots\left(2\cos\theta - 2\cos\frac{\overline{2n-1}\,\pi}{n}\right),$$

i.e.

$$\frac{S}{(-b)^n} = \left(\frac{a}{-b} - 2\right)\left(\frac{a}{-b} - 2\cos\frac{2\pi}{n}\right)\ldots\left(\frac{a}{-b} - 2\cos\frac{\overline{2n-1}\,\pi}{n}\right),$$

or

$$S = (a+2b)\left(a + 2b\cos\frac{2\pi}{n}\right)\ldots\left(a + 2b\cos\frac{\overline{2n-1}\,\pi}{n}\right)$$

(1) n *odd* $\cos\frac{2n-2}{n}\pi = \cos\frac{2\pi}{n}$, $\cos\frac{2n-4}{n}\pi = \cos\frac{4\pi}{n}$, and there is an even number of factors with cosines,

$$S(a+2b) = (a+2b)^2\left(a+2b\cos\frac{2\pi}{n}\right)^2\ldots\left(a+2b\cos\frac{n-1}{n}\pi\right)^2$$

(2) n *even* the middle factor of those which have cosines in them is $a + 2b\cos\frac{n\pi}{n}$ (occurring only once), or $a - 2b$;

$$S(a^2-4b^2) = (a+2b)^2\left(a+2b\cos\frac{2\pi}{n}\right)^2\ldots\left(a+2b\cos\frac{n\pi}{n}\right)^2$$

341 Let

$$S = x^n \sin n\alpha - nx^{n-1}\sin(n\alpha+\beta) + \ldots$$

$$C = x^n \cos n\alpha - nx^{n-1}\cos(n\alpha+\beta) + \ldots$$

and write $a = e^{\alpha\sqrt{-1}}$, $b = e^{\beta\sqrt{-1}}$, then

$$C + S\sqrt{-1} = \left(x^n - nx^{n-1}b + \frac{n\,\overline{n-1}}{1\cdot 2}x^{n-2}b^2 - \ldots\right)a^n$$

$$= (x-b)^n a^n$$

$$= (x - \cos\beta - \sqrt{-1}\sin\beta)^n(\cos n\alpha + \sqrt{-1}\sin n\alpha)$$

Put $x - \cos\beta = f\cos\psi$, $\sin\beta = -f\sin\psi$,

$$C + S\sqrt{-1} = f^n(\cos\psi + \sqrt{-1}\sin\psi)^n(\cos n\alpha + \sqrt{-1}\sin n\alpha)$$

$$= f^n\{\cos n(\psi+\alpha) + \sqrt{-1}\sin n(\psi+\alpha)\},$$

$$S = f^n \sin n(\psi+\alpha)$$

The roots of the equation $S = 0$ are therefore given by $n(\psi+\alpha) = \kappa\pi$, where κ has all integral values from 0 to $n-1$

$$\psi = \frac{\kappa}{n}\pi - \alpha = \kappa\phi - \alpha.$$

Now $$x-\cos\beta=\quad f\cos\psi=f\cos(\alpha-\kappa\phi),$$
$$\sin\beta=-f\sin\psi=f\sin(\alpha-\kappa\phi)$$
$$x=\cos\beta+f\cos(\alpha-\kappa\phi)$$
$$=\cos\beta+\frac{\cos(\alpha-\kappa\phi)\sin\beta}{\sin(\alpha-\kappa\phi)}$$
$$=\frac{\sin(\alpha+\beta-\kappa\phi)}{\sin(\alpha-\kappa\phi)}.$$

342 The equation may be written

$$(x+e^{\iota\alpha})\ +(x+e^{-\iota\alpha})^n=0$$

$$x+e^{\iota\alpha}=(x+e^{-\iota\alpha})(-1)^{\frac{1}{n}}$$
$$=(x+e^{-\iota\alpha})(\cos 2\overline{r+1}\pi+\iota\sin 2\overline{r+1}\pi)^{\frac{1}{n}}$$
$$=(x+e^{-\iota\alpha})\,e^{2\iota\phi},\ \text{where}\ \phi=\frac{2r+1}{2n}\pi$$
$$x(1-e^{2\iota\phi})=e^{\iota(2\phi-\alpha)}-e^{\iota\alpha},$$
$$x=\frac{e^{\iota\alpha}-e^{\iota(2\phi-\alpha)}}{e^{2\iota\phi}-1}$$
$$=\frac{e^{\iota(\alpha-\phi)}-e^{-\iota(\alpha-\phi)}}{e^{\iota\phi}-e^{-\iota\phi}}$$
$$=\frac{\sin(\alpha-\phi)}{\sin\phi}$$

the roots are given by $\dfrac{\sin\left(\alpha-\dfrac{2r+1}{2n}\pi\right)}{\sin\dfrac{2r+1}{2n}\pi}$, where r has all integral values from 0 to $n-1$

343 Let $P_1P_2P_3$ be the polygon inscribed in a circle whose centre is O, let A be the fixed point, 2α the angle P_1OA, so that

$$\angle P_rOA=2\alpha+\frac{2\pi}{n}(r-1)$$

Then $AP_1=2a\sin\alpha$, $AP_2=2a\sin\left(\alpha+\dfrac{\pi}{n}\right)$, and so on,

$$\Sigma l^{2m}=2^{2m}\,.\,a^{2m}\left\{\sin^{2m}\alpha+\sin^{2m}\left(\alpha+\frac{\pi}{n}\right)+\quad+\sin^{2m}\left(\alpha+\frac{r-1}{n}\pi\right)\right\}$$

Now

$$2^{2m-1}\sin^{2m}a\ (-1)^m=\cos 2ma-m\cos(2m-2)a+\ .\ +(-1)^m\ \frac{\lfloor 2m}{2(\lfloor m)^2}$$

by Art 294 Resolve each term of the series within the brackets into a series of cosines, we obtain a number of series of the form

$$\cos pa+\cos p\left(a+\frac{\pi}{n}\right)+\ .\ +\cos p\left(a+\frac{n-1}{n}\pi\right)$$

$$\text{This series}=\frac{\cos\left(p+\frac{n-1}{2n}\pi\right)\sin\frac{np\pi}{2n}}{\sin\frac{p\pi}{2n}}$$

$$=0,\ \text{since}\ \sin\frac{p\pi}{2}=0,\ p\ \text{being even}$$

$$\text{the sum of the series within brackets}=\frac{n\lfloor 2m}{2^m(\lfloor m)^2}$$

$$\Sigma l^{2m}=n\ \ a^{2m}\ \ \frac{\lfloor 2m}{(\lfloor m)^2}$$

344 From the relation $(1-c)\tan\theta=(1+c)\tan\phi$ we obtain

$$\tan(\theta-\phi)=\frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}=\frac{\tan\theta-\frac{1-c}{1+c}\tan\theta}{1+\frac{1-c}{1+c}\tan^2\theta}$$

$$=\frac{2c\tan\theta}{1+c+(1-c)\tan^2\theta}=\frac{c\sin 2\theta}{1+c\cos 2\theta} \qquad (1)$$

As in the solution of Ex. 19, Ch xxii, we obtain for the sum of the first series

$$\frac{1}{2\iota}\log\frac{(1+c\cos 2\theta)+\iota c\sin 2\theta}{(1+c\cos 2\theta)-\iota c\sin 2\theta}$$

$$=\frac{1}{2\iota}\log\frac{1+\iota\tan(\theta-\phi)}{1-\iota\tan(\theta-\phi)},\ \text{from (1)},$$

$$=\frac{1}{2\iota}\log\frac{\cos(\theta-\phi)+\iota\sin(\theta-\phi)}{\cos(\theta-\phi)-\iota\sin(\theta-\phi)}$$

$$=\frac{1}{2\iota}\log e^{2\iota(\theta-\phi)}=\theta-\phi+\kappa\pi$$

When $\theta=0$, $\phi=0$ and the series vanishes, therefore $\kappa=0$ The second series can be found from the first by changing the sign of c, writing ϕ for θ and changing the sign of the result Hence the second series $=-(\phi-\theta)=\theta-\phi$

345 $$\frac{1}{2^n}\frac{\sin\frac{\theta}{2^n}}{2\cos\frac{\theta}{2^n}+1}+\frac{1}{2^n}\frac{\sin\frac{\theta}{2^n}}{2\cos\frac{\theta}{2^n}-1}=\frac{1}{2^n}\frac{4\sin\frac{\theta}{2^n}\cos\frac{\theta}{2^n}}{4\cos^2\frac{\theta}{2^n}-1}$$

$$=\frac{1}{2^{n-1}}\frac{\sin\frac{\theta}{2^{n-1}}}{2\cos\frac{\theta}{2^{n-1}}+1}$$

Thus representing $\frac{1}{2^n}\frac{\sin\frac{\theta}{2^n}}{2\cos\frac{\theta}{2^n}+1}$ by u_n, the nth term of the given series

$$=u_{n-1}-u_n$$

Thus the sum of the series is

$$u_0-u_1+u_1-u_2+u_2-u_3+$$
$$=u_0=\frac{\sin\theta}{2\cos\theta+1}.$$

346 $\Sigma_{n=0}^{n=N}\cos(m\alpha+n\beta)$

$$=\cos m\alpha+\cos(m\alpha+\beta)+\cos(m\alpha+2\beta)+\quad+\cos(m\alpha+N\beta)$$

$$=\frac{\sin\frac{N+1}{2}\beta}{\sin\frac{\beta}{2}}\cos\left(m\alpha+\frac{N\beta}{2}\right) \qquad \text{(Art 327)}$$

$$\sum_{m=0}^{m=M}\sum_{n=0}^{n=N}\cos(m\alpha+n\beta)=\frac{\sin\frac{N+1}{2}\beta}{\sin\frac{\beta}{2}}\sum_{m=0}^{m=M}\cos\left(m\alpha+\frac{N\beta}{2}\right)$$

$$=\frac{\sin\frac{N+1}{2}\beta}{\sin\frac{\beta}{2}}\;\frac{\sin\frac{M+1}{2}\alpha}{\sin\frac{\alpha}{2}}\;\cos\left(\frac{M\alpha}{2}+\frac{N\beta}{2}\right)$$

(Art. 327)

347 The sum of the series is given in Art 333 Putting $n\beta=2\pi$ in the result we get for the sum

$$\frac{\cos\alpha-x\cos\left(\alpha-\frac{2\pi}{n}\right)-x^n\left\{\cos\alpha-x\cos\left(\alpha-\frac{2\pi}{n}\right)\right\}}{1-2x\cos\beta+x^2}$$

$$=\frac{\left\{\cos\alpha-x\cos\left(\alpha-\frac{2\pi}{n}\right)\right\}(1-x^n)}{1-2x\cos\beta+x^2}$$

Since $x^n=1$ this expression $=0$ unless x is such that $1-2x\cos\beta+x^2=0$

Solving this equation, $x=\cos\beta\pm\sqrt{-1}\sin\beta$

In the two cases arising from these values of x the series

$$=\cos\alpha+\cos(\alpha+\beta)\cos\beta+\cos(\alpha+2\beta)\cos 2\beta+$$
$$\pm\sqrt{-1}\{\cos(\alpha+\beta)\sin\beta+\cos(\alpha+2\beta)\sin 2\beta+\quad\}$$

Since $\qquad 2\cos(\alpha+r\beta)\cos r\beta=\cos(\alpha+2r\beta)+\cos\alpha,$

and $\qquad 2\cos(\alpha+r\beta)\sin r\beta=\sin(\alpha+2r\beta)-\sin\alpha,$

the sum required

$$=\frac{1}{2}\{n\cos\alpha+\cos\alpha+\cos(\alpha+2\beta)+\cos(\alpha+4\beta)+\quad\}$$
$$\pm\frac{1}{2}\sqrt{-1}\{-n\sin\alpha+\sin\alpha+\sin(\alpha+2\beta)+\sin(\alpha+4\beta)+\quad\}$$
$$=\frac{n}{2}(\cos\alpha\mp\sqrt{-1}\sin\alpha) \qquad \text{(Art 328)}$$

348 The coefficient of a^κ

$$=\cos\frac{\kappa\pi}{n}-\cos\frac{2\kappa\pi}{n}+\cos\frac{3\kappa\pi}{n}-\quad \text{to } 2n \text{ terms}$$

In the result of Art 330 put $\alpha=\beta=\frac{\kappa\pi}{n}$, the sum of the above series is therefore

$$\frac{\sin\left(\frac{2n+1}{2}\ \frac{\kappa\pi}{n}\right)\sin\kappa\pi}{\cos\frac{\kappa\pi}{2n}}$$

This expression $=0$ (since $\sin\kappa\pi=0$), unless $\cos\frac{\kappa\pi}{2n}=0$, that is, unless κ is an odd multiple of n

If κ is an odd multiple of n each term of the above series is equa to -1, and the sum is therefore $-2n$

Hence the sum of the given series

$$=-2n\{a^n+a^{3n}+a^{5n}+\quad\}$$
$$=\frac{2na^n}{a^{2n}-1}$$

349 $\frac{1}{2}\operatorname{cosec}^2\alpha\operatorname{cosec}2\alpha-\frac{1}{2}\cot^2\alpha\cot 2\alpha$

$$=\frac{1-\cos^2\alpha\cos 2\alpha}{2\sin^2\alpha\sin 2\alpha}=\frac{1-(\cos 2\alpha+\sin^2\alpha)\cos 2\alpha}{2\sin^2\alpha\sin 2\alpha}$$
$$=\frac{\sin^2 2\alpha-\sin^2\alpha\cos 2\alpha}{2\sin^2\alpha\sin 2\alpha}=\frac{2\sin 2\alpha\sin\alpha\cos\alpha-\sin^2\alpha\cos 2\alpha}{2\sin^2\alpha\sin 2\alpha}$$
$$=\cot\alpha-\frac{1}{2}\cot 2\alpha$$

Hence we have

$$\frac{1}{2}\operatorname{cosec}^2\alpha\operatorname{cosec}2\alpha=\frac{1}{2}\cot^2\alpha\cot 2\alpha+\cot\alpha-\frac{1}{2}\cot 2\alpha,$$

$$\frac{1}{2^2}\operatorname{cosec}^2 2\alpha\operatorname{cosec}2^2\alpha=\frac{1}{2^2}\cot^2 2\alpha\cot 2^2\alpha+\frac{1}{2}\cot 2\alpha-\frac{1}{2^2}\cot 2^2\alpha,$$

$$\frac{1}{2^3}\operatorname{cosec}^2 2^2\alpha\operatorname{cosec}2^3\alpha=\frac{1}{2^3}\cot^2 2^2\alpha\cot 2^3\alpha+\frac{1}{2^2}\cot 2^2\alpha-\frac{1}{2^3}\cot 2^3\alpha,$$

$$\cdot$$

$$\frac{1}{2^n}\operatorname{cosec}^2 2^{n-1}\alpha\operatorname{cosec}2^n\alpha=\frac{1}{2^n}\cot^2 2^{n-1}\alpha\cot 2^n\alpha+\frac{1}{2^{n-1}}\cot 2^{n-1}\alpha-\frac{1}{2^n}\cot 2^n\alpha$$

Therefore by adding, the sum required

$$=S+\cot\alpha-\frac{1}{2^n}\cot 2^n\alpha$$

350 By the theorem for the sum of the homogeneous products of n dimensions of x, y, z (see Smith's Algebra, Art 300, Ex 3) we have

$$-\frac{x^{n+2}(y-z)+y^{n+2}(z-x)+z^{n+2}(x-y)}{(y-z)(z-x)(x-y)}=\Sigma x^p y^q z^r \qquad (1),$$

where p, q, r are positive integral quantities such that $p+q+r=n$

Put $x=\cos\alpha+\sqrt{-1}\sin\alpha$, $y=\cos\beta+\sqrt{-1}\sin\beta$, $z=\cos\gamma+\sqrt{-1}\sin\gamma$,

$$x^p y^q z^r=\Sigma(\cos p\alpha+\sqrt{-1}\sin p\alpha)(\cos q\beta+\sqrt{-1}\sin q\beta)(\cos r\gamma+\sqrt{-1}\sin r\gamma)$$

$$=\Sigma\cos(p\alpha+q\beta+r\gamma)+\sqrt{-1}\,\Sigma\sin(p\alpha+q\beta+r\gamma) \qquad (2)$$

Now

$$y-z=2\sqrt{-1}\sin\frac{\beta-\gamma}{2}\left(\cos\frac{\beta+\gamma}{2}+\sqrt{-1}\sin\frac{\beta+\gamma}{2}\right) \qquad \text{(Ch xix Ex 10)}$$

Therefore the left-hand side of equation (1)

$$=\frac{1}{4\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\beta}{2}}$$

$$\left[-\frac{\{\cos(n+2)\alpha+\sqrt{-1}\sin(n+2)\alpha\}\sin\frac{\beta-\gamma}{2}}{\cos\left(\frac{\alpha+\beta}{2}+\frac{\alpha+\gamma}{2}\right)+\sqrt{-1}\sin\left(\frac{\alpha+\beta}{2}+\frac{\alpha+\gamma}{2}\right)}+\text{two similar expressions}\right]$$

$$=\frac{1}{4\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\beta}{2}}.$$

$$\left[\left\{\cos\left(\overline{n+1}\alpha-\frac{\beta+\gamma}{2}\right)+\sqrt{-1}\sin\left(\overline{n+1}\alpha-\frac{\beta+\gamma}{2}\right)\right\}\sin\frac{\beta-\gamma}{2}\right.$$

$$\left.+\text{two similar expressions}\right] \qquad (3)$$

Equating the imaginary parts in (2) and (3) we obtain the result required

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